Let \( R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\} \) be a rectangle in \( \mathbb{R}^2 \), and let \( f(x, y) \) be some continuous function whose graph lies above the \( x \)-\( y \) plane. We might ask: what is the volume of the shape created by its graph?

We denote this value by

\[
\iiint_{R} f(x, y) \, dA \quad \text{(or } \iint_{R} f(x, y) \, dA) \quad ,
\]

and compute it as follows: if \( f(x, y) = \text{const} \), then the shape is a rectangular prism, and its volume is

\[
\iiint_{R} M \, dA = M(b-a)(d-c)
\]

Otherwise, we partition \( R \) into a collection of smaller rectangles, approximating the shape of the graph under each one as a rectangular prism. That is: we choose

\[a = x_0 < x_1 < \cdots < x_n = b\] and \[c = y_0 < y_1 < \cdots < y_m = d\],

as well as "sample points"

\[x_i^* \in (x_{i-1}, x_i)\] and \[y_{j}^* \in (y_{j-1}, y_j)\], \( i = 1, \ldots, n \), \( j = 1, \ldots, m \),

and approximate the area by the Riemann sum

\[
\iint_{R} f(x, y) \, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_{j}^*) (x_i - x_{i-1})(y_j - y_{j-1})
\]

\[= \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_{j}^*) \Delta x_i \Delta y_j\]

\text{height of graph shape} \quad \text{length of graph shape} \quad \text{width of graph shape}
The value of the volume we care about is the limit, as $N_1, N_2 \to \infty$ and each $\Delta x_i, \Delta y_j \to 0$, of the Riemann sum approximation. Briefly, if we assume $N_1 = N_2$, we write

$$
\int_{R} f(x, y) \, dA = \lim_{N \to \infty} \sum_{i,j} f(x_i^*, y_j^*) \Delta x_i \Delta y_j
$$

A few things are clear from the definition:

**FACTS:** Let $R \subseteq \mathbb{R}^2$ be a rectangle, $R = [a, b] \times [c, d]$.

1. If $a, b, c, d \in \mathbb{R}$ and $f, g$ are continuous, then

   $$
   \int_{R} a f(x, y) + b g(x, y) \, dA = a \int_{R} f(x, y) \, dA + b \int_{R} g(x, y) \, dA
   $$

2. If $0 \leq f(x, y) \leq g(x, y)$, then

   $$
   0 \leq \int_{R} f(x, y) \, dA \leq \int_{R} g(x, y) \, dA.
   $$

**Example:** Let $R = [0, 1] \times [0, 1]$ and $f(x, y) = xy$. Find

$$
\int_{R} f(x, y) \, dA.
$$

**Solution:** We cut up the intervals $[0, 1]$ into $N$ equal pieces each, so that

$$
x_i = \frac{i}{N}, \quad y_j = \frac{j}{N}, \quad \Delta x_i = \Delta y_j = x_i - x_{i-1} = y_j - y_{j-1} = \frac{1}{N}.
$$

To keep things simple, take $x_i^* = x_i$, $y_j^* = y_j$. Then

$$
\sum_{i,j} f(x_i^*, y_j^*) \Delta x_i \Delta y_j = \frac{1}{N^2} \sum_{i,j} \left( \frac{i}{N} \right) \left( \frac{j}{N} \right) = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{i}{N} \right) \frac{1}{N} = \frac{1}{N^2} \sum_{i=1}^{N} \frac{i}{N} = \frac{1}{N^2} \left( \frac{N(N+1)}{2} \right) = \frac{1}{N^2} \left( \frac{N^2 + N}{2} \right) = \frac{1}{4}.
$$

It follows that

$$
\int_{R} f(x, y) \, dA = \frac{1}{4}.
$$
Of course, we don't actually compute our integrals this way. Instead, we observe that

\[ \iint_R f(x, y) \, dA = \lim_{N \to \infty} \lim_{N' \to \infty} \sum_{i=1}^N \sum_{j=1}^{N'} \delta_i \delta_j f(x_i^*, y_j^*) \Delta x_i \Delta y_j \]

\[ = \lim_{N \to \infty} \sum_{i=1}^N \left( \int_c^d f(x_i^*, y) \, dy \right) \Delta x_i \]

\[ = \lim_{N' \to \infty} \sum_{j=1}^{N'} \left( \int_a^b f(x, y_j^*) \, dx \right) \Delta y_j . \]

We introduce some notation: let

\( f^*(y) = \int_a^d f(x, y) \, dx \) and \( f^*(x) = \int_c^b f(x, y) \, dy \).

Then we have shown that

\[ \iint_R f(x, y) \, dA = \int_a^b f^*(x) \, dx = \int_c^d f^*(y) \, dy = \int_a^b \left( \int_c^d f(x, y) \, dy \right)dx . \]

Remarks: (1) When we write \( f^*(y) = \int_a^d f(x, y) \, dx \), we mean that we integrate the terms involving \( y \) as though \( y \) were constant. After all, for any fixed \( y \), we should think of \( f(x, y) \) as a function of \( x \).

(2) Note that

\[ \int_a^b \left( \int_c^d f(x, y) \, dy \right)dx \neq \int_c^d \left( \int_a^b f(x, y) \, dx \right)dy . \]

Example: Let \( R = [-2, 1] \times [1, 3] \) and \( f(x, y) = x^2/y^2 \). Then

\[ f^*(x) = \int_1^3 f(x, y) \, dy = \int_1^3 x^2/4 \, dy = x^2 \int_1^3 y^{-2} \, dy = x^2 \left[ -y^{-1} \right]_{y=1}^{3} = x^2 \left( -1 + 1 \right) = \frac{2x^2}{3} . \]

(see above)
At the same time,
\[ f(x,y) = \int_{-2}^{2} \int_{-2}^{2} f(x,y) \, dx \, dy = \frac{1}{2} \int_{-2}^{2} x^2 \, dx = \frac{1}{2} \left[ \frac{1}{3} x^3 \right]_{x=-2}^{x=2} = \frac{1}{3} \left( 1 - (-8) \right) \]
\[ = \frac{3}{3} = 1. \]

Finally, we have
\[ \int_{-2}^{2} \frac{2x^2}{3} \, dx = \left[ \frac{2x^3}{9} \right]_{x=-2}^{x=2} = \frac{2}{9} - \left( -\frac{16}{9} \right) = 2 \]
and
\[ \int_{-2}^{2} \frac{3}{7} \, dy = \left[ \frac{3y^2}{7} \right]_{y=-2}^{y=2} = -1 - (-3) = 2. \]

The technique we have just seen is called iterated integration. Whenever \( f(x,y) \) is continuous on the closed rectangle \([a,b] \times [c,d] \), we have
\[ \int_{R} f(x,y) \, dA = \int_{a}^{b} \left( \int_{c}^{d} f(x,y) \, dy \right) \, dx = \int_{c}^{d} \left( \int_{a}^{b} f(x,y) \, dx \right) \, dy. \]

The order of the "product" \( dx \, dy \) or \( dy \, dx \) is significant and cannot be switched freely!

Example: Let \( f(x,y) = e^{x+y} \) and let \( R = [0,1] \times [0,2]. \)

Example: Let \( f(x,y) = x^2 \cos y + y^2 \), and let \( R = [0,1] \times [0,\pi]. \)

Then
\[ \int_{R} \int_{0}^{\pi} (x^2 \cos y + y^2) \, dy \, dx = \int_{0}^{\pi} \left[ \frac{1}{2} x^2 \sin y + \frac{1}{3} y^3 \right]_{y=0}^{y=\pi} \, dx = \int_{0}^{\pi} \left( \frac{1}{2} \pi^2 \right) \, dx = \frac{1}{2} \pi^3. \]

(cont.)
Or, we could choose to integrate the opposite order:

\[ \iint_R f(x, y) \, dA = \int_0^\pi \int_0^{\frac{1}{3} \cos y + y^2} x^2 \cos y + y^2 \, dx \, dy = \int_0^\pi \left[ \frac{1}{3} x^3 \cos y + x y^2 \right]_x=0^1 \, dy \]

\[ = \int_0^\pi \left( \frac{1}{3} \cos y + y^2 \right) \, dy = \left[ \frac{1}{3} \sin y + \frac{1}{3} y^3 \right]_0^\pi = \frac{\pi^3}{3}. \]

In the case of rectangles, it doesn't matter so much if we integrate \( dx \, dy \) or \( dy \, dx \). As we will see next time, it will matter much more when we integrate over more interesting regions.