Today: critical points, second derivative test.

Goal: Find max/min values of \( f(x, y) \) or \( f(x, y, z) \) in a region.

\[ \text{ex: } f(x, y) \text{ is some function.} \]

Find max/min of \( f(x, y) \) on the square.

* two sources of difficulty: - \( f(x, y) \)
  - the region.

Today: Step 1: finding critical points of \( f(x, y) \).

Def: \((a, b)\) is a critical point for \( f(x, y) \)
if \( f_x(a, b) = f_y(a, b) = 0 \)
"or at least one of \( f_x, f_y \) is undefined."
(better: \( f \) is not differentiable at \((a, b)\)).

Def: \((a, b)\) is a local max for \( f(x, y) \)
if there is a disc around \((a, b)\) s.t.
\[ f(a, b) \geq f(x, y) \text{ for all } (x, y) \text{ in this disc} \]

Example: Grouse Mtn peak is a local max.
for the altitude as a function of NS/EW coordinates.

Theorem: a local max/min can occur only at a critical point.
Worksheet 7.

Find and classify the critical points of \( f(x,y) = x^3 + x^2y^2 - y^4 \).

Critical point: \( f_x = 0 \) and \( f_y = 0 \).

\[ f_x = 3x^2 + 2xy^2 \]
\[ f_y = 2x^2y - 4y^3 \]

Need to solve:

\[
\begin{align*}
3x^2 + 2xy^2 &= 0 \\
2x^2y - 4y^3 &= 0
\end{align*}
\]

Factor both equations:

\[
\frac{x}{2}(3x + 2y^2) = 0 \\
2y(2x^2 - 2y^2) = 0
\]

\[
(x - \sqrt{2}y)(x + \sqrt{2}y) = 0
\]

\( f_x = 0 \) means:

\[
\begin{align*}
x &= 0 & \text{or} & \quad 3x + 2y^2 &= 0 \\
& & \quad y &= \pm \sqrt{\frac{3x}{2}}
\end{align*}
\]

\( f_y = 0 \):

\[
\begin{align*}
y &= 0 & \text{or} & \quad x - \sqrt{2}y &= 0 \\
& & \quad x + \sqrt{2}y &= 0
\end{align*}
\]

Get: \((0,0)\) and: intersections of the green parabola with the red lines.

Get:

\[
\begin{align*}
y^2 &= -3x/2 & \text{or} & \quad y^2 &= -3x/2 \\
x - \sqrt{2}y &= 0 & \text{or} & \quad x + \sqrt{2}y &= 0
\end{align*}
\]

\[
\begin{align*}
x &= -3 \\
y &= -\frac{3}{\sqrt{2}} \\
x^2 &= -3x \\
x = 0 \text{ or } x = -3
\end{align*}
\]

\[
\begin{align*}
y^2 &= -3x/2 \\
y &= -\frac{x}{\sqrt{2}} \\
x^2/2 &= -\frac{3x}{2}, \text{ again } x = -3 \\
& & \text{but } y &= \frac{3}{\sqrt{2}}
\end{align*}
\]
Critical points: $\,(0, 0), \, (-3, -\frac{3}{12}), \, (-3, \frac{3}{12})\,$

Classification of critical points

- a critical point can be:
  - local max
  - local min
  - saddle point — for functions of 2 variables.
  - neither

- What is a saddle point?

  when you walk along one direction, you find that your point is a local max.
  but if you walk in some other direction, it is a local min! (imagine the Lions)

Why can a local max/min occur only at a critical point?

- whichever way you go, directional derivative at the local max should be 0 (or undefined)

  \[
  \text{D}u = \bar{v} \cdot \bar{u}, \quad \text{so} \quad \bar{v} \bar{u} = 0 \\
  \text{which means} \quad f_x = f_y = 0.
  \]