Math 200 Midterm II (November 1, 2012)
Sections 107. Instructor: Julia Gordon

Name:

Student Number:

Problem 1:

(a) [*2 points*] Let $F(x,y,z) = \sin(xz+y) - x^2 + z$. Find the expression for ∇F – the gradient of F at a point (x,y,z).

$$\bar{\nabla} F = \langle \cos(xz+y)\cdot z - 2x,\ \cos(xz+y),\ \cos(xz+y)\cdot x + 1 \rangle$$

(b) [*3 points*] For the same function $F(x,y,z)$, find the directional derivative $D_{\bar{u}}F$ at the point $(1, \pi/2, 0)$ in the direction of the vector $\langle 1,2,3\rangle$.

The unit vector in the direction of $\langle 1,2,3\rangle$ is $\bar{u} = \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$

$\bar{\nabla} F \big|_{(1,\frac{\pi}{2},0)} = \langle -2, 0, 1 \rangle$ $\qquad D_{\bar{u}} F = \langle -2,0,1\rangle \cdot \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$

(compute: $\cos(1\cdot 0 + \frac{\pi}{2}) = 0$) $\qquad = -\frac{2}{\sqrt{14}} + \frac{3}{\sqrt{14}} = \boxed{\frac{1}{\sqrt{14}}}$

(c) [*4 points*] Find the equation of the tangent plane to the surface defined by the equation $x^2 - z = \sin(xz+y)$ at the point $(1, \pi/2, 0)$.

This surface is the level surface of
$F(x,y,z) = \sin(xz+y) - x^2 + z$.
Then $\bar{\nabla} F \big|_{(1,\frac{\pi}{2},0)}$ is normal to the tangent plane.

Answer: $\quad -2(x-1) + 0\cdot(y-\frac{\pi}{2}) + 1\cdot(z-0) = 0$

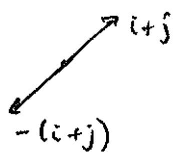
Problem 2: The temperature T at a point on a metal plate depends on the coordinates x, y. We do not know what the temperature function is, but the following information is given: at the point $P(10, 11)$, the temperature does not change in the direction of the vector $\mathbf{i} + \mathbf{j}$; and the rate of change of the temperature at P in the direction of the vector $\mathbf{i} - \mathbf{j}$ equals $-3\sqrt{2}$ degrees per centimeter.

(a) [*2 points*] If an ant is crawling through the point P in the direction of the vector $-\mathbf{i} - \mathbf{j}$ at the speed of 4 cm/s, what is the rate of change of temperature that the ant is experiencing?

(b) [*4 points*] Find the gradient of the temperature function at the point P.

(c) [*3 points*] If a beetle is crawling through the point P at the speed of 4 cm/s in the direction of the vector $\mathbf{i} + 2\mathbf{j}$, what is the rate of change of temperature that the beetle is experiencing?

(a)



We are given that $D_{\bar{u}} T = 0$ where $\bar{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ (the unit vector in the direction of $i+j$)

Then $D_{-\bar{u}} T = -D_{\bar{u}} T = \boxed{0}$.

(b). Let $\bar{\nabla} T|_P = \langle a, b \rangle$.

We are given: $D_{\bar{u}_1} T = 0$ where $\bar{u}_1 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$D_{\bar{u}_2} T = -3\sqrt{2}$ where $\bar{u}_2 = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

Then
$$\begin{cases} \langle a, b \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0 \\ \langle a, b \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = -3\sqrt{2} \end{cases}$$

$$\begin{cases} a + b = 0 \\ a - b = 6 \end{cases} \qquad a = 3, \quad b = -3 \qquad \boxed{\bar{\nabla} T|_P = \langle 3, -3 \rangle}$$

2 (c) The velocity vector of the beetle is $4\bar{u}$, where $\bar{u}$ is the unit vector in the direction $i+2j$

$\bar{u} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$. Suppose $x(t)$, $y(t)$ are coordinates of the beetle.

$$\frac{dT}{dt} = \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt} = \bar{\nabla} T \Big|_p \cdot \underbrace{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}_{\bar{v}}$$

$$= \langle 3, -3 \rangle \cdot \langle \frac{4}{\sqrt{5}}, \frac{2 \cdot 4}{\sqrt{5}} \rangle$$

$$= \frac{12}{\sqrt{5}} - \frac{24}{\sqrt{5}} = \boxed{-\frac{12}{\sqrt{5}} \ {}^\circ/\text{sec}}$$

Problem 3: A hiker is going down a hill whose shape is given by $z = e^{4-x^2-2y^2}$.

(a) [*2 points*] Find the direction of the steepest descent when the hiker is at the point $P(1, 1/2, e^{2.5})$. State your answer in terms of the directions of the compass (N, S, W, E, NW, etc.); you can assume that East is the positive direction of the x-axis, and North is the positive direction of the y-axis.

(b) [*3 points*] Suppose that the trail the hiker is on follows the path of the steepest descent from P. Find the angle the trail is descending at (compared to the horizontal plane).

(c) [*3 points*] For the same trail as in (b), find a tangent vector to the trail at P. (It should be a vector with three components).

(a) $\bar{\nabla} f = \langle -2xe^{4-x^2-2y^2}, \; -4y\, e^{+4-x^2-2y^2} \rangle$

(where $f(x,y) = e^{4-x^2-2y^2}$ ← the graph of this function is our hill).

$\bar{\nabla} f \big|_{(1,\frac{1}{2})} = \langle -2e^{2.5}, -2e^{2.5} \rangle$

This vector points SW; the steepest descent is opposite to the gradient, so it'll be $\boxed{NE}$

(b) Let α be the angle of the trail.

$\tan(\alpha) = D_{\bar{u}} f$, where $\bar{u}$ is the unit vector defining the direction of the trail.

In our case, $\bar{u}$ is opposite to $\bar{\nabla} f$, so

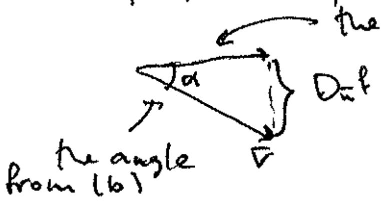
$D_{\bar{u}} f = -|\bar{\nabla} f|. = -\sqrt{4(e^{2.5})^2 + 4(e^{2.5})^2} = \boxed{-2e^{2.5}\sqrt{2}}$

So, $\boxed{\alpha = \tan^{-1}(-2e^{2.5}\sqrt{2})}$

Note that α being negative indicates that the hiker is going down.

(c) This vector should have the projection onto the xy-plane that is opposite to $\bar{\nabla} f$, and "slope" as in (b).

the unit vector in the direction of $-\bar{\nabla} f$



we get: $\bar{v} = \langle \underbrace{\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}}_{\bar{u} = \text{unit vector in NE direction}}, \underbrace{-2e^{2.5}\sqrt{2}}_{D_{\bar{u}} f \text{ for this } \bar{u}} \rangle$

Problem 4: [*5 points*]

A function $f(x,y)$ satisfies: $\frac{\partial f}{\partial x}\big|_{(0,\cancel{2})}^{200} = a$ and $\frac{\partial f}{\partial y}\big|_{(0,\cancel{2})}^{200} = 3$. The variables x, y, s, and w satisfy the relations:

$$se^{x+w} = 10$$
$$2s^2 + w^2 = y.$$

Find the value of a such that $\frac{\partial f}{\partial s}$ is zero at the point $s = 10, w = 0$.

we have (from Chain Rule):

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Note that when $(s,w) = (10,0)$, we have $(x,y) = (0,2)$, because:

$$y = 2 \cdot 10^2 + w^2 = 200$$

$$10\,e^{x+0} = 10, \quad \text{so} \quad e^x = 1 \text{ and } x = 0.$$

Then $$\frac{\partial f}{\partial s}\Big|_{(10,0)} = \frac{\partial f}{\partial x}\Big|_{(0,200)} \cdot \frac{\partial x}{\partial s}\Big|_{(10,0)} + \frac{\partial f}{\partial y}\Big|_{(0,200)} \cdot \frac{\partial y}{\partial s}\Big|_{(10,0)}$$

$$= a\,\frac{\partial x}{\partial s}\Big|_{(10,0)} + 3\,\frac{\partial y}{\partial s}\Big|_{(10,0)}$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(2s^2+w^2) = 4s, \quad \text{so} \quad \boxed{\frac{\partial y}{\partial s}\Big|_{(10,0)} = 40}$$

It remains to find $\frac{\partial x}{\partial s}$. For that we need implicit differentiation, since x is defined implicitly.

Differentiate $se^{x+w} = 10$ with respect to s.

Get: $\frac{\partial}{\partial s}(se^{x+w}) = 0$. Use product rule (treat x as a function of s)

$$e^{x+w} + se^{x+w}\frac{\partial x}{\partial s} = 0$$

$$\frac{\partial x}{\partial s} = -\frac{e^{x+w}}{se^{x+w}} = -\frac{1}{s}. \quad \boxed{\frac{\partial x}{\partial s}\Big|_{(10,0)} = -\frac{1}{10}}$$

Putting it all together, get:

$$\left.\frac{\partial f}{\partial s}\right|_{(10,0)} = a \cdot \left(-\frac{1}{10}\right) + 3 \cdot 40 = 120 - \frac{a}{10}$$

We want: $120 - \frac{a}{10} = 0$, so $\boxed{a = 1200}$

Problem 5: All parts of this problem are about the function $f(x,y) = x^2 + y^3 - y^2x - y^2$.

(a) [*4 points*] Find all critical points of f.

(b) [*4 points*] Classify the critical points of f using the second derivative test.

(c) [*6 points*] Find the list of points where you need to compare the values of f in order to find its absolute minimum on the closed bounded domain bounded by the curve $x = y^2/4$ and the vertical line $x = 1$.

$$f_x = 2x - y^2$$
$$f_y = 3y^2 - 2xy - 2y$$

$$\begin{cases} 2x - y^2 = 0 \\ 3y^2 - 2xy - 2y = 0 \end{cases} \qquad \begin{cases} y^2 = 2x \\ \left[\begin{array}{l} y = 0 \\ \text{or} \\ 3y - 2x - 2 = 0 \end{array}\right. \end{cases}$$

If $y = 0$, get $x = 0$

If $y \neq 0$, we have $y^2 = 2x = 3y - 2$

$y^2 - 3y + 2 = 0 \qquad y_{1,2} = 1$ and 2

$x = \frac{y^2}{2}$

Get: ~~[illegible]~~ $(\frac{1}{2}, 1)$ and $(2,2)$

$\boxed{\text{Answer: } (0,0),\ (\tfrac{1}{2}, 1),\ (2,2)}$

(b) $f_{xx} = 2$

$f_{xy} = -2y$

$f_{yy} = 6y - 2x - 2$

at $(0,0)$: $D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4$

$(0,0)$ is ~~[illegible]~~ a saddle point

at $(\frac{1}{2}, 1)$:

$$D = \begin{vmatrix} 2 & -2 \\ -2 & 6-1-2 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 3 \end{vmatrix} = 6 - 4 > 0$$

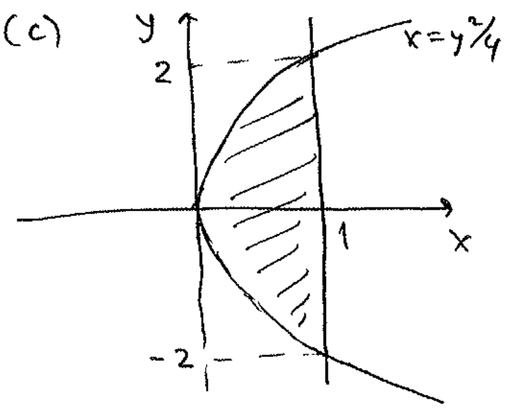
$f_{xx} > 0$

$\Rightarrow$ $(\frac{1}{2}, 1)$ is a local min

The point $(2,2)$: $D = \begin{vmatrix} 2 & -4 \\ -4 & 6\cdot 2 - 2\cdot 2 - 2 \end{vmatrix}$

$= \begin{vmatrix} 2 & -4 \\ -4 & 6 \end{vmatrix} = 12 - 16 < 0$, so it is a saddle point.

(c)



Inside the domain, we have the critical pt $(\frac{1}{2}, 1)$; also, $(0,0)$ happens to be on the boundary.
Now we have to look for possible abs. min points on the boundary.

1) The line $x = 1$, $-2 \le y \le 2$.
Plug in $x=1$ into $f(x,y)$, get:
$f(1,y) = 1 + y^3 - y^2 - y^2 = 1 + y^3 - 2y^2$
$f'(y) = 3y^2 - 4y$ $\qquad 3y^2 - 4y = 0 \quad y = 0$ or $y = 4/3$
Get the points $(1,0)$, and $(1, 4/3)$

2) $x = y^2/4$. Plug this into $f(x,y)$.

Get: $f(\frac{y^2}{4}, y) = \left(\frac{y^2}{4}\right)^2 + y^3 - y^2 \cdot \frac{y^2}{4} - y^2$

$= -\frac{3}{16} y^4 + y^3 - y^2 =: g(y)$

$g'(y) = -\frac{3}{4} y^3 + 3y^2 - 2y$

$= y\left(-\frac{3}{4} y^2 + 3y - 2\right)$

$g'(y) = 0$: $y = 0$ or

$-3y^2 + 12y - 8 = 0$

$3y^2 - 12y + 8 = 0$

$y_{1,2} = \frac{1}{3}\left(6 \pm \sqrt{36 - 24}\right)$

$= 2 \pm \frac{2\sqrt{3}}{3}$ ← one of them is in $[-2, 2]$

Answer: $(\frac{1}{2}, 1)$; $(0,0)$; $(1, 4/3)$; $\left(\frac{1}{4}\left(2 - \frac{2}{\sqrt{3}}\right)^2, 2 - \frac{2}{\sqrt{3}}\right)$

Problem 6: [*5 points*] A fly is zooming around a room. Fix one corner of the room, call it the point O. Prove that at the moment when the fly is at the maximal distance from O, its velocity is perpendicular to the line that connects it to O.

Hint: Recall that if the coordinates of a fly at a time t are $(x(t), y(t), z(t))$, then its velocity at the time t is the vector of derivatives $\langle x'(t), y'(t), z'(t)\rangle$.

Let O be the origin.

Then $(x(t), y(t), z(t))$ are the coordinates of the fly at the time t.

Then the square of its distance from the origin is $f(t) = x^2(t) + y^2(t) + z^2(t)$

At the time t_0 when $f(t)$ is maximal, we must have $f'(t_0) = 0$.

By chain rule,

$$f'(t) = 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)$$

$$= 2\langle x(t), y(t), z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle$$

So, we get:

$$0 = f'(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle \cdot \bar{v}$$

Then $\bar{v} \perp \langle x(t_0), y(t_0), z(t_0)\rangle$, which is a vector connecting O to the fly.