

- Today :
- symmetric equations for lines in \mathbb{R}^3
 - distances
 - quadric surfaces.

Lecture 6.
Thursday Jan. 23

Recall: last time we developed parametric equations for a line:



(x, y, z) on the line through P
parallel to $v = \langle a, b, c \rangle$
(along)

$$P(x_0, y_0, z_0)$$

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

Symmetric equation of the same line:

in fact, two equations:

take the parametric equation and "try to solve for t ":

$$t = \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

← symmetric
equation
for the line.

Example: convert the ~~given~~ parametric equation

$$\begin{cases} x = 1 + 2t \\ y = 3 + t \\ z = 4 + 5t \end{cases}$$

to symmetric form:

Answer: $\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-4}{5}$ — symmetric equation.

plane parallel
to the z -axis

plane parallel to the x -axis

(represents our line as
the intersection of two
specific planes: one
parallel to z -axis, one to x -axis).

Trick example: what if our plane is parallel
to one of the coordinate planes:

i.e. its direction vector has some ~~a~~
component(s) = 0:

$$\begin{cases} x = 3 \\ y = 4 + t \\ z = 7 + 2t \end{cases}$$
 direction vector: $\vec{v} = \underline{\langle 0, 1, 2 \rangle}$

Then symmetric equation has the form:

$$\begin{cases} x = 3 \\ \frac{y-4}{1} = \frac{z-7}{2} \end{cases}$$
 — system of equations
(do Not divide by 0!).

Worksheet 3: Distances to lines and planes

- (1) Let the lines L_1 and L_2 be given by the parametric equations

$$\mathbf{r}_1(t) = t\mathbf{i} + (1 - 2t)\mathbf{j} + (2 + 3t)\mathbf{k},$$

$$\mathbf{r}_2(s) = (3 - 4s)\mathbf{i} + (2 + 3s)\mathbf{j} + (1 - 2s)\mathbf{k}.$$

*(if the same
names
rename t
+ s)*

*make sure
parameters have
different names!*

common
point?

- (a) Do these lines intersect?
 (b) Find an equation of the plane containing the line L_2 and parallel to the line L_1 . *L₁, L₂ are "skew"*
 (c) Find the distance between the lines L_1 and L_2 .

(a) Step 1: Write coordinate parametric equations.
 Look for a common point!

$$\begin{array}{l} \text{Line 1 : } \\ \left\{ \begin{array}{l} x = t \\ y = 1 - 2t \\ z = 2 + 3t \end{array} \right. \end{array}$$

$$\begin{array}{l} \text{Line 2 : } \\ \left\{ \begin{array}{l} x = 3 - 4s \\ y = 2 + 3s \\ z = 1 - 2s \end{array} \right. \end{array}$$

Common point would have:

$$\left\{ \begin{array}{l} t = 3 - 4s \\ 1 - 2t = 2 + 3s \\ 2 + 3t = 1 - 2s \end{array} \right.$$

try solving these:
 plug in t from
 line 1 to
 lines 2, 3:

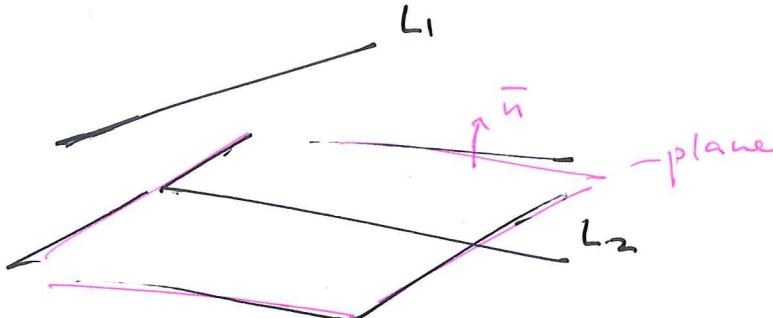
$$\left\{ \begin{array}{l} t = 3 - 4s \\ 1 - 2(3 - 4s) = 2 + 3s \\ 2 + 3(3 - 4s) = 1 - 2s \end{array} \right.$$

$$\begin{aligned} & \Rightarrow \begin{cases} t = 3 - 4s \\ 8s - 5 = 2 + 3s \\ -12s + 11 = 1 - 2s \end{cases} \\ & \Rightarrow \begin{cases} 5s = 7 \\ -10s = -10 \end{cases} \end{aligned}$$

no common solution!

The lines do not intersect.

(b)



Everything about
planes is ~~solved~~
solved
using normal
vectors.

\vec{n} - normal to my plane.

want: $\vec{n} \perp$ both L_1, L_2

parallel
to L_1

contains L_2

Let $\vec{v}_1 = \langle 1, -2, 3 \rangle$

$\vec{v}_2 = \langle -4, 3, -2 \rangle$ - direction vectors

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ -4 & 3 & -2 \end{vmatrix} = \langle -5, -10, -5 \rangle \parallel \langle 1, 2, 1 \rangle$$

Answer: (use any point on L_2 : want the plane to contain L_2).

let $P = (3, 2, 1)$ - point on L_2 .

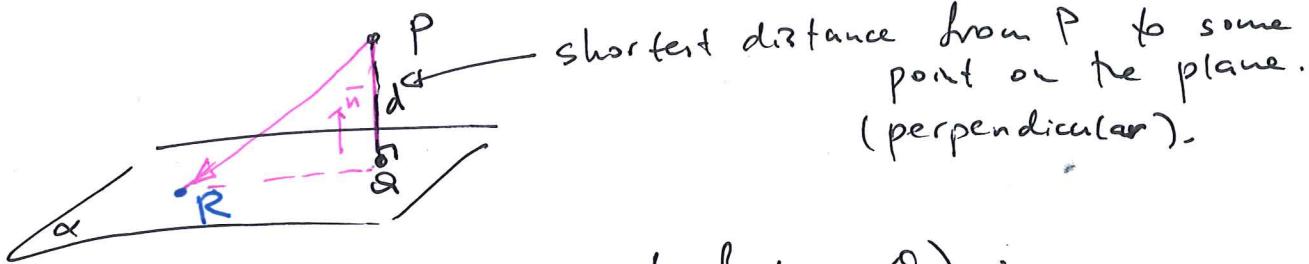
$$1 \cdot (x-3) + 2 \cdot (y-2) + 1 \cdot (z-1) = 0$$

$$x + 2y + z = 8 \text{ - also fine.}$$

$$-5(x-3) - 10(y-2) - 5(z-1) \text{ - also fine.}$$

Distances in space :

- point -to -plane

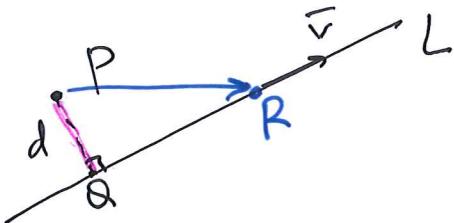


shortcut : (instead of finding Q) :

take any point R in the plane
and compute $|\text{proj}_{\vec{n}} \vec{PR}| = |\vec{PR} \cdot \frac{\vec{n}}{\|\vec{n}\|}| = d$

↑
normal
to the plane ↑
unit
normal
vector

- Point -to -line :



$$d = \left\| \vec{PR} - \text{proj}_{\vec{v}} \vec{PR} \right\|$$

↑
component
of \vec{PR}
orthogonal to the line L .
(to \vec{v})

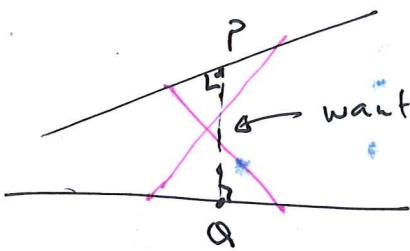
$$d = \left\| \vec{PR} - \frac{\vec{PR} \cdot \vec{v}}{\|\vec{v}\|^2} \cdot \vec{v} \right\|$$

↑
old formula for $\text{proj}_{\vec{v}} \vec{PR}$

Further shortcut:

$$\vec{PR} \times \vec{v} \leftarrow \text{can you use this?} \\ (\text{answer: in the book})$$

(c) distance between skew lines:



want it to be perp.
to both lines
(at the closest two points)

Our trick: we already have a plane containing L_2
and parallel to L_1 .

Then the distance between these lines
= distance from any point on L_1 to the
plane containing L_2 .

So, all we have to do: take a point on L_1 ,
use the formula for the distance to the plane in (b):

Part (c): take a point on L_1 :

$P = (0, 4, 2)$ ← the point used to write
the equation of L_1 .
distance from P to the plane we found in (b):
our plane B :

$$(x-3) + 2(y-2) + (z-1) = 0$$

take R ← the point used to write the
equation of the plane

$$R = (3, 2, 1)$$

$$\vec{PR} = \langle 3, 1, -1 \rangle$$

$$\vec{n} = \text{normal of the plane} = \langle 1, 2, 1 \rangle$$

$$d = \left| \vec{PR} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right| = \left| \frac{\langle 3, 1, -1 \rangle \cdot \langle 1, 2, 1 \rangle}{\sqrt{1+4+1}} \right| \\ = \left| \frac{4}{\sqrt{6}} \right| = \boxed{\frac{4}{\sqrt{6}}}$$