Worksheet 7: more on chain rule

1. Let \( f(x, y, z) = x^3 + \ln(yz) \), let \( x = u^2 + v^2 \), \( y = 5v \), \( z = uv \). Find \( \frac{\partial f}{\partial u} \) and evaluate it at \((u, v) = (1, 2)\).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2, \quad \text{not needed} \\
\frac{\partial f}{\partial y} &= \frac{1}{y}, \quad y = \frac{z}{v} \\
\frac{\partial f}{\partial z} &= \frac{1}{z}, \quad z = uv
\end{align*}
\]

\[
\begin{align*}
\frac{\partial x}{\partial u} &= 2u \\
\frac{\partial y}{\partial u} &= 0 \\
\frac{\partial z}{\partial u} &= v
\end{align*}
\]

when \((u, v) = (1, 2)\)

\[
\begin{align*}
x &= 1^2 + 2^2 = 5 \\
y &= 5 \cdot 2 = 10 \\
z &= 1 \cdot 2 = 2
\end{align*}
\]

So \((x, y, z) = (5, 10, 2)\).

2. The wave equation in one space dimension is:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

where \(c\) is a constant (for electromagnetic waves, it is the speed of light).

Prove that any function of the form \( u(x, t) = f(x-ct) + g(x+ct) \), where \( f \) and \( g \) are functions of a single variable, satisfies this equation.

Steps:

1. Understand the question:

   \( f, g \) are functions

2. As a warm-up, we can do example: let \( f(s) = e^s \), let \( g(s) = \cos(s) \)

   Then \( u(x, t) = e^{x-ct} + \cos(x+ct) \)

3. Using the functions \( f(s) \) and \( g(s) \)

   Now can compute \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2} \),
General solution

\[ S = x - ct \]
\[ S = x + ct \]

How to find \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial t} \):

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( f(x-ct) + g(x+ct) \right) = \frac{df}{ds} \cdot \frac{\partial s}{\partial x} + \frac{dg}{ds} \cdot \frac{\partial s}{\partial x},
\]

\[
= f'(x-ct) \cdot 1 + g'(x+ct) \cdot 1.
\]

\[
\frac{\partial^2 u}{\partial x^2} = f''(x-ct) \cdot 1 + g''(x+ct) \cdot 1 \tag{1}
\]

Now the \( t \)-partials:

\[
\frac{\partial u}{\partial t} = \frac{df}{ds} \cdot \frac{\partial s}{\partial t} + \frac{dg}{ds} \cdot \frac{\partial s}{\partial t} = f'(x-ct) \cdot (-c) + g'(x+ct) \cdot c,
\]

\[
u = f + g \quad s = x - ct \quad s = x + ct
\]

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( f'(x-ct) \cdot (-c) + g'(x+ct) \cdot c \right)
\]

\[
= (-c) f''(x-ct) \cdot \frac{\partial}{\partial t} (x-ct) + c \cdot g''(x+ct) \cdot \frac{\partial (x+ct)}{\partial t}
\]

Apply chain rule again to each term:

\[
= (-c)^2 \cdot f''(x-ct) + c^2 g''(x+ct)
\]

\[
= c^2 \left( f''(x-ct) + g''(x+ct) \right) \tag{2}
\]

Comparing (1) with (2), we see that

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

as required to show.