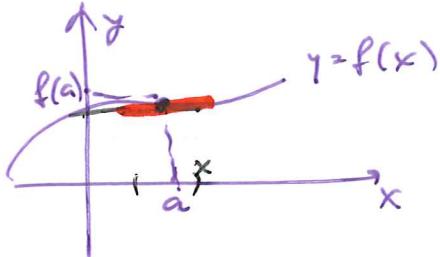


To day: Linear approximations, tangent planes.

Recall: $f(x)$ - function of one variable



Try to approximate $f(x)$ near a by something simpler.

$f(x) \approx f(a)$ - the crudest approximation. (constant)

$f(x) \approx$ tangent line approximation

$$= \underbrace{f(a) + f'(a)(x-a)}_{\text{slope;}}$$

$$y = f(a) + f'(a)(x-a)$$

\Rightarrow an equation of the tangent line at $(a, f(a))$

$$\text{slope} = f'(a)$$

at $x=a$, get $y = f(a)$,

so this is a line through $(a, f(a))$ with correct slope.

Note: we build an approximation term-by-term:

$$f(a) + f'(a)(x-a)$$

start

linear approx. -
best of all linear.

Note: best quadratic approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2 + \dots$$

$(x \text{ near } a)$

higher-order terms
get smaller
and the sum
gets closer
and closer
to $f(x)$.

Side note :

equation of a line:

$$y = \underline{kx} + b \quad - \text{slope-intercept form.}$$

$$k = f'(a). \quad \text{We know: } b = \underline{y(0)}.$$

has to contain $(a, f(a))$

could plug it in and solve for b :

$$x = a$$

$$y = f(a)$$

$$f(a) = \underbrace{f'(a) \cdot a}_k + b \quad \text{solve :}$$

$$b = f(a) - f'(a) \cdot a$$

the same as
we get from $y = \underline{f(a) + f'(a)(x-a)}$

Why do we care for a linear approximation if we know a formula for $f(x)$?

• sometimes want to reduce computation



• sometimes, don't have a formula for $f(x)$

but can measure: $f(a)$
 $f'(a)$.

In 2 variables :

Linear approximation for $f(x,y)$ near (a,b) :

$$f(x,y) \approx \underbrace{f(a,b)}_{\substack{\uparrow \\ \text{crudest approximation}}} + \frac{\partial f}{\partial x} \Big|_{(a,b)} (x-a) + \frac{\partial f}{\partial y} \Big|_{(a,b)} (y-b)$$

$L(x,y)$ linear approximation (linearization)
of $f(x,y)$ near (a,b) .

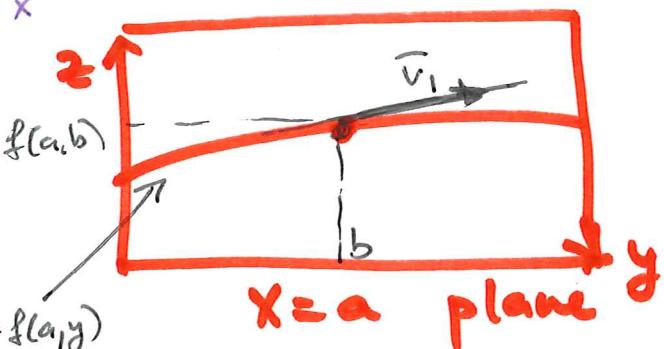
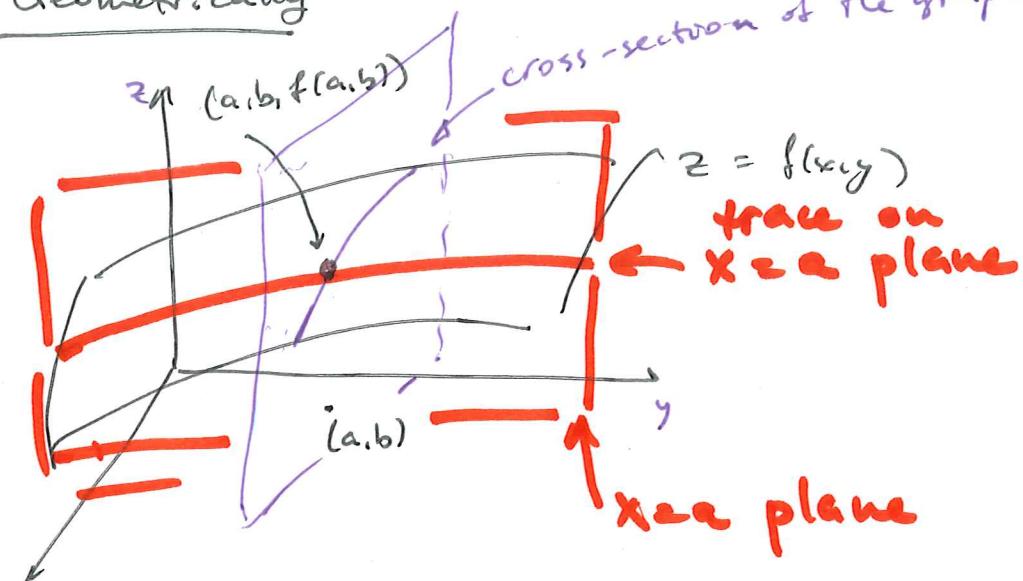
Set

$z = L(x,y)$ ← graph of the linearization.

• it is a plane.

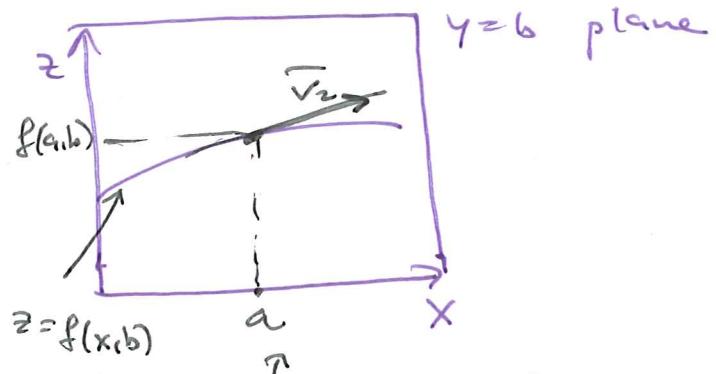
• it is the tangent plane at $(a,b, f(a,b))$
to the graph $z = f(x,y)$.

Geometrically



slope of the tangent vector is $\frac{d}{dy} \underbrace{f(a,y)}_{y=b \text{ fn of } y}$

$$= \frac{\partial f}{\partial y} \Big|_{(a,b)}$$



$$\text{slope} = \frac{\partial f}{\partial x} \Big|_{(a,b)}$$

The tangent plane to our graph contains both of these tangent vectors.

As we think of them as 3d vectors, we get:

$$\vec{v}_1 = \langle 0, 1, f_y(a,b) \rangle \leftarrow \text{vector in the } yz\text{-plane (red plane)}$$

$$\vec{v}_2 = \langle 1, 0, f_x(a,b) \rangle \text{ - vector in the } xz\text{-plane}$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & f_y(a,b) \\ 1 & 0 & f_x(a,b) \end{vmatrix} = \langle f_x(a,b), f_y(a,b), -1 \rangle$$

↑
normal to the tangent plane

This gives equation of the plane:

$$\boxed{f_x(a,b)(x-a) + f_y(a,b)(y-b) - 1 \cdot (z - f(a,b)) = 0}$$

$$\Leftrightarrow \boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

- so our equation is correct!

Note: You do not need to do this every time you need an equation of the tangent plane. Just use the linearization

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(no cross products!)

(so this gives a shortcut for 2.5)

What if we have more variables?

$$f(x_1, x_2, x_3, \dots, x_n)$$

Linearization: near (a_1, a_2, \dots, a_n)

$$\begin{aligned} f(x_1, \dots, x_n) \approx f(a_1, \dots, a_n) &+ \left. \frac{\partial f}{\partial x_1} \right|_{(a_1, \dots, a_n)} (x_1 - a_1) \\ &+ \dots + \left. \frac{\partial f}{\partial x_n} \right|_{(a_1, \dots, a_n)} (x_n - a_n) \end{aligned}$$

For $f(x, y, z)$:

$$\begin{aligned} f(x, y, z) \approx f(a, b, c) &+ \left. \frac{\partial f}{\partial x} \right|_{(a, b, c)} (x - a) + \left. \frac{\partial f}{\partial y} \right|_{(a, b, c)} (y - b) \\ &+ \left. \frac{\partial f}{\partial z} \right|_{(a, b, c)} (z - c) \end{aligned}$$

Note: Question 2 in the worksheet:

see posted solutions.

Read 2.5, 2.6, 2.4