Triple integrals

\[ \text{solid } E \text{ (filled in 3-dim object)} \]
\[ f(x, y, z) \text{ - function of 3 variables defined on } E \]

(examples: \( f(x, y, z) \) - density of the material in the room, \( T(x, y, z) \) - temperature, solid = all the air)

Define \( \iiint_{E} f(x, y, z) \, dV \) \text{ with respect to volume}.

**Definition:** using Riemann sums:

Chop \( E \) (approximately) into small boxes.

\[ \sum_{ijjk} f(x^*, y^*, z^*) \Delta x \Delta y \Delta z \text{ volume of the box} \]

Riemann sum for our integral.

(when \( f \) is continuous)

Riemann sums have a limit as the boxes get smaller

the limit is called \( \iiint_{E} f(x, y, z) \, dV \).
To compute: set up as an iterated integral

(1) the shape of $E$ is encoded in the limits of integration.

Example: Find the average $A_f(x_1, x_2) = x + y - z$
over the tetrahedron bounded by the coordinate planes and the plane $3x + 2y + z = 6$

Note: average of a function:
- over a 3rd solid: $\frac{1}{\text{Vol}(E)} \iiint_E f(x, y, z) \, dx \, dy \, dz$
- over a 2nd region: $\frac{1}{\text{area}(D)} \iint_D f(x, y) \, dx \, dy$
- over an interval $[a, b]$: $\frac{1}{b-a} \int_a^b f(x) \, dx$.
1. Volume of E: 2 ways: 1) think of $E$ as the solid under the graph of $z = h(1,y)$
or 2) $\iiint_E 1 \, dV$.

Drawing planes:
- $3x + 2y + z = 6$
- look for intercepts with axes
- think of whether the plane is parallel to something.

$3x = 6 \quad (x-axis)$
$2y = 6$
$z = 6$

**Our plane:**
graph of $z = 6 - 2y - 3x$.

**Our solid:** under the graph of $z = 6 - 2y - 3x$

over the triangle
\[ V = \iiint_T (6-3x-2y) \, dA \]

\[ = \int_0^2 \int_{3-3/2x}^{3-3/2} \int_0^{2-3/2x} (6-3x-2y) \, dy \, dx \]

If we were computing the volume by a triple integral, the plane should be the red triangle \( T \) evaluated from inside out.

\[ = \int_0^2 \int_0^{3-3/2x} (6-3x-2y) \, dy \, dx \]

as above.
We have our integral:
\[ \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} 1 \, dz \, dy \, dx \]

inside an integral with respect to x, so can use x for limits, but outside limits have to be numbers.

For the average:
\[ \text{Average } (f) = \frac{1}{V} \int \int \int (x + y - z) \, dz \, dy \, dx \]

see above. From the problem:

\[ = \frac{1}{V} \int_0^2 \int_0^{3-\frac{3}{2}x} (x^2 + y^2 - \frac{1}{2}z^2 \bigg|_0^{6-3x-2y}) \, dy \, dx \]

\[ = \frac{1}{V} \int_0^2 \int_0^{3-\frac{3}{2}x} (x(6-3x-2y) + y(6-3x-2y) - \frac{1}{2}(6-3x-2y)^2) \, dy \, dx \]