Worksheet Problem 3 (from last time)

Recall: \( f(x,y) \), linearization near \((a,b)\) is given by the formula

\[
L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)
\]

Tangent plane: graph of \( L(x,y) \).
(to graph of \( f(x,y) \)
at \((a,b, f(a,b))\))

\[z = f(x,y)\]
\[\text{trace on y} = b\]
\[\text{trace on x} = a\]

Solution of Worksheet Problem 3: \( f(x,y) = x^2 - 3xy \).

Plug in \((1,2)\):

\( f(1,2) = 1^2 - 3 \cdot 2 \cdot 1 = -5 \)

So the point \((1,2,-5)\) is on the graph.

\( f_x = 2x - 3y \)
\( f_y = -3x \)

Evaluate at \((1,2)\):

\( f_x(1,2) = 2 - 3 \cdot 2 = -4 \)
\( f_y(1,2) = -3 \)

Answer: tangent plane is
\[
z = -5 + (-4)(x-1) + (-3)(y-2)
\]

Sanity check: does \((1,2,-5)\) lie on the plane?
Discussion of why this method of finding the tangent plane agrees with other methods:

We got: \( z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \)

We also know: tangent vectors to the traces of the graph on \( x=a \) and \( y=b \) are the two vectors making the plane.

We also know:

Slope of the tangent vector to the trace on \( y=b \) is:
\[
\frac{\partial f}{\partial x}(a,b) = f_x(a,b)
\]

(fixed \( y = b \), think \( f \) as a function of \( x \))

Slope of the tangent line to the red curve is:
\[
\frac{\partial f}{\partial y}(a,b)
\]
So we get that: the vector lying in the plane \( y=b \) with slope \( f_x(a,b) \)
and the vector in the plane \( x=a \) with slope \( f_y(a,b) \)
lie in our tangent plane.

These vectors can be expressed as:

\[ \begin{align*}
\mathbf{v}_1 &= \langle 1, 0, f_x(a,b) \rangle \\
\mathbf{v}_2 &= \langle 0, 1, f_y(a,b) \rangle
\end{align*} \]

Then the normal to the tangent plane is:

\[ \mathbf{n} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 0 & f_x(a,b) \\
0 & 1 & f_y(a,b)
\end{vmatrix} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle \]

The equation of the plane:

\[ -f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0. \]

\[ z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]

Got the same formula as before.
The total differential

\[ f(x,y) - f_a \] for 2 variables,

The total differential at \((a,b)\) is:

\[ df = f_x(a,b) \, dx + f_y(a,b) \, dy \]

for \( f(x,y,z) \) = function of 3 variables,

\[ df = f_x(a,b,c) \, dx + f_y(a,b,c) \, dy + f_z(a,b,c) \, dz \]

**Interpretation:**

\[ \Delta f \approx f_x(a,b,c) \, \Delta x + f_y(a,b,c) \, \Delta y + f_z(a,b,c) \, \Delta z \]

small change in \( f \)
that corresponds to a small change in the inputs
(same as linearization)

**Example:** volume of a box, measured length up to 1mm
width up to 2mm, height up to 0.3mm.

Got: \( l = 10 \text{ mm}, \ w = 15 \text{ mm}, \ h = 12 \text{ mm} \).

Estimate the error in the volume. (in \text{mm}^3).

**Answer:** \( V = l \cdot w \cdot h \)

\[ dV = \frac{\partial V}{\partial l} \, dl + \frac{\partial V}{\partial w} \, dw + \frac{\partial V}{\partial h} \, dh \]

approximate error \( \frac{\partial V}{\partial w} = l \cdot h \), \( \frac{\partial V}{\partial h} = l \cdot w \)

\( \frac{\partial V}{\partial l} = w \cdot h \) evaluate: \( \frac{\partial V}{\partial l} \bigg|_{(10,15,12)} \)

Get: \( dV = 180 \cdot 1 + (10 \cdot 12) \cdot 2 + (10 \cdot 15) \cdot 0.3 \)

error in \( l \) error in \( w \) error in \( h \)