Tangent planes to surfaces in space

For a graph of a function of two variables, we have two main approaches to writing the equation of the tangent plane at a given point. It is important to understand that these two ways agree. (There is also a third way, summarized at the end, which also gives the same answer, of course).

Here is a table summarizing the situation:

<table>
<thead>
<tr>
<th>equation of the surface</th>
<th>point on it</th>
<th>A normal vector to the tangent plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>graph of ( f(x, y) : z = f(x, y) )</td>
<td>((a, b, f(a, b)))</td>
<td>((-f_x(a, b), -f_y(a, b), 1))</td>
</tr>
<tr>
<td>equation ( F(x, y, z) = 0 )</td>
<td>((a, b, c)) with ( F(a, b, c) = 0 )</td>
<td>(\nabla F \big</td>
</tr>
</tbody>
</table>

The point is that the first line is a special case of the second line. If our surface is the graph of \( f(x, y) \), then it has the equation \( z = f(x, y) \). Let us make a new function: \( F(x, y, z) = z - f(x, y) \). \textbf{Note:} \( F(x, y, z) \) is a function of 3 variables, while \( f(x, y) \) is a function of 2 variables!

Then our graph is also given by the equation \( F(x, y, z) = 0 \). The gradient of \( F \) is exactly \( \nabla F = \langle -f_x, -f_y, 1 \rangle \).

This leads to the same equation of the tangent plane that we get from linearization: we learned earlier in this course that the tangent plane to the graph of \( f(x, y) \) is given by:

\[
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).
\]

A normal vector to this plane is: \( \langle -f_x(a, b), -f_y(a, b), 1 \rangle \), which (of course!) agrees with the gradient \( \nabla F \).

\textbf{The same equation using the cross product.} We also discussed that if you take the graph \( z = f(x, y) \) and consider its cross-sections with the planes \( x = a \) and \( y = b \), you get two curves, called the \textit{traces} of \( f(x, y) \) on these planes. The tangent vectors to these curves are: \( \mathbf{v}_1 = \langle 1, 0, f_x(a, b) \rangle \) and \( \mathbf{v}_2 = \langle 0, 1, f_y(a, b) \rangle \). Why so: consider for example the plane \( x = a \). The trace of \( f(x, y) \) on this plane is the graph of the function of \( y \), let us call it \( h(y) := f(a, y) \) that we obtain by plugging in \( x = a \) into \( f \). The slope of the tangent line to this graph at \( y = b \) is \( h'(b) = f_y(a, b) \). The slope is the ratio of the \( z \)-component to the \( y \)-component here; we can choose the \( y \)-component to be 1. Then we get that our tangent vector should have the \( z \)-component equal to \( f_y(a, b) \). Its \( x \)-component is 0 because it lies in the plane \( x = a \) (a vertical plane parallel to the \( y \)-axis; all vectors in it have zero \( x \)-component). Thus we get the vector \( \langle 0, 1, f_y(a, b) \rangle \).

Now the tangent plane to the graph \( z = f(x, y) \) at \((a, b)\) must contain both these tangent vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), so we can use the usual method for finding an equation of a plane containing a given point and parallel to two given vectors. Its normal is \( \mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 \). Computing the cross product, we
get:

\[ \mathbf{n} = (1, 0, f_x(a, b)) \times (0, 1, f_y(a, b)) = (-f_x(a, b), -f_y(a, b), 1). \]

This gives us the same equation of the plane as line 1 in the table above.