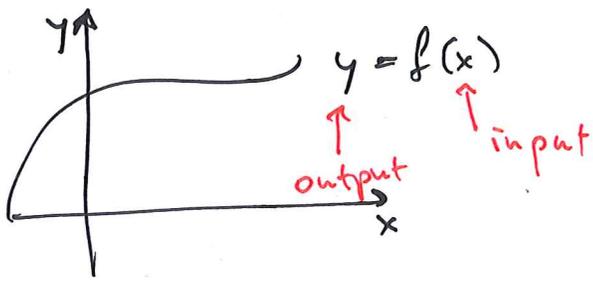


Today: functions of several variables.
(finish quadric surfaces).

1 variable

- Think of a function as a graph:



- Functions of 2 variables:

input comes from some region in the plane \mathbb{R}^2

output : a number.

Need 3 dimensions for a graph.

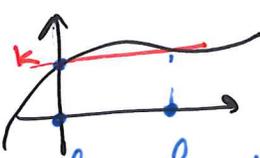
- often cannot plot without a computer.

- Think of different ways to use our geometric intuition to think about functions.

Contour plot: plot of level curves

Level curve:

the set of all points (x,y) in the domain such that $f(x,y) = k$.



for functions of 1 variable it would be (usually finitely many) solutions to $f(x) = k$.

Lecture 8, Jan. 30

Note: I was using various graphs and contourplots during lecture.

In Wolframalpha, type:
`plot(z = x^2 - y^2)`.

Then type
`plot(z = 1/sqrt(x^2 - y^2))`
for illustration of the worksheet problem.

Look at both the graphs and contourplots.

in 2 variables:

ex: $x^2 - y^2 = k$

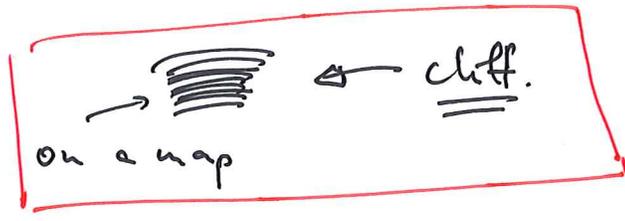
Get a whole curve in the plane!

function:
 $f(x,y) = x^2 - y^2$

graph:
 $z = x^2 - y^2$
for z

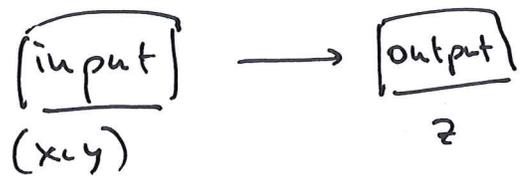
level curves close together:

function changes fast (don't have to go far to get to the next level)



level curves far apart - fairly flat terrain.

Note: function: doesn't always mean a formula!



- function.
Don't have to have a formula!

Contour plot gives us a lot of information about a function, even if we are not given a formula.

Domain: \leftarrow (subset of \mathbb{R}^2) \leftarrow picture.
The set of values (x, y) where $f(x, y)$
is defined. (domain is a region in \mathbb{R}^2 !)

Range: the set of all possible values that
 $f(x, y)$ can take:

$$\text{Range} = \{ k \in \mathbb{R} : \exists x, y : f(x, y) = k \}$$

= set of values that give
non-empty level curve.

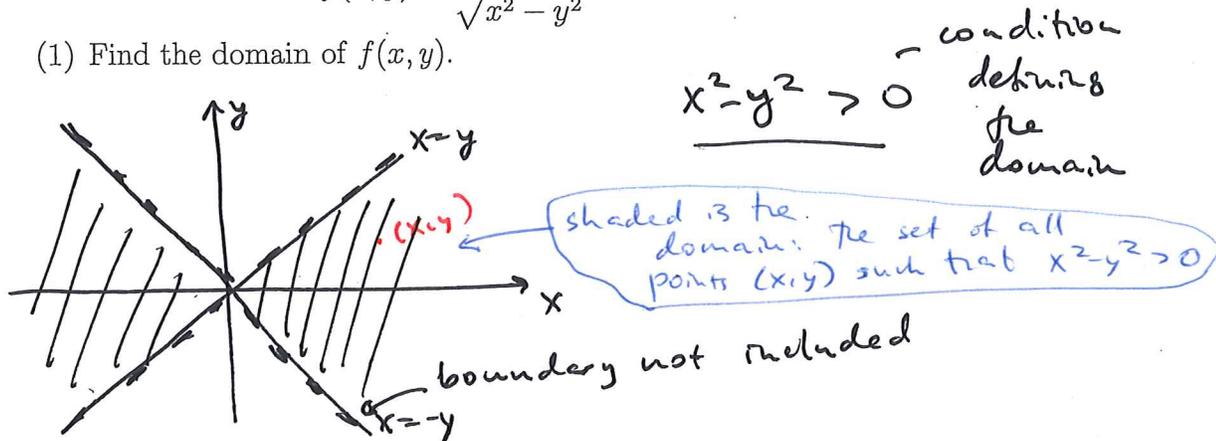
- expect an interval of values.

(subset of \mathbb{R})

Worksheet 4: Functions of two variables: domain, range, level curves. All the questions on this worksheet are about the function

$$f(x, y) = \frac{1}{\sqrt{x^2 - y^2}}$$

(1) Find the domain of $f(x, y)$.



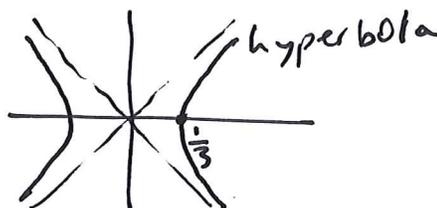
(2) sketch the set of all (x, y) such that $f(x, y) = 3$.

$$f(x, y) = 3$$

$$x^2 - y^2 = \frac{1}{9}$$

$$\frac{1}{\sqrt{x^2 - y^2}} = 3$$

$$1 = 3^2(x^2 - y^2)$$



(3) Find the range of $f(x, y)$. = $(0, \infty)$

Look for all k such that: $f(x, y) = k$ has a solution.

$$\frac{1}{\sqrt{x^2 - y^2}} = k \leftarrow \text{careful when squaring: } k \leq 0: \text{ no solution.}$$

$$\left\{ \begin{array}{l} 1 = k^2(x^2 - y^2) \\ k > 0 \end{array} \right. \rightarrow \text{there will be solutions.}$$

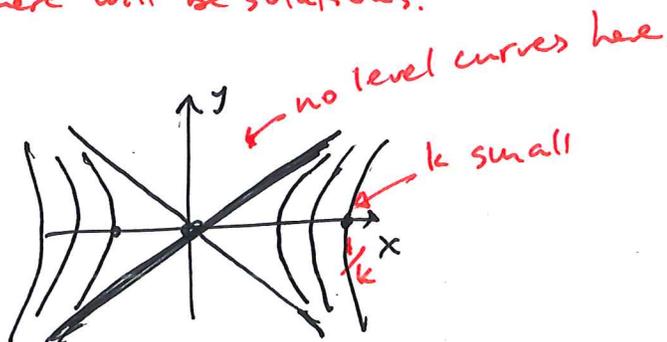
(4) Sketch the level curves of $f(x, y)$.

- try to get the shape right.

Get some intercepts right ...

- idea of spacing

further apart: k is changing slowly.

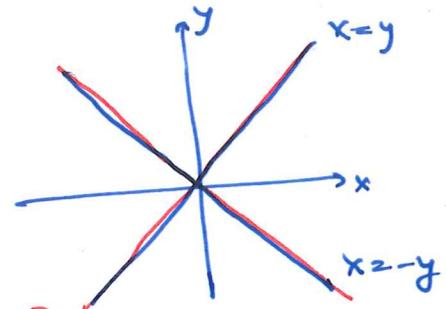


Our example: $x^2 - y^2 = f(x, y)$

Take $k=0$: get 2 lines

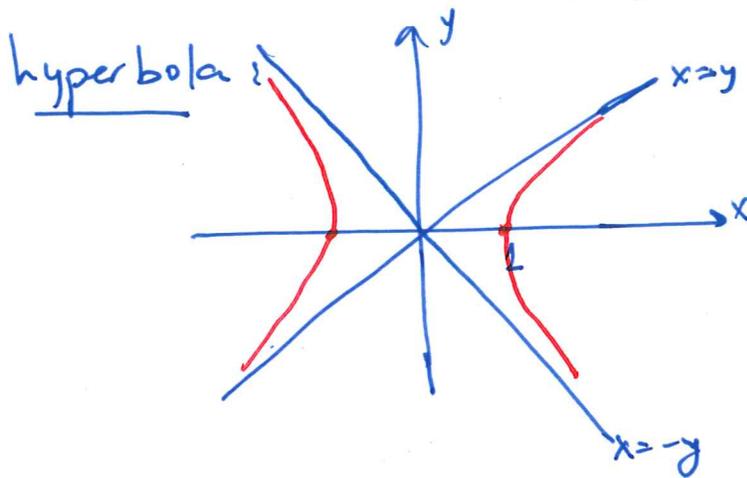
$$x^2 - y^2 = 0$$

$$(x-y)(x+y) = 0 \Leftrightarrow \begin{cases} x=y \\ x=-y \end{cases}$$

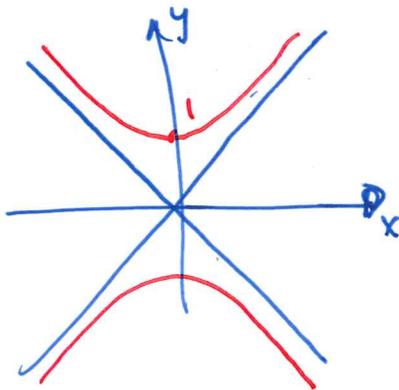


level curve $k=0$

level curve $k=1$: $x^2 - y^2 = 1$
 $(x-y)(x+y) = 1$



level curve $k=-1$:



Contourplot: you take equally spaced values of k .

$k = -1, 0, 1, 2, 3, 4, \dots$

or $k = 100, 110, 120, \dots$

or $k = -0.1, -0.2, -0.3, \dots$

(computer chooses some set of k 's).

Topo map: contourplot of the function:

(x, y) - coordinates $\begin{matrix} E/N \\ \uparrow \quad \uparrow \\ x \quad y \end{matrix}$

output: altitude above sea level

Mountain = graph of the altitude function.

Function of 3 variables

(x, y, z) \mapsto $f(x, y, z)$ - output.
input

Can I have its graph? \leftarrow could make an animation!
(use time as the 4th axis :))

domain uses up our 3 dimensions.

Need another axis to make a graph.

Can make a graph: subset of \mathbb{R}^4 , can talk about it.
but cannot visualize.

So, if we want to "visualize" a function of 3 variables, at best we can think of its level surfaces

Contourplot - will be several surfaces in \mathbb{R}^3 ,

defined by the equations:

~~$f(x, y, z) = k$~~ $f(x, y, z) = k$ \leftarrow pick a few values of k

for each k ,

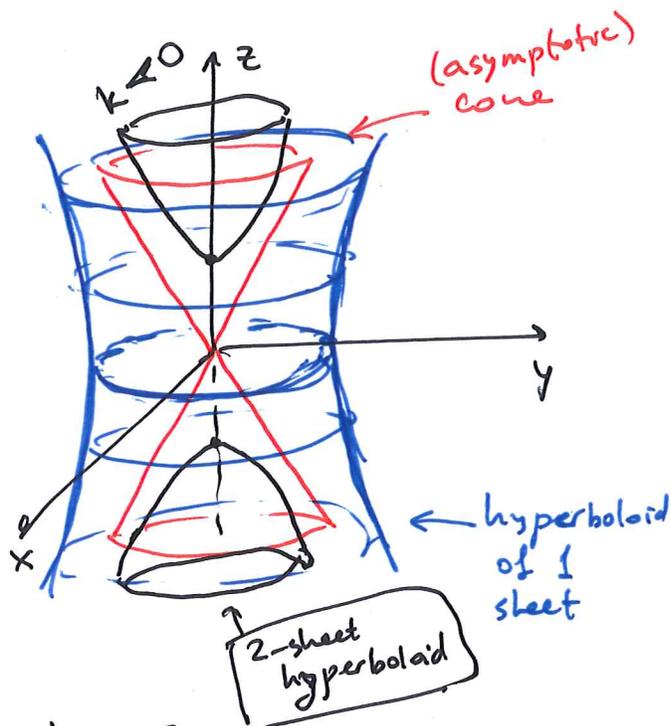
defines a surface in \mathbb{R}^3 .

Our quadric surfaces:

example: level surfaces of $f(x,y,z) = x^2 + y^2 - z^2$

These level surfaces will be given by the equations:

$$x^2 + y^2 - z^2 = k$$



when $k=0$

when $k > 0$

$$x^2 + y^2 - z^2 = k > 0$$

level surface

To analyze it,
slice it with:
horizontal planes.
say, xy-plane: $z=0$

$$x^2 + y^2 = k$$

if $z=c$

$$x^2 + y^2 - c^2 = k$$

$$x^2 + y^2 = k + c^2 \leftarrow \text{bigger circle}$$

when $k < 0$

$$x^2 + y^2 - z^2 = k$$

negative

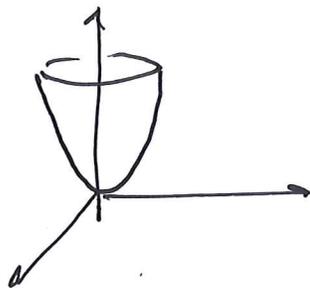
$$x^2 + y^2 = k + z^2$$

need this ≥ 0 .

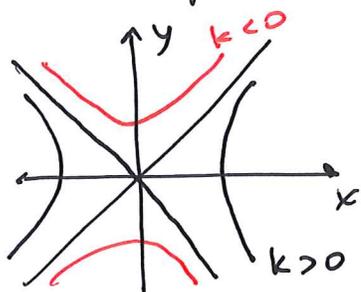
No solutions if $z^2 + k < 0$.

Back to paraboloids:

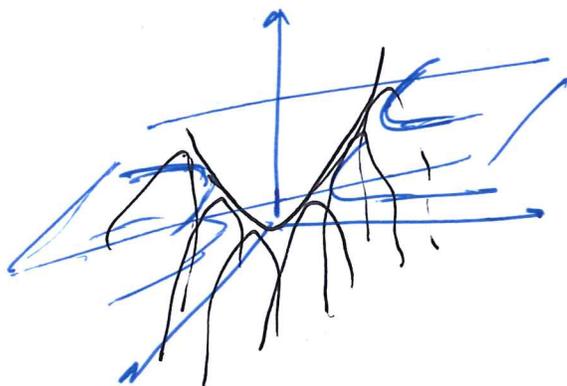
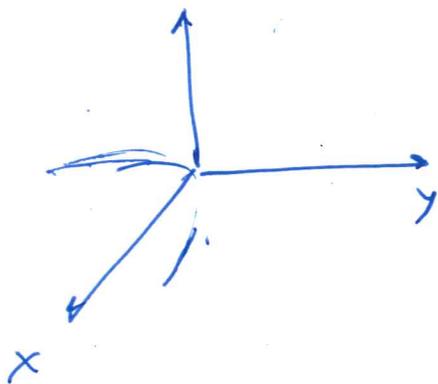
They are graphs: $z = x^2 + y^2$



$z = x^2 - y^2$ = hyperbolic paraboloid:
(looks like a saddle)



Why hyperbolic: level curves: $x^2 - y^2 = k$ - hyperbolas!
(cross-sections with horizontal planes)



We covered: quadric surfaces,
12.1 of Apex Calc!

Next: continuity, partial derivatives!
