Math 263 Midterm II

Problem 1: Consider the function \( z = f(x, y) \) defined via intermediate variables \( u, v \) and \( w \):
\[
z = e^w \sin(u + 2v), \quad u = xy, \quad v = x - y, \quad w = \sin(x + y).
\]
(a) [10 points] Use the chain rule to evaluate the partial derivatives \( f_x, f_y \) at \((x, y) = (\pi, 0)\). Compute the gradient vector \( \nabla f(x, y) \) at the same point.

(b) [10 points] Calculate the directional derivative of \( f(x, y) \) in the direction of the vector \( 3\mathbf{i} + 4\mathbf{j} \) at the point \((x, y) = (\pi, 0)\).

(c) [10 points] What is the equation of the tangent plane to \( z = f(x, y) \) at \((x, y) = (\pi, 0)\)?

Solution:

(a) By the chain rule,
\[
f_x = f_u u_x + f_v v_x + f_w w_x, \quad f_y = f_u u_y + f_v v_y + f_w w_y.
\]
Now,
\[
u_x = y, \quad u_y = x, \quad v_x = -v_y = 1, \quad w_x = w_y = \cos(x + y),
\]
and
\[
f_u = e^w \cos(u + 2v), \quad f_v = 2e^w \cos(u + 2v), \quad f_w = e^w \sin(u + 2v).
\]
At \( x = \pi \) and \( y = 0 \), we have
\[
u = 0, \quad v = \pi, \quad w = 0,
\]
\[
u_x = 0, \quad u_y = \pi, \quad v_x = -v_y = 1, \quad w_x = w_y = -1,
\]
and
\[
f_u = 1, \quad f_v = 2, \quad f_w = 0.
\]
Hence, \( f_x = 2 \) and \( f_y = \pi - 2 \), giving
\[
\nabla f = <2, \pi - 2>.
\]

(b) The unit vector in the direction of the vector, \( 3\mathbf{i} + 4\mathbf{j} \) is \( \frac{3}{5} \), and so the directional derivative is
\[
\nabla f \cdot \frac{1}{5} <3, 4> = \frac{6 + 4(\pi - 2)}{5} = \frac{4\pi - 2}{5}.
\]

(c) The tangent plane at \((x_0, y_0, z_0)\) is defined by \( z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \). At our point, \( f = 0 \), and so
\[
z = 2(x - \pi) + (\pi - 2)y.
\]
Problem 2:
(a) [10 points] Find the critical points of the function \( f(x, y) = e^x (x - y^2) \).

(b) [10 points] Use the second-derivative test to determine whether \( f \) attains a local maximum, minimum or saddle point at the critical points. Evaluate \( f(x, y) \) at the critical points.

(c) [15 points] Determine the absolute maximum of \( f(x, y) \) over the closed domain

\[-2 \leq x \leq 0, \quad -1 \leq y \leq 1.\]

Solution:
(a) First partial derivatives:

\[
 f_x = e^x (x + 1 - y^2), \quad f_y = -2e^x y.
\]

The only critical point is therefore given by \((x, y) = (-1, 0)\).

(b) The second partial derivatives are:

\[
 f_{xx} = e^x (2 - y^2), \quad f_{yy} = -2e^x, \quad f_{xy} = -2e^x y.
\]

At the critical points,

\[
 f_{xx} = e^{-1}, \quad f_{yy} = -2e^{-1}, \quad f_{xy} = 0.
\]

And the discriminant,

\[
 D = f_{xx}f_{yy} - f_{xy}^2 = -2e^{-2} < 0.
\]

Thus the critical point is a saddle point: \( f(-1,0) = -e^{-1} \).

(c) Since the critical point is a saddle, the absolute maximum must occur on the boundary of the domain.

Along the top and bottom boundaries, \( y = \pm 1 \) and the function value is \( g(x) = e^x (x - 1) \).

\[
 g'(x) = xe^x \leq 0 \quad \text{over} \quad -2 \leq x \leq 0 \quad \text{and} \quad g \text{attains its maximum at} \quad x = -2: \quad f(-2, \pm 1) = -3e^{-2}.
\]

At the left boundary, \( x = -2 \) and the function reduces to \( h(y) = -e^{-2}(y^2 + 2) \), which has its maximum at \( y = 0, \quad h(0) = -2e^{-2} \).

At the right boundary, \( x = 0 \) and the function reduces to \( p(y) = -y^2 \) whose maximum is \( p(0) = 0 \).

Thus, the absolute maximum of \( f(x, y) \) is 0, and it occurs at \((0,0)\).

Note: Full points if one writes that \( f(x, y) = e^x (x - y^2) \) is non-positive over the closed domain, and thus its absolute maximum occurs at the origin, and is 0.
Problem 3:

(a) [10 points] Consider the three-dimensional region bounded by the surface

\[ x^2 + y^2 + z^2 = a^2, \]

and the planes: \( y = 0, \) \( z = 0 \) and \( x = y. \) Set up an iterated double integral in Cartesian coordinates for calculating the volume of the region. Do not evaluate this integral!

(b) [15 points] Calculate the above volume using a double integral in polar coordinates.

(c) [10 points] Evaluate the integral of \( g(y) = (\sin y)/y \) over the triangular domain on the \( xy\)-plane bounded by \( x = 0, \) \( y = x \) and \( y = \pi/4. \) **Hint:** the domain can be seen either as type I or type II, but the iterated integral is much easier in one way than the other.

Solution:

(a) Since the curved surface can be rewritten in the form \( z = \sqrt{a^2 - x^2 - y^2}, \) the volume of the region is given by

\[ \int \int_D \sqrt{a^2 - x^2 - y^2} \, dxdy, \]

where \( D \) is the region in the first quadrant of the \((x, y)\)-plane below the line \( x = y, \) to the left of the curve, \( x^2 + y^2 = a^2. \) By integrating first in \( x, \) we may write the iterated integral,

\[ \int_0^a \left[ \int_y^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2} \, dx \right] dy, \]

(b) In polar coordinates, the region of integration is simply \( 0 \leq \theta \leq \pi/4 \) and \( 0 \leq r \leq a. \) Hence, we must evaluate

\[ \int_0^{\pi/4} d \theta \int_0^a \sqrt{a^2 - r^2} \, rdr = \frac{\pi}{8} \int_0^{a^2} \sqrt{a^2 - u} \, du = \frac{\pi}{12} \left[ -(a^2 - u)^{3/2} \right]_0 = \frac{\pi a^3}{12}. \]

(c) The region of integration is the wedge above \( y = x \) and below \( y = \pi/4. \) Hence,

\[ \int_0^{\pi/4} \left[ \int_0^y \frac{\sin y}{y} \, dx \right] dy \]

\[ = \int_0^{\pi/4} \left[ \int_x^{\pi/4} \frac{\sin y}{y} \, dy \right] dx \]

The first integral of the original double integral evaluates to give

\[ \int_0^{\pi/4} \sin y \, dy = [- \cos y]_{y=0}^{y=\pi/4} = 1 - \frac{\sqrt{2}}{2}. \]