Math 200, Section 105, Supplement to the lectures on October 15-17, 2013.

Goals for this week: geometric meaning of the gradient in two and three dimensions; tangent lines and tangent planes; all instances of chain rule; implicit differentiation.

Required pre-reading: Sections 14.5 and 14.6.

1. GEOMETRIC MEANING OF THE GRADIENT

Recall from the last class: for a function of any number of variables, the directional derivative in the direction of a unit vector $\bar{u}$, is computed by the formula $D_{\bar{u}}f = \nabla f \cdot \bar{u}$. The geometric meaning of the gradient follows from this formula.

We finished with the two questions: for a differentiable function $f(x, y)$ of two variables, at a point (a, b),

1. In what direction $\bar{u}$ is the directional derivative $D_{\bar{u}}f$ maximal?
2. Does there always exist a direction $\bar{u}$ such that the directional derivative in this direction is 0?

Let us consider an example:

Example 1. $f(x, y) = 3x^2 \cos(y)$; find a direction $\bar{u}$ such that $D_{\bar{u}}|_{(2,\pi)}f$ is (a) maximal; (b) zero. How many unit vectors $\bar{u}$ satisfy these conditions?

Solution. To compute directional derivatives, it is helpful to first find the gradient vector at $(2, \pi)$. We have $f_x = 6x \cos(y)$, $f_y = -3x^2 \sin(y)$; plug in $(2, \pi)$, and get

$$\nabla|_{(2,\pi)}f = \langle 12, 0 \rangle.$$

Now, for any vector $\bar{u}$, we have $D_{\bar{u}}f = \nabla f \cdot \bar{u} = \langle 12, 0 \rangle \cdot \bar{u}$. Let θ be the angle between $\bar{u}$ and ∇f. Then $\nabla f \cdot \bar{u} = |\nabla f||\bar{u}| \cos(\theta) = |\nabla f| \cos(\theta)$, and so it is maximal when $\cos(\theta)$ is maximal, that is, $\theta = 0$, and it is zero when $\cos(\theta) = 0$, that is, when $\bar{u}$ is perpendicular to ∇f. Thus, in our example, $D_{\bar{u}}f$ is maximal when $\bar{u} = \langle 1, 0 \rangle$ (the unit vector that points in the same direction as ∇f), and $D_{\bar{u}}f = 0$ when $\bar{u} = \langle 0, \pm 1 \rangle$ (these are the two unit vectors that are perpendicular to ∇f).

We can see that there was nothing special in this example: because of the formula $D_{\bar{u}}f = \nabla f \cdot \bar{u}$, in any situation, the directional derivative of f at a given is maximal in the direction of the unit vector that points in the same direction as the gradient, and is zero in the two directions that are perpendicular to the gradient.

1.1. Tangent lines to level curves.

The second property (about zero directional derivative) has an important application: to finding tangent lines to level curves, and tangent planes to level surfaces.

Indeed, let $f(x, y)$ be any function, and let $\bar{u}$ be a tangent vector to the level curve $f(x, y) = k$ at a point (a, b) that lies on this level curve. What should the directional derivative $D_{\bar{u}}f$ be? (remember, the meaning of directional derivative is the rate of change of the function f in the direction $\bar{u}$, at (a, b)). So, since the function is *constant* along the level curve, the rate of change should be zero. Thus, the gradient vector $\nabla|_{(a,b)}f$ should be *perpendicular to the tangent line to the level curve of f that passes through the point (a, b).*

This gives us a very easy way of writing down an equation of the tangent line to the level curve passing through a given point (see "additional examples" at the end).

1.2. Tangent planes to level surfaces.

Now, what happens for a function of three variables? We get a convenient way of finding the equations of tangent planes to surfaces.

For a function of any number of variables, the directional derivative $D_{\bar{u}}f$ will be zero if $\bar{u}$ is perpendicular to the gradient. Let us see what this says for a function of three variables: the equation $f(x, y, z) = k$ determines a surface in space (a level surface of f). Then the gradient at a point P is perpendicular to the tangent plane to the level surface passing through P. This can be used to find the tangent planes to surfaces (an easier way than implicit differentiation+linearization).

Example 2. Find the tangent plane to the ellipsoid $x^2 + 3y^2 + 2z^2 = 6$ at $(1, 1, 1)$.
Solution: As usual, to find an equation of a plane we need to find a normal vector. Let us think of the ellipsoid as a level surface of the function $f(x, y, z) = x^2 + 3y^2 + 2z^2$ (corresponding to the value 6). Then the normal vector to the tangent plane is the gradient of this function. $\nabla f = \langle 2x, 6y, 4z \rangle$.

$$\nabla f|_{(1,1,1)} = \langle 2, 6, 4 \rangle.$$

Then the tangent plane has the equation $2(x-1) + 6(y-1) + 4(z-1) = 0$.

1.3. Gradient as the direction of the fastest increase.

Let us now discuss the geometric meaning of the first property we discovered today: that the directional derivative is maximal when the direction vector $\bar{u}$ points in the same direction as the gradient. Turning this phrase "upside-down", we can say that the gradient points in the direction of the fastest increase of the function from a given point.

Example 3. Concentration of salt in a tidal pool is given by $c(x, y, z) = 0.2 - ze^{-x-2y}$. A salt-loving bacterium is at the point $(0, 1, 3)$. In which direction should the bacterium move from this point so that the salinity increases the fastest? (Find the unit vector in that direction). What if the bacterium wanted the salinity to *decrease* the fastest?

    Solution:

$$\bigtriangledown f = \langle ze^{-x-2y}, 2ze^{-x-2y}, e^{-x-2y} \rangle.$$

At $(0, 1, 1)$ it is: $\langle 3e^{-2}, 6e^{-2}, e^{-2} \rangle$.

    The direction of the fastest increase is along the gradient. The unit vector in that direction is $\langle \frac{3}{\sqrt{46}}, \frac{6}{\sqrt{46}}, \frac{1}{\sqrt{46}} \rangle$.

    For the fastest decrease, we need to take the direct opposite to the gradient (makes the directional derivative "as far negative as possible".

    This type of interpretation of the gradient works for a function of any number of variables.

    Important formulas: The rate of change of f in the direction of the gradient equals $|\bigtriangledown f|$, and in the direction opposite to the gradient it is $-|\bigtriangledown f|$. You should be able to see why!

1.4. "Gradient points in the direction of the steepest ascent".

For a function of *two* variables, it is possible to imagine its graph. Also, some surfaces in three dimensions, such as mountains, can be thought of as a graph of a function $z = f(x, y)$.

    So, suppose you are hiking on a mountain defined by the equation $z = f(x, y)$. Then what is the direction of the steepest ascent?

    Here we have to be careful about what we mean by "direction". Since you are hiking, your velocity is a vector (tangent to the trail you are hiking along) with three components (let us think that the x-axis points East, the y-axis points North, and the z-axis points up. Then every step you take changes your coordinate along each axis). Now, what does "direction of the steepest ascent" mean, mathematically? Suppose you look at the map, and decide that you are going North-West. Then the rate at which your altitude is increasing is $D_{\bar{u}}z$, where $\bar{u}$ is the unit vector in the North-West direction, that is, $\bar{u} = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. Note that once you have chosen the direction (that is, a vector $\bar{u}$ with *two* components, which you could draw on the map, but not on the mountain), the steepness of your ascent (or descent) as you move in that direction is determined by the shape of the mountain. Thus, the question about the direction of the steepest ascent from a given point is a question about *directions of the compass that you need to follow* (that is, it's a direction vector with *two* components, on the map), and the vector along which you are actually moving on the mountain will

have three components, with the third component prescribed by the shape of the mountain. With this understanding, we can say that the gradient of the function $f(x, y)$ points in the direction of the steepest ascent.

The gradient points in the direction that you should aim the compass in order to find the direction of the steepest way up (the fastest increase of the altitude z); the opposite, $-\bigtriangledown f$, points in the direction of the steepest way down.

Example 4. You are on a hill the shape of which is described by $z = 1 - 0.03x^2 - 0.1y^4$, where z is the altitude above sea level (in kilometers), and x and y measure the distances on the map (in kilometers) East and North, respectively, from some fixed point, e.g. the wind turbine at the top of the hill.

  (a) When your GPS tells you that you are at the point $(2, 1)$ on the map, what is your altitude above the sea level? What are your coordinates our xyz-coordinate system then?
  (b) What direction should you follow for the steepest descent from this point? (The answer should have two components).
  (c) Suppose you are on a mountain bike, and you go in the direction of the steepest descent that you found in part (b), with speed 15m/s. Find your actual velocity vector. (The answer should have 3 components, and the first two should form a vector that is proportional to the answer from (b)).

Solution. Part (a): Plug in $x = 2, y = 1$ into the equation for z, and get: $1 - 0.03 \cdot 4 - 0.1 \cdot 1 = 0.78km$. Then your coordinates are $(2, 1, 0.78)$.

Part (b). Find the gradient:

$$\bigtriangledown f = \langle -0.06x, -0.4y \rangle.$$

Plug in our point $(2, 1)$:

$$\bigtriangledown|_{(2,1)} f = \langle -0.12, -0.4 \rangle.$$

For the steepest descent, we need the unit vector opposite to the gradient. It is the vector $\bar{u} = \langle \frac{3}{\sqrt{109}}, \frac{10}{\sqrt{109}} \rangle$. Note that this vector points approximately North-North-East.

Part (c). The velocity vector has to be tangent to the trail (on the hill) that you are following. We are given that the trail follows the direction of the steepest descent, that is, on the map it goes from the point $(2, 1)$ along the vector $\bar{u}$ from part (b). At what slope does it descend? The rate of change of z in the direction $\bar{u}$ is

$$D_{\bar{u}}z = \bigtriangledown f \cdot \bar{u} = -|\bigtriangledown f| = -\sqrt{0.12^2 + 0.4^2} \simeq -0.417.$$

This means, that the slope of the trail in the direction of the vector $\bar{u}$ equals -0.417. Note that $\bar{u}$ is the unit vector; then (this needs a picture which I cannot draw in this file; draw it on a separate sheet of paper), the vector

$$\bar{w} = \langle \frac{3}{\sqrt{109}}, \frac{10}{\sqrt{109}}, -0.417 \rangle$$

is tangent to the trail at the point $(2, 1, 0.78)$ (this is the point you are at, in space – see part (a)). Note that the units of the components of $\bar{w}$ are m/m, that is, no units. Finally, your velocity should be the vector of length 15 proportional to w. This is the vector

$$\bar{v} = \frac{15}{|\bar{w}|}\bar{w} \simeq \frac{15}{\sqrt{1 + 0.417^2}}\langle \frac{3}{\sqrt{109}}, \frac{10}{\sqrt{109}}, -0.417 \rangle.$$

2. Back to chain rule, Section 14.5

2.1. **The most general case of Chain Rule.** So far, we know the following chain rule:

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t).$$

This, of course, works for f – a function of any number of variables:

$$\frac{d}{dt}f(x(t), y(t), z(t), w(t)) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) + \frac{\partial f}{\partial z}z'(t) + \frac{\partial f}{\partial w}w'(t).$$

(The reason is the same – linearization.)

What if the inputs of f depend on more than one parameter: what if we have $f(x, y, z)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$. Then we can track down the rate of change of the output with respect to u, and to v, separately. Imagine that we fixed v, and compute the change with respect to u. Then we are in the previous situation. Get:

$$\frac{\partial}{\partial u}f(x(u, v), y(u, v), z(u, v)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$

The only information we need in order to use chain rule in this form is all the relevant partials at the relevant points.

Example 5. Suppose ∇f at a point (a, b, c) is $\langle 1, 2, 3 \rangle$. Suppose $x(10, 15) = a$, $y(10, 15) = b$, $z(10, 15) = c$, and $\frac{\partial x}{\partial u}|_{(10,15)} = 11.1$, $\frac{\partial y}{\partial u}|_{(10,15)} = 3.4$, $\frac{\partial z}{\partial u}|_{(10,15)} = 2.6$. Find $\frac{\partial f}{\partial u}|_{(10,15)}$.

(Note: I am using different-looking numbers for the things that have different meanings, so that it is easier to track what to plug in where).

Example of a situation where applications to the chain rule arise: 100 solar cells are connected together; let f be the total output of this array of solar cells; then f is a function of 100 variables, where each variable is the energy production of each cell. Each of these variables is a function of two variables: the intensity of light, and its wavelength. One might be interested in how the output of the array changes with respect to changes in the intensity and wavelength of light; this is what chain rule can be used to measure.

Important: Please see the textbook for a picture that helps memorize how to write down the Chain rule.

2.2. **Implicit differentiation.** As in calculus 1, if the function is given implicitly, we can solve for its derivative (now, for all its partial derivatives).

Example 6.
$$x^2 + 3y^2 + 2z^2 = 6.$$
This is intentionally the same example we've seen already. Find $\partial z/\partial x$, $\partial z/\partial y$, and then the equation of the tangent plane, at the point $(1, 1, 1)$.

Solution. Differentiate with respect to x, treating z as a function of x and y 9and therefore using Chain Rule every time we see a z). Get:

$$2x + 4z\frac{\partial z}{\partial x} = 0,$$

so $\frac{\partial z}{\partial x} = -\frac{2x}{4z}$. Plugging in $(1,1,1)$, get that $\frac{\partial z}{\partial x}|_{(1,1,1)} = -\frac{1}{2}$. Similarly can find $\frac{\partial z}{\partial y}|_{(1,1,1)} = -\frac{6y}{4z}|_{(1,1,1)} = -\frac{3}{2}$.

Now we can use these partial derivatives to find the linearization $L(x, y)$ of z as a function of (x, y) near the point $(1, 1)$ (with the values of z also being close to 1): $L(x, y) = 1 - \frac{1}{2}(x - 1) - \frac{1}{2}(y - 1)$.

Then the equation of the tangent plane to our ellipsoid at $(1, 1, 1)$ is

$$z - 1 = -\frac{1}{2}(x - 1) - \frac{1}{2}(y - 1).$$

Important: compare this with the equation we got in Example 2! Is it the same plane?

The general formula for partial derivatives of an implicit function:

Suppose z is given implicitly as a function of x and y by the equation $F(x, y, z) = k$, where F is some function of 3 variables, and k is a constant. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}; \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Exercise: prove this general formula.

3. Additional examples

Example 7. With the same f as in Example 1, find the tangent line to the level curve of f that passes through the point $(1, \pi/3)$. Let us compute the gradient at that point:

$$\nabla|_{(2,\pi/3)} f = \langle 6{\cdot}1 \cos(\pi/3), -3{\cdot}1^2 \sin(\pi/3) \rangle = \langle 6/2, -3\sqrt{3}/2 \rangle = \langle 3, 3\sqrt{3}/2 \rangle.$$

Now we know the normal vector, and know the point. Then (similarly to writing the equation of the tangent plane in space when the normal vector and a point are given), get the answer:

$$3(x - 1) + \frac{3\sqrt{3}}{2}(y - \pi/3) = 0.$$

Note that the other method of doing it would be to find the slope of the tangent line to this level curve would be to write iota equation:

$$3x^2 \cos(y) = k$$

(first question: what is the value of k, if we know that the point $(1, \pi/3)$ lies on this level curve?). Then use implicit differentiation (from Calculus 1) to find the slope of the tangent line to this level curve. It would be harder to do this than to use our new method with the gradient. **Good exercise for better understanding:** try to see why both methods should always give the same answer!).