Today: continuity, limits  
- partial derivatives.

for functions of several variables.

Continuity: Recall in Calc I, \( f(x) \) is continuous if you can draw its graph without lifting a pen.

\[
\lim_{{x \to a}} f(x) = L
\]

or \( \text{values of } f(x) \text{ approach } L \) as \( x \) approaches \( a \).

in many variables, you need to be much more precise about what continuity means:

**Definition:** \( f(x,y) \) is continuous at \((a,b)\)

if \( \forall \varepsilon > 0 \), exists \( \delta > 0 \) such that

if distance from \((x,y)\) to \((a,b)\) is less than \( \delta \),

then \[ |f(x,y) - f(a,b)| < \varepsilon. \]

(\( \varepsilon, \delta \) - Greek letters traditionally used to mean small numbers.

This says: "when \((x,y)\) is close to \((a,b)\),

\( \delta \) measures how close the input \((x,y)\)

needs to be to \((a,b)\)

in order for the value \( f(x,y) \) to be no more than \( \varepsilon \) far from \( f(a,b) \)."
Example: Suppose you are trying to measure the volume of a box with sides $x, y, z$.

$$V(x, y, z) = x \cdot y \cdot z$$

Maybe want the error in our volume measurement to be less than 1 cm$^3$. \[\text{choose } \varepsilon = 1 \text{ cm}^3\]

Note: every measurement tool has its own "built-in error". (ruler: up to 1 mm precise).

Example: Last fall, bike-to-work.

Math Dept: logged 1893.7 km

Chemistry: 1893.4 km

Same, not significant because measurements of distance are not as precise.

$S(x, y, z) = x \cdot y \cdot z$ is continuous.

So: exists a precision (call it $\delta$) for the measurements $x, y, z$, such that if the error on $x, y, z$ is $\leq \delta$, then the volume is guaranteed to have the error $< \varepsilon = 1 \text{ cm}^3$.

This $\delta$ will depend on how big the box is!
Example (on the exam).

\[ f(x, y) = \frac{x^2 y}{x^4 + y^2} \]

Can define using "\( \varepsilon \) and \( \delta \)."

Does it have a limit at \((0, 0)\)?

Looks like it is \( \frac{0}{0} \) - indeterminate.

In one variable, there was L'Hôpital's rule to determine the limit.

Do not have it in several variables!

We show the limit does not exist here:

Try and see what the limits are as \((x, y)\) approaches \((0, 0)\) along different paths:

Picture of the domain (Not a graph of \( f(x, y) \)!

Try: \((x, y)\) approaches \((0, 0)\) along the line \( y = mx \).
Plug \( y = mx \) into \( f \):

\[
\frac{f(x, mx)}{x^2 + (mx)^2} = \frac{mx^3}{x^4 + m^2 x^2} = \frac{mx}{x^2 + m^2}
\]

\( m \) is fixed \( (m \neq 0) \)

let \( x \to 0 \).

\[
\frac{0}{m^2} = 0.
\]

So, along every line \( y = mx \) (except \( x\)-axis),
we get \( f(x, y) \to 0 \), as \((x, y) \to 0\).

So maybe \( \lim_{(x, y) \to (0, 0)} f(x, y) = 0 \)? \( \text{No} \):

Try approaching differently:

\[
f(x, mx^2) = \frac{x^2 \cdot (mx^2)}{x^4 + m^2 x^4} = \frac{mx}{1 + m^2}
\]

These parabolas are the level curves
of \( f(x, y) \)!

(Valued on each parabola turned out to be constant!)

We get different limits at \((0, 0)\) depending on \( m \)!

So there is no limit at \((0, 0)\) !

What about approaches: \( \text{?} \)

That's why we need the \( \varepsilon - \delta \) definition!
Upshot: you can use approach along different lines /parabolas/.

To show that $f(x, y)$ does NOT have a limit at a particular point.

But this will not be useful for proving that $f(x, y)$ has a limit (when it has a limit).

The only way to prove it has a limit is using the $\varepsilon-\delta$ definition (not on the exam).

We will take on faith that $\hat{5}$ functions obtained by arithmetic operations and compositions from continuous functions of a single variable are continuous.

Addition, mult.

and division when denominator $\neq 0$ on the whole domain.

Example: \[ \frac{\cos^3(x^2y - e^y)}{1 + e^{xy} - \sin(x+y)} \]

- continuous on $\mathbb{R}^2$.

In practice, for us issues with continuity /differentiability will only arise on the boundary of the domain and we will ignore them (will work inside the domain where the functions are continuous).
Partial derivatives:

\[ f(x, y) \] will have two partial derivatives!

\[ f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = \frac{\partial f}{\partial y} \]

- Keep y constant, think of it as a function of x, and differentiate.
- Keep x constant, think of it as a function of y, and differentiate.

**Def:** Let \((a, b)\) be a point in the domain of \(f(x, y)\).

Then \( f_x (a, b) = \frac{\partial f}{\partial x} \bigg|_{(a, b)} = \lim_{t \to 0} \frac{f(a + t, b) - f(a, b)}{t} \)

**Note:**

\[ f_y (a, b) = \frac{\partial f}{\partial y} \bigg|_{(a, b)} = \lim_{t \to 0} \frac{f(a, b + t) - f(a, b)}{t} \]

\( y = b \) is fixed

There are single-variable definitions!
Recall: for a function \( f(x) \) of one variable,
\[
 f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}
\]

measure the rate of change of \( f \) between \( a \) and \( a+t \).

Figured out the formulas for the derivatives of all the basic functions: for \( \cos(x) \), \( e^x \) was hard!
- rules for \( + \) sums, products, quotients, compositions.
  - chain rule

Then differentiate automatically.

A way to think about partial derivatives:

- start with \( f(x,y) \).
- Plug in \( y = b \) (restrict \( f \) to the horizontal line \( y = b \)).
- Get: \( f(x,b) \), a function of one variable \( x \).
  - and we differentiate it as usual!
Example: \( f(x, y) = x^2 e^{\cos(y)} \)

\[
\frac{\partial f}{\partial x} = 2x e^{\cos y}
\]

\[
\frac{\partial f}{\partial y} = x^2 \cdot \frac{\partial}{\partial y}(e^{\cos y})
\]

Treat \( x \) as constant.

Partial derivative with respect to \( y \)

= \( x^2 \cdot e^{\cos y} \cdot (-\sin y) \)

- Geometric meaning

Recall: in one variable, \( f'(a) = \text{slope of the tangent line to the graph of } f(x) \) at \((a, f(a))\).
In 2 variables:

\[ z = f(x, y) \]

Trace of graph of \( f(x, y) \) on the plane \( x = a \)

\[ \frac{\partial f}{\partial x} \bigg|_{(a, b)} = \text{slope of the tangent line to this trace} \]

In 2 variables:

\[ y = b - \text{vertical plane} \]
Draw tangent lines to the traces.
The partial derivatives measure the slopes of these tangent lines.

Example: let \( f(x,y) \) be some function such that
\[
f(3,4) = 5
\]
and \( \frac{\partial f}{\partial x} \bigg|_{(3,4)} = 2 \) and \( \frac{\partial f}{\partial y} \bigg|_{(3,4)} = -1 \).

Find the equation of the tangent plane to the graph of \( f(x,y) \) at the point \((3,4,5)\).

We are given: \( \vec{v}_1 = \langle 1, 0, 2 \rangle \)

- tangent vector to the trace of \( f \) on the plane \( y = 4 \)

We choose it

We consider the plane \( y = 4 \) and the trace of \( f \)

On this plane \( \Delta x = 1 \).

- tangent vector to the trace on the plane \( x = 3 \)

\[ \vec{v}_2 = \langle 0, 1, -1 \rangle \]

The plane must contain the point itself: \((3,4,5)\)

Our point on the graph.

Then we get the equation:

\[ -2(x-3) + (y-4) + (z-5) = 0 \]