We continue with Julia’s example from Friday. Let’s recall the setup: we wanted to find the absolute extrema of the function

$$f(x, y) = 2x + xy$$

on the domain

$$D = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4 \}.$$ 

We already know how to do this on the interior

$$\text{int}(D) = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4 \},$$

and we now learn a new method for the boundary, namely the method of Lagrange multipliers. We will find the extrema of $$f(x, y)$$ subject to the constraint

$$x^2 + y^2 = 4 = 0$$

by assuming that such points occur as critical numbers of the function

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

Then $$\nabla L = 0$$ is equivalent to the system of equations

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad g(x, y) = 0.$$ 

In our case, we get

$$1) \ 2x + y = 2\lambda x, \quad 2) \ x = 2\lambda y, \quad 3) \ x^2 + y^2 = 4.$$ 

We plug (1) into (3), and then (2) into (3), to get

$$4) \ y^2(1 + 4\lambda^2) = 4, \quad \text{and} \quad 5) \ x^2(1 + 4\lambda^2) = 8\lambda x.$$ 

Isolating for $$y^2$$ and $$x^2$$, respectively, and plugging back into (3) gives

$$\frac{8\lambda x}{1 + 4\lambda^2} + 4 = 4 \quad \text{hence} \quad 2\lambda x = 4\lambda^2.$$ 

If $$\lambda \neq 0$$ then (2) implies $$x = 0$$ so $$f(x, y) = 0$$. Otherwise, $$\lambda = 0$$ so

$$6) \ x = 2\lambda.$$ 

(2) and (6) together imply $$y = 1$$; then (4) implies $$\lambda^2 = \frac{3}{4}.$$

(See over)
Finally, \( x = \pm \sqrt{3} \) and \( y = 1 \) give, from (2), \( x = \pm \sqrt{3} \).

We have \( f(\pm \sqrt{3}, 1) = \pm 3 \sqrt{3} \). These are, respectively, the maximum and minimum value of \( f \) on \( D \); and, hence, on all of \( \mathbb{R} \).

The constant \( \lambda \) from the above calculation is called the Lagrange multipliers. Notice that its value doesn’t actually affect the max or min of \( f \).

Why it works: The assumption we make is that, when \( f(x, y) \) is at a maximum (or minimum) subject to the constraint \( g(x, y) = 0 \), we have that \( \nabla f \) and \( \nabla g \) are parallel (similarly for \( f(x, y, z) \)).

Geometrically: suppose \( f(x, y) = M \) is the desired maximum. Then we’re claiming that the tangent lines to the level curves \( f = M \) and \( g = 0 \) are parallel at these points. If not, then (see picture, left) we could move along \( g = 0 \) toward \( \nabla f \), and therefore increasing the value of \( f \).

Algebraically: if at the maximum (say), \( \mathbf{u} \in \mathbb{R}^2 \) is tangent to the level curve \( g = 0 \), then \( \nabla g \cdot \mathbf{u} = 0 \).

Because \( g \) is constant on the curve, and \( \nabla f \cdot \mathbf{u} = 0 \) because the value of the restriction of \( f \) to \( g = 0 \) is assumed to be extreme here. Thus \( \nabla f \cdot \mathbf{u} = \nabla g \cdot \mathbf{u} = 0 \iff \mathbf{u} \perp \nabla f \) and \( \mathbf{u} \perp \nabla g \).

The minimum/tangent planes of \( f \) and \( g \) both therefore are perpendicular to \( \mathbf{u} \), and so are parallel to each other, \( \nabla f \parallel \nabla g \).

In practice, our task will be to find extrema of \( f \) given the constraint \( g = 0 \), and to do this by finding local extrema of \( L(f, g, X) = f - \lambda g \).

This works when \( f, g \) are functions of 2, or 3, variables.
Example: Find the maximum and minimum of \( f(x, y, z) = xyz \) on the ellipsoid
\[
x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1.
\]

Solution: Let \( g(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 1 \), and let \( L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z) \). Then
\[
\nabla L = \begin{pmatrix}
\frac{2x}{1} & \frac{2y}{4} & \frac{2z}{9} \\
0 & \frac{y}{2} & 0 \\
0 & 0 & \frac{z}{3}
\end{pmatrix}
\]
and so \( \nabla L = 0 \) only if

(1) \( yz = 2\lambda x \), (2) \( x^2 = 9y/2 \), (3) \( xy = 2\lambda z/9 \), (4) \( x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1 \).

First of all: if \( \lambda = 0 \) then (1)–(4) imply that exactly one of \( x, y, z \) is nonzero, so \( f(x, y, z) = 0 \).

Otherwise we can assume \( \lambda \neq 0 \), eqn (4) implies one of \( x, y, z \) nonzero, then by (1), (2), or (3) respectively we have that all \( x, y, z \) are nonzero, so we can divide by these quantities.

Dividing eqn (2) by (1) gives
\[
\frac{x^2}{y} = \frac{y}{2} \iff \frac{x}{y} = \frac{y}{2} \iff x^2 = \frac{y^2}{4}.
\]
Similarly, dividing (3) by (1) gives
\[
\frac{xy}{yz} = \frac{2\lambda z}{18\lambda x} \iff \frac{x}{z} = \frac{z}{9} \iff x^2 = \frac{z^2}{9}.
\]
Thus by (4) we have
\[
1 = x^2 + \frac{y^2}{4} + \frac{z^2}{9} = x^2 + x^2 + x^2 = 3x^2, \ \text{so} \ x^2 = \frac{1}{3}.
\]
From this we deduce \( y^2 = 4x^2 = 4/3 \) and \( z^2 = 9x^2 = 3 \). Thus
\[
x = \pm \frac{1}{\sqrt{3}}, \ y = \pm \frac{2}{\sqrt{3}}, \ z = \pm \sqrt{3},
\]
and so
\[
f(x, y, z) = \left( \pm \frac{1}{\sqrt{3}} \right) \left( \pm \frac{2}{\sqrt{3}} \right) \left( \sqrt{3} \right) = \pm \frac{2}{\sqrt{3}}.
\]
(4) The positive value is our desired maximum value of $f(x, y, z)$, and the negative value its minimum. Our calculations show that

$$f(x, y, z) = \frac{2}{\sqrt{3}}$$
when $(x, y, z) = (\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

and that

$$f(x, y, z) = -\frac{2}{\sqrt{3}}$$
when $(x, y, z) = (\frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}), (\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

and that these are the absolute maximum & minimum, respectively, of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

Next time, we will begin our discussion of two-dimensional integrals.