Worksheet 16: Triple integrals in cylindrical coordinates

1. Find the $z$-coordinate of the centroid of the solid that is bounded by the two paraboloids: $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

That: Think of it as $\iiint (f \, dV) \, dA$  

Want: only set up the numerator:

\[
\bar{z} = \frac{1}{M} \iiint z \, dV
\]

set up.

Step 1: Need the intersection.

\[
z = x^2 + y^2 = 8 - x^2 - y^2
\]

\[
\begin{align*}
2(x^2 + y^2) &= 8 \\
x^2 + y^2 &= 4
\end{align*}
\]

- circle in the plane $z = 4$
Step 2: Write the integral as a single integral inside a double:

\[ \iiint \left( \int_{2=2x^2+y^2}^{z=8-x^2-y^2} z \, dz \right) \, dA \]

What is \( D \)? - projection of our solid to the \( xy \)-plane.
Given by: \( x^2 + y^2 = 4 \) (see Step 1).

Now: set up \( \iiint \, dA \) in polar coordinates.

\[ D: x^2 + y^2 \leq 4, \]
\[ \text{in polar: } r \leq 2 \]

Get:

\[ \int_0^{2\pi} \int_0^2 \left( \int_{r^2}^{8-r^2} z \, dz \right) \, r \, dr \, d\theta = 1 \]

Note: we had \( x^2 + y^2 \leq z \leq 8 - x^2 - y^2 \)
in polar:
\[ r^2 \leq z \leq 8 - r^2 \]
\[ M = \int_0^{2\pi} \int_0^2 \left( \int_{r^2}^{8-r^2} 1 \, d\theta \right) r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^2 \left( 8 - r^2 - r^2 \right) \cdot r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^2 \left( \frac{8 - r^2}{r} \right) \, dr \, d\theta \]

\[ = \int_0^{2\pi} \left( \frac{8 - r^2}{r} \right) \, dr \]

\[ \bar{z} = \frac{1}{M} \cdot \overline{I} \quad \text{from prev. page.} \]

\[ = \int_0^{2\pi} \int_0^2 \left( 8 - r^2 - r^2 \right) \cdot r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^2 \left( \text{function of } r \right) \, dr \, d\theta \]

\[ = 2\pi \int_0^2 \left( 8 - 2r^2 \right) \cdot r \, dr \]

\[ = 2\pi \int_0^2 \left( 8 - 2r^2 \right) \, dr \]
We could also (just for practice) do it in the other order:

\[
\int \int_{\text{over the cross-section with fixed } z} \frac{2}{\pi z} \, dA \, dz =
\]

- red disk when \( z \leq 4 \)
- green disk when \( 4 \leq z \leq 8 \).

What is the radius?

red: \( z = x^2 + y^2 \)
\( 0 \leq r \leq \sqrt{z} \)

\[
= 4 \left( \int_0^{\sqrt{4}} \int_0^{\sqrt{z}} \right. 1 \cdot r \, dr \, d\theta \left. \right) \, dz
\]

\[
+ \int_4^8 \int_0^{\sqrt{8-z}} r^2 \, dr \, d\theta \, dz
\]

Note: When computing total mass, you could get away with integrating over just the bottom and multiply by 2. But when integrate \( z \), cannot.
Cylindrical coordinates:

1) Leave $z$ alone, convert $x,y$ to polar: $x = r \cos \theta$, $y = r \sin \theta$.

2) Leave $y$ alone, convert $x,z$ to polar: $x = r \cos \theta$, $z = r \sin \theta$.

3) Leave $x$ alone, convert $y,z$ to polar: $y = r \cos \theta$, $z = r \sin \theta$.

$$dV = r \, dr \, d\theta \cdot \begin{vmatrix} dx \\ dy \end{vmatrix}$$

← whichever variable was unchanged.
Spherical coordinates

Conversion formulas:

\[ z = \rho \cos \varphi \]
\[ r = \rho \sin \varphi \]

Then:

\[ x = (\rho \sin \varphi) \cos \theta \]
\[ y = (\rho \sin \varphi) \sin \theta \]

Defining points in space:

\((x, y, z) \leftrightarrow (\rho, \varphi, \theta)\)

- \(\rho\): longitude
- \(\varphi\): latitude
- \(z\): distance from the origin

Intersection of xy-plane with the vertical plane containing \((x, y, z)\).

\[ r = \text{distance from } (x, y, z) \text{ to the } z \text{-axis.} \]
\[ r^2 = x^2 + y^2 \]
Evaluation of the integral and center of mass from Problem 1:

\[ n = \int_0^{2\pi} \int_0^2 \int_0^{8-r^2} 1 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \frac{z}{r^2} \cdot \frac{8-r^2}{r} \, r \, dr \, d\theta \]

\[ = 2\pi \int_0^2 (8-r^2-r^2) \cdot r \, dr = 2\pi \int_0^2 8r - 2r^3 \, dr \]

\[ = \left[ 4\pi r^2 - \frac{2}{4} r^4 \right]_0^2 = 32\pi - 16\pi = 16\pi. \]

Centre of mass (the z-coordinate):

\[ \bar{z} = \frac{1}{M} \int_0^{2\pi} \int_0^2 \int_0^{8-r^2} \left( \int \frac{z}{r^2} \cdot 1 \, dz \right) \, r \, dr \, d\theta \]

\[ = \frac{1}{M} \int_0^{2\pi} \int_0^2 \frac{z^2}{2} \cdot \frac{8-r^2}{r^2} \, r \, dr \, d\theta \]
\[
\frac{2\pi}{2M} \int_0^2 \left( \frac{(8-r^2)^2 - r^4}{64 - 16r^2 + r^4 - r^4} \right) \cdot r \, dr = \frac{\pi}{M} \left( \frac{64 \frac{r^2}{2}}{16} \right) - 16 \frac{r^4}{4} \bigg|_0^2
\]

\[
= \frac{\pi}{M} \cdot (128 - 64) = \frac{64\pi}{16\pi} = 4
\]

plug \[ M=16\pi \text{ from above} \]

**Answers:** \[ M=16\pi, \bar{z}=4 \]

**Reality check:** the solid is symmetric about the plane \[ z=4 \], so we could have guessed that its centre of mass has to lie on this plane.

**Note:** when computing the mass (volume), some people just computed the mass of the bottom part and multiplied by \[ 2 \]. This is correct, but this doesn’t work when computing \[ \bar{z} \] (because now you are integrating the function \[ z \] over this solid, and its values are not the same on the top and on the bottom).