Worksheet 7: more on chain rule

1. Let $f(x,y,z) = x^3 + \ln(yz)$, let $x = u^2 + v^2$, $y = 5v$, $z = uv$. Find $\frac{\partial f}{\partial u}$ and evaluate it at $(u,v) = (1,2)$.

$$\frac{\partial f}{\partial x} = 3x^2$$

not needed.

$$\frac{\partial f}{\partial y} = \frac{1}{yz}\cdot z = \frac{1}{y}$$

$$\frac{\partial f}{\partial z} = \frac{1}{yz}\cdot y = \frac{1}{z}$$

$$\frac{\partial x}{\partial u} = 2u$$

$$\frac{\partial y}{\partial u} = 0$$

$$\frac{\partial z}{\partial u} = v$$

Important step:

when $(u,v) = (1,2)$

$x = 1^2 + 2^2 = 5$

$y = 5\cdot 2 = 10$

$z = 1\cdot 2 = 2$

So $(x,y,z) = \boxed{(5,10,2)}$

plug it into $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + 0 + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u} = 3x^2\cdot 2u + \frac{1}{z}\cdot v$$

What to plug in? values of x, y, z into $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}$:

$$\frac{\partial f}{\partial u}\Big|_{(1,2)} = 3x^2\Big|_{(5,10,2)}\cdot 2 + \frac{1}{z}\Big|_{(5,10,2)}\cdot 2 = 3\cdot 25\cdot 2 + \frac{1}{2}\cdot$$

$$= 151$$

2. The *wave equation in one space dimension* is:

$$\frac{\partial^2 u}{\partial^2 t} = c^2\frac{\partial^2 u}{\partial^2 x},$$

where c is a constant (for electromagnetic waves, it is the speed of light). Prove that any function of the form $u(x,t) = f(x-ct)+g(x+ct)$, where f and g are functions of a single variable, satisfies this equation.

Steps: 1) Understand the question:

f, g, u are functions

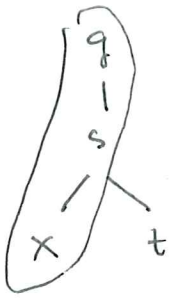
some functions of a single variable, call this variable s.

As a warm-up, on can do example: let $f(s) = e^s$, let $g(s) = \cos(s)$

Then $u(x,t) = e^{x-ct} + \cos(x+ct)$ ← we make the function $u(x,t)$ using the functions $f(s)$ and $g(s)$

tional

Now can compute $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial t^2}$.



General solution

$\boxed{s = x - ct}$ $\boxed{s = x + ct}$

How to find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(f(x-ct) + g(x+ct)\right) = \frac{df}{ds}\cdot\frac{\partial s}{\partial x} + \frac{dg}{ds}\cdot\frac{\partial s}{\partial x}$$

here $\frac{\partial s}{\partial x} = \frac{\partial(x-ct)}{\partial x} = 1$

$$= f'(x-ct)\cdot 1 + g'(x+ct)\cdot 1$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = f''(x-ct)\cdot 1 + g''(x+ct)\cdot 1} \quad (*)$$

Now the t-partials:

$$\frac{\partial u}{\partial t} = \frac{df}{ds}\cdot\frac{\partial s}{\partial t} + \frac{dg}{ds}\cdot\frac{\partial s}{\partial t} = f'(x-ct)\cdot(-c) + g'(x+ct)\cdot c$$

$u = f+g$ $\quad$ $s = x-ct$ $\quad$ $s = x+ct$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(f'(x-ct)\cdot(-c) + g'(x+ct)\cdot c\right)$$

$$= (-c)\, f''(x-ct)\cdot\frac{\partial(x-ct)}{\partial t} + c\cdot g''(x+ct)\cdot\frac{\partial(x+ct)}{\partial t}$$

apply chain rule again to each term

$$= (-c)^2\cdot f''(x-ct) + c^2 g''(x+ct)$$

$$= \boxed{c^2\left(f''(x-ct) + g''(x+ct)\right)} \quad (**)$$

Comparing $(*)$ with $(**)$, we see that

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$$ – as required to show.