Worksheet 15: Triple integrals

1. Set up, in any order, the integral representing the x-coordinate of the centroid of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, $y + x = 1$, and the parabolic cylinder $z = 1 - x^2$.

Inequalities: $x, y \geq 0$, $z \geq 0$

$\int_{y=0}^{y=1-x} \int_{x=0}^{x=1-y} \int_{z=0}^{z=1-x^2} x \, dz \, dy \, dx$

2. Change the order of integration so that the integral from the previous problem is written as

$\iiint x \, dx \, dy \, dz$

(\text{All inequalities need to be satisfied!})

(see next page for the inequalities defining the solid)
Inequalities:

\[
\begin{align*}
&x \geq 0 \\
&y \geq 0 \\
&z \geq 0
\end{align*}
\]

\[y \leq 1 - x \quad \text{vertical plane } x + y = 1\]
\[z \leq 1 - x^2 \quad \text{parabolic cylinder } z = 1 - x^2\]

Decide on the order.

Convenient to put \( x \) on the outside.

(then we can use \( x \) for limits of the inside integrals)

What are the limits for \( x \)?

- from our partial sketch: \( 0 \leq x \leq 1 \).

or: (if no sketch): \( x > 0 \).

how big can \( x \) get?

we have: \( y \leq 1 - x \).

when \( x \) is the largest, it should meet \( y = 0 \).

looks like \( 0 < x \leq 1 \).

Check against \( z \): \( z = 1 - x^2 \) meets \( z = 0 \)
at \( x = 1 \).

Answer:

\[
\int_0^1 \int_0^{1-x} \int_0^{1-x^2} dy \\ dx
\]

Computing M

\[x \text{ if } x \text{.} \]
Check: outside double integral: \[ \iiint_{V} 1 \, dy \, dx \, dz \]

\[ \int_{0}^{1} \int_{0}^{1-x} dy \, dx \]

← agrees with the bottom face of our solid.

- Looks like no limits depend on \( y \)?
  What does this mean?

Geometrically this says, for every \( x \), the cross section of our solid by the plane \( x = x_0 \), is a rectangle \( (1-x_0) \times (1-x_0^2) \).

Go back to the sketch.

- Can improve the sketch: missing: the intersection of the red parabolic cylinder with the blue plane \( x+y = 1 \).

Look for common points.

Draw the common curve.

Shading: vertical plane by vertical lines.

- We see: cross sections with fixed \( x \) are indeed rectangles:
  square \( [0,1] \times [0,1] \) on the \( yz \)-plane \((x=0)\)
  and get smaller and collapse to a point when \( x = 1 \),

\[ \int_{0}^{1} \int_{1-x}^{1} \int_{0}^{z} \, dy \, dx \, dz \]

\[ z = 1 - x^2 \]

\[ x = \sqrt{1 - z} \]
We need:

\[ y + x \leq 1, \quad x \leq 1 - y. \]

but:

\[ x^2 + 2 \leq 1, \quad \text{so } x \leq \sqrt{1 - 2}. \]

Both have to hold.

There is a competition between \( 1 - y \) and \( \sqrt{1 - 2} \):

Where is \( 1 - y \leq \sqrt{1 - 2} \)?

\[ 1 - y = \sqrt{1 - 2} \]

\[ (1 - y)^2 = 1 - 2 \]

\[ 2 = 1 - (1 - y)^2 \]

\[ y = 1 - \sqrt{1 - 2} \]

Equation of the border of "competing bounds for x".
Our integral splits:

\[ \int_0^1 \int_0^{1-x} \int_{1-x}^{\sqrt{1-x^2}} x \, dx \, dy \, dz + \int_0^1 \int_{1-x}^1 \int_0^{\sqrt{1-x^2}} x \, dx \, dy \, dz \]

\[ = \int_0^1 \int_0^{1-x} \int_0^{\sqrt{1-x^2}} x \, dx \, dy \, dz + \int_0^1 \int_0^{1-x} \int_0^{\sqrt{1-x^2}} x \, dx \, dy \, dz \]