

## Chain Rule and Implicit Differentiation

Now that we know Chain Rule, we can derive a simpler method for implicit differentiation:  
suppose we have a relation  $F(x,y,z) = 0$ , where  $F$  is a function of 3 variables.

(For example,  $F(x,y,z) = \sin(xz+y)$ )

This relation defines  $z$  as an implicit function of  $x$  and  $y$ . (It also defines  $x$  as an implicit function of  $y$  and  $z$ , or  $y$  as an implicit function of  $x$  and  $z$ . You can pick any two variables to be the independent variables).

Then there is a simple formula for the partial derivatives of the implicit function.  
If we decide that  $x, y$  are independent variables and  $z$  is the implicit function, then:

$$\boxed{\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z}, \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z}\end{aligned}}$$

where  $F_x, F_y, F_z$  are the usual partial derivatives of  $F(x,y,z)$ .

In our example

$$F(x,y,z) = \sin(xz+y)$$

$$F_x = \cos(xz+y) \cdot z$$

$$F_y = \cos(xz+y)$$

$$F_z = \cos(xz+y) \cdot x$$

$$\text{Then } \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z}{x}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{1}{x}$$

(Note: if we did this differentiation the old way - differentiating  ~~$\sin(xz+y)$~~   $\sin(xz+y)$  and thinking of  $z$  as a function of  $x, y$  and then solving for  $\frac{\partial z}{\partial x}$ , we would get the same answer — this formula just gives the answer faster).

Why this formula works:

$$\text{we have } F(x, y, z) = 0.$$

Think of all three variables  $x, y, z$  as functions of  $x$  and  $y$ :  $x(x, y) = x$   
 $y(x, y) = y$   
and  $z$  is the unknown implicit function  $z(x, y)$ .

Now differentiate  $F(x, y, z) = 0$  with respect to  $x$ , using Chain Rule:

$$F_x \cdot \underbrace{\frac{\partial x}{\partial x}}_1 + F_y \cdot \underbrace{\frac{\partial y}{\partial x}}_0 + F_z \cdot \underbrace{\frac{\partial z}{\partial x}}_0 = 0$$

since  
 $y$  doesn't  
depend on  $x$ .

Now solve for  $\frac{\partial z}{\partial x}$ :

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$