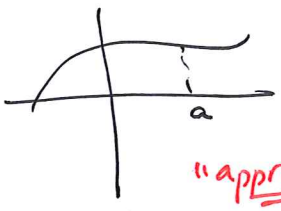
Lecture 9
February 4

Today: - continuity, limits
- partial derivatives.

for functions of several variables.

Continuity: Recall in Calc-1, "$f(x)$ is continuous if you can draw its graph without lifting a pen"



or "values of $f(x)$ approach $f(a)$ as x approaches a" ← continuous at a

"approaches" is bad. →

...

in many variables, you need to be much more precise about what continuity means:

not on the exam

Definition: $f(x,y)$ is continuous at (a,b) if $\forall\, \varepsilon > 0$, exists $\delta > 0$ such that if distance from (x,y) to (a,b) is less than δ, then $|f(x,y) - f(a,b)| < \varepsilon$.

(ε, δ - Greek letters traditionally used to mean small numbers).

This says: "when (x,y) is close to (a,b) then $f(x,y)$ is close to $f(a,b)$" ← ε measures how close to $f(a,b)$ we need to get

δ measures how close the input (x,y) needs to be to (a,b) in order for the value of $f(x,y)$ to be no more than ε far from $f(a,b)$.

Explaining the meaning of the definition of continuity

Example: suppose you are trying to measure the volume of a box with sides x, y, z.

$$V(x,y,z) = x \cdot y \cdot z$$

Maybe want the error in our volume measurement to be less than $1\,cm^3$. ← $\boxed{\text{choose } \varepsilon = 1\,cm^3}$

Note: every measurement tool has its own "built-in error". (ruler: up to 1 mm precise).

example: last fall, bike-to-work.

Same — Math Dept logged 1893.7 km
Chemistry: 1893.4 km

not significant because measurements of distances are not as precise)

$f(x,y,z) = x \cdot y \cdot z$ is continuous.

so: exists a precision (call it δ) for the measurements x, y, z, such that if the error on x, y, z is $\leq \delta$, then the volume is guaranteed to have the error $< \varepsilon = 1\,cm^3$.

This δ will depend on how big the box is!

Example (on the exam).

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}$$

can define using "ε and δ"

Does it have a limit at $(0,0)$?

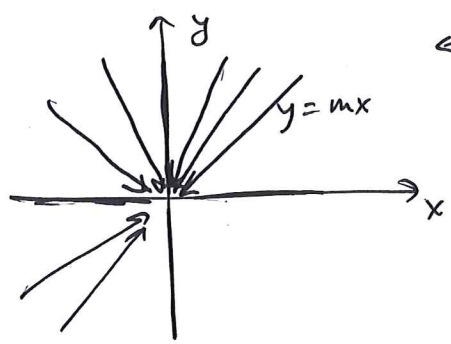
looks like it is "$\frac{0}{0}$" - indeterminate.

in one variable, there was L'Hôpital's rule to help determine the limit.

do not have it in several variables!

We show the limit does not exist here:

try and see what the limits are as (x,y) approaches $(0,0)$ along different paths:



← picture of the domain (NOT a graph of $f(x,y)$!)

Try: (x,y) approaches $(0,0)$ along the line $y = mx$.

Plug in $y = mx$ into f:

$$f(x, \underbrace{mx}_{y=mx}) = \frac{x^2 \cdot (mx)}{x^4 + (mx)^2} = \frac{m x^3}{x^4 + m^2 x^2} = \frac{mx}{\underbrace{x^2}_{\to 0} + m^2} \to \frac{0}{m^2} = 0.$$

m is fixed ($m \neq 0$)

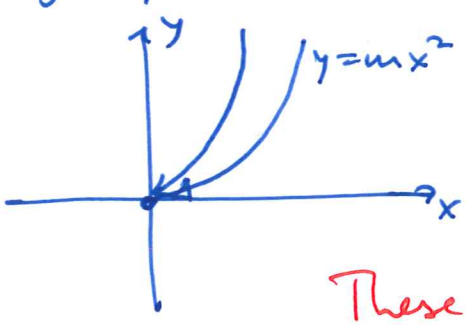
let $x \to 0$.

function of only x.

So, along <u>every line</u> $y = mx$ (except x-axis), we get $f(x,y) \to 0$, as $(x,y) \to 0$.

So maybe $\lim_{(x,y)\to(0,0)} f(x,y) = 0$? <u>No:</u>

Try approaching differently:



$$f(x, mx^2) = \frac{x^2 \cdot (mx^2)}{x^4 + m^2 x^4} = \frac{m}{1+m^2}$$

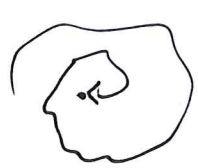
These parabolas are the <u>level curves</u> of $f(x,y)$!

(value on each parabola turned out to be constant!)

We get different limits at $(0,0)$ depending on m!

So <u>there is no limit at $(0,0)$</u>!

What about approaches:



That's why we need the ε-δ definition!

Upshot: you can use approach along different lines / parabolas / ...

on the exam. to show that $f(x,y)$ does <u>NOT</u> have a limit at a particular point.

But this will not be useful for proving that $f(x,y)$ <u>has</u> a limit (when it has a limit).

The only way to prove it <u>has</u> a limit is using the $\varepsilon-\delta$ definition (not on the exam).

<u>We will take on faith that</u>: functions obtained by arithmetic operations and compositions from continuous functions of a single variable are continuous.

addition, mult. and division when denominator $\neq 0$ on the whole domain.

<u>Example</u>

$$\frac{\cos^3(x^2y - e^y)}{1 + e^{xy - \sin(x+y)}}$$ – continuous on $\mathbb{R}^2$.

In practice, for us issues with continuity / differentiability will only arise on the boundary of the domain and we will ignore them (will work <u>inside</u> the domain where the functions are continuous).

Partial derivatives:

~~Def~~ $f(x,y)$ will have two partial derivatives:

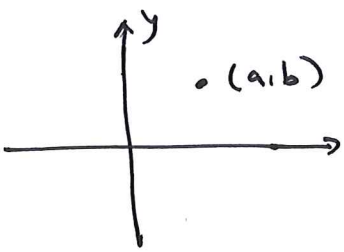
$f_x = \frac{\partial f}{\partial x}$ and ← keep y constant, think of it as a function of x, and differentiate

$f_y = \frac{\partial f}{\partial y}$ ← keep x constant, think of it as a function of y and differentiate.

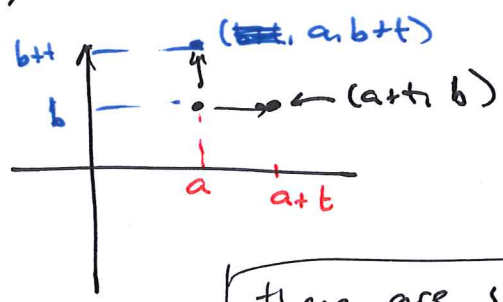
Def: Let (a,b) be a point in the domain of $f(x,y)$. $y=b$ is fixed

Then $\underbrace{f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}}_{\text{notation}} = \lim_{t\to 0} \frac{f(a+t,\, b) - f(a,b)}{t}$

$$f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{t\to 0} \frac{f(a,\, b+t) - f(a,b)}{t}$$



domain of $f(x,y)$



$\boxed{\text{these are single-variable definitions!}}$

Recall: for a function $f(x)$ of one variable,

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}$$

measure the rate of change of f between a and $a+t$.

Figured out the formulas for the derivatives of all the basic functions: for $\cos(x)$, e^x - was hard!

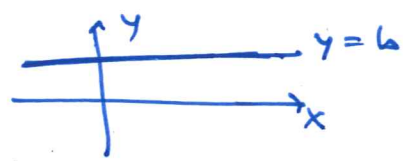
- rules for sums, products, quotients, compositions.
  chain rule

Then differentiate automatically.

A way to think about partial derivatives:

start with $f(x,y)$.

Plug in $y=b$ (restrict f to the horizontal line



)

Get: $f(x,b)$

- function of one variable x

and we differentiate it as usual!

Example: $f(x,y) = x^2 e^{\cos(y)}$

$$\frac{\partial f}{\partial x} = 2x\, e^{\cos y}$$

new function of x, y

$$\frac{\partial f}{\partial y} = x^2 \cdot \frac{\partial}{\partial y}\left(e^{\cos y}\right)$$

treat x as constant

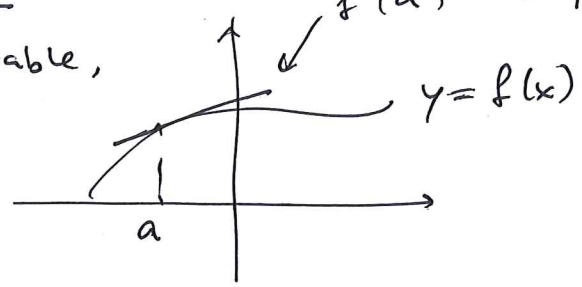
partial derivative with respect to y

Treat y as a constant!

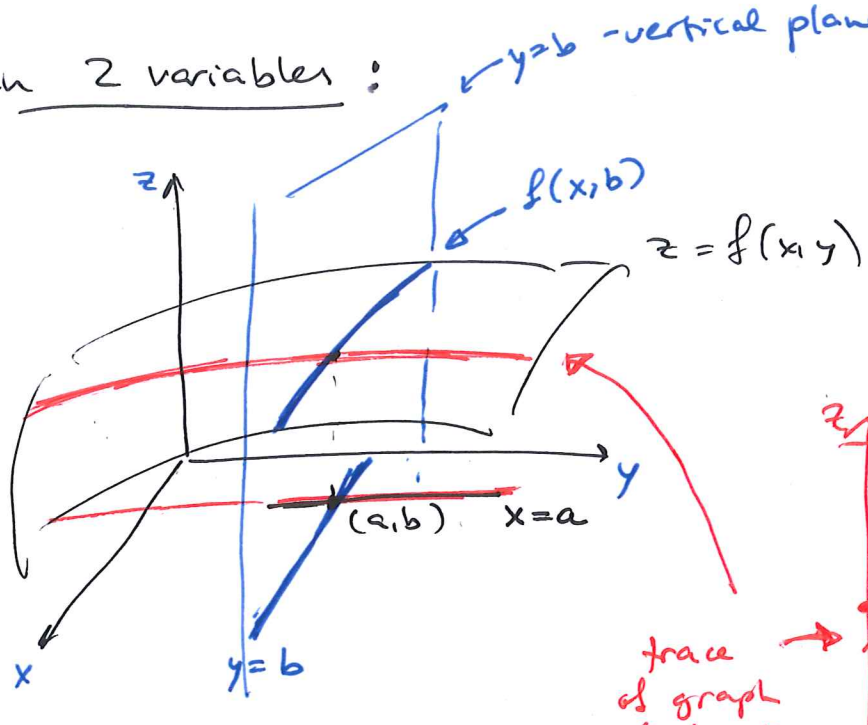
$$= x^2 \cdot e^{\cos y} \cdot (-\sin y)$$

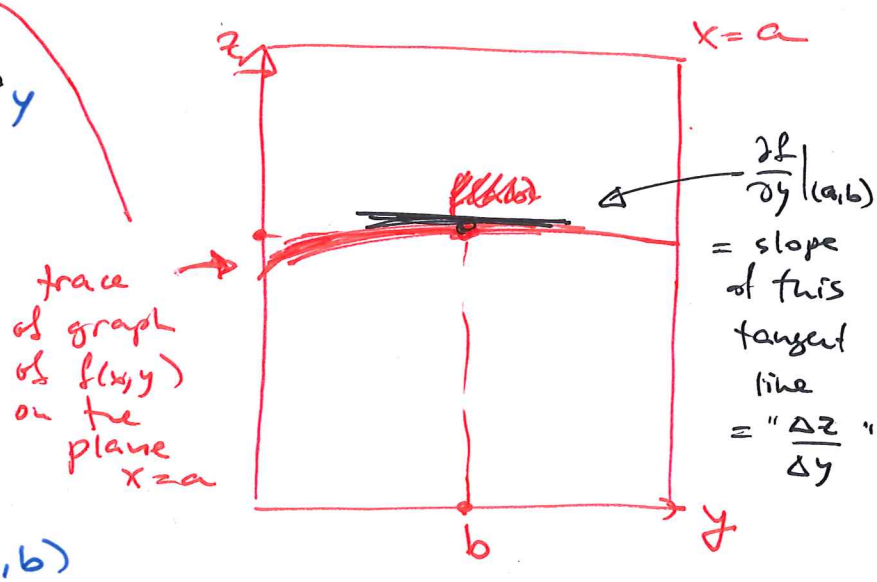
- Geometric meaning

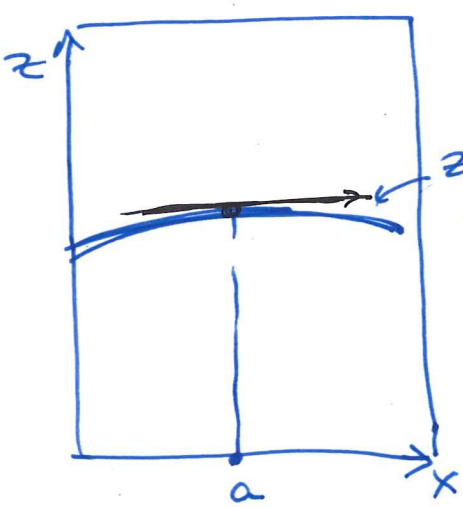
recall: in one variable,



In 2 variables:







← trace of our graph of f(x,y) on the plane y=b

$\frac{\partial f}{\partial x}\Big|_{(a,b)}$ = slope of the tangent line to this trace (in xz variables) = "$\frac{\Delta z}{\Delta x}$"

Draw tangent lines to the traces.
The partial derivatives measure the slopes of these tangent lines.

<u>Example</u>: let $f(x,y)$ be some function such that

$$f(3,4) = 5$$

and $\left.\frac{\partial f}{\partial x}\right|_{(3,4)} = 2$ and $\left.\frac{\partial f}{\partial y}\right|_{(3,4)} = -1.$

Find the equation of the tangent plane to the graph of $f(x,y)$ at the point $(3,4,5)$.

We <u>are given</u>: $\bar{v}_1 = \langle 1, 0, 2 \rangle$ — tangent vector to this trace of f.

choose it

$\frac{\Delta z}{\Delta x}$ our $\Delta x = 1$.

consider the plane $y = 4$ and the trace of f on this plane

$\bar{v}_2 = \langle 0, 1, -1 \rangle$ — tangent vector to the trace on the plane $x = 3$

x fixed $\quad \frac{\Delta z}{\Delta y}$.

$$\bar{n} = \bar{v}_1 \times \bar{v}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix} = -2\bar{i} + \bar{j} + \bar{k}$$

The plane must contain the point itself: $(3,4,5)$
is our point on the graph.
Then we get the equation:

$$\boxed{-2(x-3) + (y-4) + (z-5) = 0}$$