Worksheet 1: skew lines in space

Let the lines L_1 and L_2 be given by the parametric equations

$$\mathbf{r_1}(t) = t\mathbf{i} + (1-2t)\mathbf{j} + (2+3t)\mathbf{k},$$
$$\mathbf{r}_2(s) = (3-4s)\mathbf{i} + (2+3s)\mathbf{j} + (1-2s)\mathbf{k}.$$

Question 1: do these lines intersect?

Solution: These two lines intersect if there exist values of t and s such that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. Equating each of the three components, we get three equations, which we try to solve for t and s. If a simultaneous solution exists, the lines intersect; if not, they don't.

$$\begin{cases} t = 3 - 4s \\ 1 - 2t = 2 + 3s \\ 2 + 3t = 1 - 2s. \end{cases}$$

Plugging in $t = 3 - 4s$ from the first equation into the second one, we get $1 - 2(3-4s) = 2 + 3s$, so $5s = 7$, so $s = 7/5$; then $t = 3 - 4s = 3 - 28/5 = -13/5$. So $t = -13/5$, $s = 7/5$ is the only solution to the first two equations; plugging it in, we see that the third equation is not satisfied, so the lines do not intersect.

Note that the direction vector of the first line is $\mathbf{v}_1 = \langle 1, -2, 3\rangle$, and the direction vector of the second line is $\mathbf{v}_2 = \langle -4, 3, -2\rangle$. Since as we see, $\mathbf{v}_1$ and $\mathbf{v}_2$ are not proportional, the lines are not parallel. Thus, the two lines are skew.

Question 2: Find an equation of the plane containing the line L_2 and parallel to the line L_1.

Any plane parallel to L_1 has to have a normal vector that is perpendicular to $\mathbf{v}_1$. Similarly, if it contains L_2, then its normal vector has to be perpendicular to $\mathbf{v}_2$. Thus, a normal vector to our plane should be perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$. To find such a vector, we use the cross product:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -4 & 3 & -2 \end{vmatrix} = \langle -5, -10, -5\rangle.$$

Now, take any point on L_2, and use that point and the normal vector we just found to write an equation of this plane. Let us use the point $P = (3, 2, 1)$ that corresponds to $s = 0$. Let's rescale $\mathbf{n}$ by $\frac{1}{-5}$ so that it's easier to deal with; then we get the equation of a plane with the normal vector $\mathbf{w} = \frac{1}{-5}\mathbf{n} = \langle 1, 2, 1\rangle$ and containing P:

$$(x-3) + 2(y-2) + (z-1) = 0.$$

Question 3: Find the distance between the lines L_1 and L_2. To find this distance, all we need to do is find the distance from any point on L_1 to the plane from the previous question (imagine the picture, with the plane containing L_2 being the floor, and L_1 – any line on the ceiling. Since the distance between the floor and the ceiling is always the same, you see that it doesn't matter which point on L_1 we take. Imagine this picture and think about it!)

Take the point on L_1 corresponding to $t = 0$ – this is the point $A = (0, 1, 2)$. Now find the distance from the point $(0, 1, 2)$ to the plane $(x - 3) + 2(y - 2) + (z - 1) = 0$. Recall that to do it, we just have to take any point in the plane (we take P), and then find the magnitude of the projection (or component) of the vector $\overline{AP}$ onto $\mathbf{n}$ (which is the same as the magnitude of the projection onto $\mathbf{w}$). We have: $\overline{AP} = \langle 3, 1, -1\rangle$. Then

$$|\mathrm{comp}_{\mathbf{w}}\overline{AP}| = \frac{|\langle 3, 1, -1\rangle \cdot \langle 1, 2, 1\rangle|}{|\mathbf{w}|} = \frac{|3 + 2 - 1|}{\sqrt{1 + 2 + 1}} = \frac{4}{\sqrt{6}}.$$

Answer: $\frac{4}{\sqrt{6}}$**.**