Today: Meaning of 2-d integrals

- Centre of mass

Definition: Centre of mass of a lamina is a point such that you can balance the lamina supporting it just at that point.

How to find it:

Warm-up:

\[ m_1 \quad \quad \quad m_2 \]

\[ 0 \quad x_1 \quad \quad x_2 \]

Observation: If \( m_1 = m_2 \), then it should be at the midpoint:

\[ \frac{x_1 + x_2}{2} \]

Two masses connected by a weightless rod.
General: \[
\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \bar{x}
\] coordinate of the centre of mass.

Why: the two ends need to have the same moment about axis 3.

Remark: \(\bar{x}\) is "weighted average" of \(x_1\) and \(x_2\).

Another example of weighted average: voting on a board of directors of a company:
10 members of the board; member 1 has \(m_1\) shares
2 has \(m_2\) shares

They are trying to decide what the price of their product should be:
\[
\bar{p} = \frac{m_1 p_1 + \cdots + m_{10} p_{10}}{m_1 + \cdots + m_{10}}
\]
weighted average of their wishes.

\(p_1\) - 1st member wants\(p_1\)
\(p_2\) - 10th member wants\(p_{10}\)
One more warm-up!

$m_1, \ldots, m_n$ - point masses at $(x_i, y_i)$, $1 \leq i \leq n$

rigid weightless connectors.

Centre of mass:

$$
\bar{x} = \frac{m_1 x_1 + \ldots + m_n x_n}{m_1 + \ldots + m_n}
\quad \text{"weighted average" of } x
$$

$$
\bar{y} = \frac{m_1 y_1 + \ldots + m_n y_n}{m_1 + \ldots + m_n}
\quad \text{weighted average of } y
$$

(total mass) in the denominator.
Continuous situation

Density \( f(x,y) \)

Centre of mass:

\[
\bar{x} = \frac{M_y}{M} = \frac{\iint_D f(x,y) \cdot x \, dA}{M}
\]

\[
\bar{y} = \frac{M_x}{M} = \frac{\iint_D f(x,y) \cdot y \, dA}{M}
\]

\[M = \text{total mass} = \iint_D f(x,y) \, dA\]
Worksheet 14: Centre of mass; improper integrals

1. Find the $x$-coordinate of the centre of mass of the lamina that occupies the disk $(x - 1)^2 + y^2 \leq 1$ and has density $\rho(x, y) = \sqrt{x^2 + y^2}$.

In polar:

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

\[ (x - 1)^2 + y^2 = 1 \]

\[ \rho(x, y) = \sqrt{x^2 + y^2} \]

1) Need $M = \text{total mass}$.

\[
M = \iint_D \rho(x, y) \, dA = \iint_D \sqrt{x^2 + y^2} \, dA = \iint_D r \, r \, dr \, d\theta
\]

2) Need to find limits of integration.
To find limits:

\[ M = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \cdot r \, dr \, d\theta \]

\[ = \int_{-\pi/2}^{\pi/2} \left[ \frac{r^3}{3} \right]_0^{2\cos\theta} \, d\theta \]

\[ = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^3\theta \, d\theta \]

\[ = \frac{8}{3} \int_{-\pi/2}^{\pi/2} (1-u^2) \, du \]

\[ u = \sin\theta \]

\[ = \frac{16}{3} \left( 1 - \frac{1}{3} \right) = \frac{32}{9} \]

So, we have

\[ M = 32/9. \]
\[ \bar{x} = \frac{M_y}{M} \]

\[ M_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{r} x \cdot f(x, y) \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{r} r^2 \cos \theta \, dr \, d\theta \]

\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \cdot \frac{r^4}{4} \bigg|_0^1 \, d\theta \]

\[ = \frac{2}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \, d\theta = 4 \int_{-1}^{1} (-u^2) \, du \]

\[ \text{change of variable: } u = \sin \theta \]

\[ \text{then } \, du = \cos \theta \, d\theta \]

\[ \cos^5 \theta = \cos^4 \theta \cdot \cos \theta \, d\theta \]

\[ (\cos^2 \theta)^2 = (1-u^2)^2 = (1-u^2) \]

\[ = 4 \int_{-1}^{1} (1-u^2) \, du = 8 \int_{0}^{1} (-u^2) \, du \]

\[ \text{function is even} \]

\[ = 8 \int_{0}^{1} (1+u^4-2u^2) \, du = 8 \left( 1 + \frac{1}{5} - \frac{2}{3} \right) = \frac{64}{15} \]

\[ \text{Answer: } \bar{x} = \frac{64/15}{32/9} = \frac{6}{5} \]
2. Classic example: the 'total mass' of the Gaussian distribution. Find

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx. \]

**Hint:** we write:

\[ (\int_{-\infty}^{\infty} e^{-x^2} \, dx)^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \int_{-\infty}^{\infty} e^{-y^2} \, dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy. \]

Now change to polar coordinates.

\[ \text{can use integrals over large circles} \]

\[ \text{let } R \to \infty. \]

Will do next time