Math 263 Midterm II

Problem 1: Consider the function $z = f(x, y)$ defined via intermediate variables u, v and w:

$$z = e^w \sin(u + 2v), \quad u = xy, \quad v = x - y, \quad w = \sin(x + y).$$

(a) [*10 points*] Use the chain rule to evaluate the partial derivatives f_x, f_y at $(x, y) = (\pi, 0)$. Compute the gradient vector $\nabla f(x, y)$ at the same point.

(b) [*10 points*] Calculate the directional derivative of $f(x, y)$ in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$ at the point $(x, y) = (\pi, 0)$.

(c) [*10 points*] What is the equation of the tangent plane to $z = f(x, y)$ at $(x, y) = (\pi, 0)$?

Solution:

(a) By the chain rule,

$$f_x = f_u u_x + f_v v_x + f_w w_x, \qquad f_y = f_u u_y + f_v v_y + f_w w_y.$$

Now,

$$u_x = y, \quad u_y = x, \quad v_x = -v_y = 1, \quad w_x = w_y = \cos(x + y),$$

and

$$f_u = e^w \cos(u + 2v), \qquad f_v = 2e^w \cos(u + 2v), \qquad f_w = e^w \sin(u + 2v).$$

At $x = \pi$ and $y = 0$, we have

$$u = 0, \qquad v = \pi, \qquad w = 0,$$

$$u_x = 0, \quad u_y = \pi, \quad v_x = -v_y = 1, \quad w_x = w_y = -1,$$

and

$$f_u = 1, \qquad f_v = 2, \qquad f_w = 0.$$

Hence, $f_x = 2$ and $f_y = \pi - 2$, giving

$$\nabla f =< 2, \pi - 2 > .$$

(b) The unit vector in the direction of the vector, $3\mathbf{i} + 4\mathbf{j}$ is $< 3, 4 > /5$, and so the directional derivative is

$$\nabla f \cdot \frac{1}{5} < 3, 4 >= \frac{6 + 4(\pi - 2)}{5} = \frac{4\pi - 2}{5}.$$

(c) The tangent plane at (x_0, y_0, z_0) is defined by $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. At our point, $f = 0$, and so

$$z = 2(x - \pi) + (\pi - 2)y.$$

Problem 2:

(a) [*10 points*] Find the critical points of the function $f(x, y) = e^x(x - y^2)$.

(b) [*10 points*] Use the second-derivative test to determine whether f attains a local maximum, minimum or saddle point at the critical points. Evaluate $f(x, y)$ at the critical points.

(c) [*15 points*] Determine the absolute maximum of $f(x, y)$ over the closed domain

$$-2 \leq x \leq 0, \quad -1 \leq y \leq 1.$$

Solution:

(a) First partial derivatives:

$$f_x = e^x(x + 1 - y^2), \quad f_y = -2e^x y.$$

The only critical point is therefore given by $(x, y) = (-1, 0)$.

(b) The second partial derivatives are:

$$f_{xx} = e^x(x + 2 - y^2), \quad f_{yy} = -2e^x, \quad f_{xy} = -2e^x y.$$

At the critical points,

$$f_{xx} = e^{-1}, \quad f_{yy} = -2e^{-1}, \quad f_{xy} = 0.$$

And the discriminant,

$$D = f_{xx}f_{yy} - f_{xy}^2 = -2e^{-2} < 0.$$

Thus the critical point is a saddle point: $f(-1, 0) = -e^{-1}$.

(c) Since the critical point is a saddle, the absolute maximum must occur on the boundary of the domain.

Along the top and bottom boundaries, $y = \pm 1$ and the function value is $g(x) = e^x(x-1)$. $g'(x) = xe^x \leq 0$ over $-2 \leq x \leq 0$ and g attains its maximum at $x = -2$: $f(-2, \pm 1) = -3e^{-2}$.

At the left boundary, $x = -2$ and the function reduces to $h(y) = -e^{-2}(y^2 + 2)$, which has its maximum at $y = 0$, $h(0) = -2e^{-2}$.

At the right boundary, $x = 0$ and the function reduces to $p(y) = -y^2$ whose maximum is $p(0) = 0$.

Thus, the absolute maximum of $f(x, y)$ is 0, and it occurs at $(0, 0)$.

Note: Full points if one writes that $f(x, y) = e^x(x - y^2)$ is non-positive over the closed domain, and thus its absolute maximum occurs at the origin, and is 0.

Problem 3:

(a) [*10 points*] Consider the three-dimensional region bounded by the surface

$$x^2 + y^2 + z^2 = a^2,$$

and the planes: $y = 0$, $z = 0$ and $x = y$. Set up an iterated double integral in Cartesian coordinates for calculating the volume of the region. Do not evaluate this integral!

(b) [*15 points*] Calculate the above volume using a double integral in polar coordinates.

(c) [*10 points*] Evaluate the integral of $g(y) = (\sin y)/y$ over the triangular domain on the xy-plane bounded by $x = 0$, $y = x$ and $y = \pi/4$. **Hint**: the domain can be seen either as type I or type II, but the iterated integral is much easier in one way than the other.

Solution:

(a) Since the curved surface can be rewritten in the form $z = \sqrt{a^2 - x^2 - y^2}$, the volume of the region is given by

$$\int\int_D \sqrt{a^2 - x^2 - y^2}\, dxdy,$$

where D is the region in the first quadrant of the $(x, y)-$plane below the line $x = y$, to the left of the curve, $x^2 + y^2 = a^2$. By integrating first in x, we may write the iterated integral,

$$\int_0^a \left[\int_y^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx \right] dy,$$

(b) In polar coordinates, the region of integration is simply $0 \le \theta \le \pi/4$ and $0 \le r \le a$. Hence, we must evaluate

$$\int_0^{\pi/4} d\theta \int_0^a \sqrt{a^2 - r^2}\, rdr$$

$$= \frac{\pi}{8} \int_0^{a^2} \sqrt{a^2 - u} du = \frac{\pi}{12} \left[-(a^2 - u)^{3/2} \right]_0^{a^2} = \frac{\pi a^3}{12}.$$

(c) The region of integration is the wedge above $y = x$ and below $y = \pi/4$. Hence,

$$\int_0^{\pi/4} \left[\int_0^y \frac{\sin y}{y} dx \right] dy$$

$$= \int_0^{\pi/4} \left[\int_x^{\pi/4} \frac{\sin y}{y} dy \right] dx$$

The first integral of the original double integral evaluates to give

$$\int_0^{\pi/4} \sin y dy = [-\cos y]_{y=0}^{y=\pi/4} = 1 - \frac{\sqrt{2}}{2}.$$