Today: Integrals over regions.

1) Interchanging the order of integration.

Example: \( \int_{-2}^{0} \int_{-2}^{2x} e^{y^2} \, dy \, dx \) - evaluate this integral.

Catch: \( e^{y^2} \): its antiderivative does not have a formula in terms of powers, exponentials, (elementary functions).

Try changing the order!

\[ y = 2x \]

\[ y = -2 \]

Answer:

\[ \int_{-2}^{0} \int_{\frac{y}{2}}^{1} e^{y^2} \, dx \, dy \]

Recall: \( y = 2x \)

Solve for \( x \):

\[ x = \frac{y}{2} \]

(want \( x \) to be a function of \( y \))
\[ = \int_{-2}^{0} \int_{-\sqrt{y}}^{\sqrt{y}} e^{y^2} \, dx \, dy = \int_{-2}^{0} e^{y^2} \left( 0 - \frac{y}{2} \right) \, dy \]

\[ = -\frac{1}{2} \int_{-2}^{0} y \cdot e^{y^2} \, dy = -\frac{1}{2} \int_{0}^{4} \frac{1}{2} e^{u} \, du \]

\[ \text{this helps! Now we can do substitution: } u = y^2 \]
\[ du = 2y \, dy \]

\[ = \frac{1}{4} \int_{0}^{4} e^{u} \, du = \frac{1}{4} (e^{4}-1) \]

Note: Variation of this example: \( \text{improper integral} \):

\[ \int_{-1}^{3} \int_{-2}^{2x} \frac{e^{y}}{y} \, dy \, dx \]

\( e^{y} \) cannot be integrated in elementary functions.

If we switch the order of integration, get extra "y", all seems fine.

But this solution would be incorrect!

But this integral is improper \( \left( \frac{e^{y}}{y} \to -\infty \text{ at } y \to 0 \right) \)

Integral doesn't converge you cannot change the order!
Polar coordinates

- So far, we can deal with domains bounded by graphs of functions:

\[
\int_{a}^{b} \int_{g(x)}^{h(x)} f(x,y) \, dy \, dx
\]

or

\[
\int_{c}^{d} \int_{h(y)}^{g(y)} f(x,y) \, dx \, dy
\]

- What if the domain is bounded by arcs of circles? It might be inconvenient to deal with the functions:

\[
\iint_{D} f(x,y) \, dA
\]

- Can be done in Cartesian (xy-) coordinates, but complicated.
Polar coordinates (Read 9.4 ) and 13.3

**idea:**

A point on the plane can be specified by:
- its \((x,y)\) coords.
- distance from \((0,0)\) and direction:
  - \(\theta\) - angle from
    - the positive \(x\)-axis
  - \(r\) = distance from \((0,0)\)

\(0 \leq \theta \leq 2\pi\)
\(r \geq 0\).

**Example:** \((-1,-1)\) in polar coordinates has expressions:

\[
\theta = \frac{5\pi}{4},
\]
\[
r = \sqrt{2}
\]

**Conversion formulas**

\[
X = r\cos\theta
\]
\[
y = r\sin\theta
\]
\[
r = \sqrt{x^2 + y^2}
\]

\[
\theta = \arctan \frac{y}{x}
\] if you are in the upper half plane
\[
\theta = \pi + \arctan \frac{y}{x}
\] if you are in the lower half plane.
Area in polar coordinates

(Want to change (x, y) to polar: (r, θ) in our double integrals).

For this, need to express "dA" in polar coords:

Recall: the det. of integral was based on cutting the domain into small rectangles.

Now we want to cut into "wedges":

\[
\frac{\Delta A}{\Delta x} = \frac{\Delta x \Delta y}{\Delta x} = \Delta y
\]

\[
\Delta A = \Delta x \Delta y
\]

Need to express \( \Delta A \) in terms of \( r, \Delta r, \Delta \theta \).

Turns out: \( \Delta A \approx r \cdot \Delta r \cdot \Delta \theta \)