Equations of lines and planes in $\mathbb{R}^3$

- **Recall**: last time we discussed how to make a normal vector $\bar{n}$ to a plane

- **How to write an equation of a plane in $\mathbb{R}^3$**

  - use normal vector (a vector perpendicular to your plane)

$\bar{n} = \langle a, b, c \rangle$ – normal vector

$P = (x_0, y_0, z_0)$ – a point in the plane

"x-not" ("x-zero") – means some fixed constant. (fixed x).

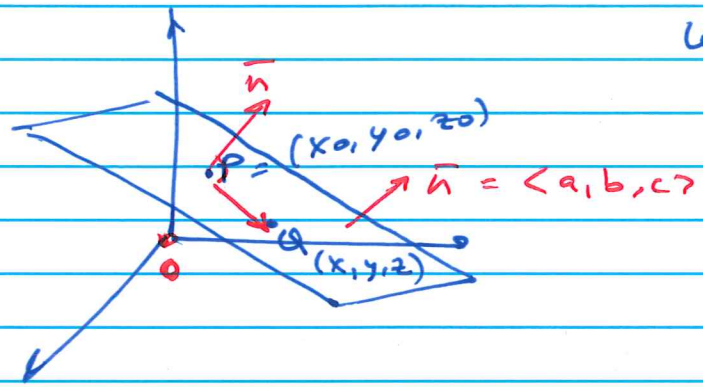
Easy to write the equation of the plane:

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.}$$

– plane normal to $\langle a, b, c \rangle$ containing P.

components of the normal vector.

Why this works:



Let $Q = (x, y, z)$ be some point in our plane

Then $\overline{PQ}$ is a vector in our plane, so $\bar{n} \perp \overline{PQ}$

so $\bar{n} \cdot \overline{PQ} = 0$

$$\| $$

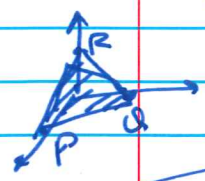
$$\underbrace{\langle a, b, c \rangle}_{\bar{n}} \cdot \underbrace{\langle x - x_0, \; y - y_0, \; z - z_0 \rangle}_{\overline{PQ}} = 0$$

$$\|$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \quad (*)$$

(The converse also works)

$Q = (x, y, z)$ lies in the plane if and only if $(*)$ is satisfied.



Example: from last class, we ~~had~~ had P, Q, R

$P = (1,0,0) \qquad Q = (0,3,0) \qquad R = (0,0,1)$

$\bar{n} = \langle 3, 1, 3 \rangle$ - normal to the plane PQR. (last time)

this was $\overline{PQ} \times \overline{PR}$

$$\boxed{3(x-1) + 1\cdot(y-0) + 3\cdot(z-0) = 0}$$ - equation of PQR

coordinates of P

Recall: we expected an equation of the form

$$3x + y + 3z + d = 0$$

here $d = 3\cdot(-1) + 0 + 0 = \boxed{-3}$

comes from coordinates of some P in the plane.

Way to check: what if we use Q?

$$3\cdot 0 + 1\cdot(-3) + 3\cdot 0 = \boxed{-3}$$ — the same!

(so our $\bar{n}$ is likely correct)

i) Find the equation of the plane ~~not~~ perpendicular to $\bar{n} = \langle 1, 2, 3 \rangle$ and containing the point $(4, 5, 6)$.

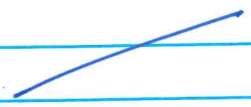
Answer: $\boxed{1 \cdot (x-4) + 2 \cdot (y-5) + 3 \cdot (z-6) = 0}$

Note: automatically get that our point satisfies the equation.

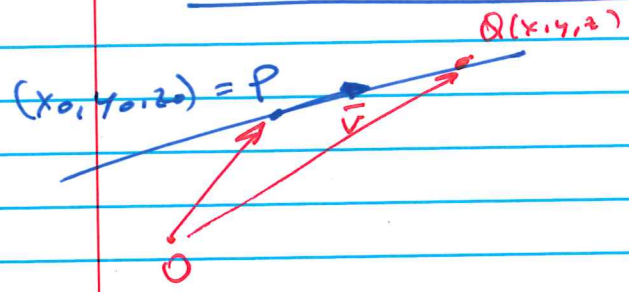
Lines in $\mathbb{R}^3$

- cannot expect to have a single equation like for the planes:

  in $\mathbb{R}^3$, need to have 2 constraints to specify a line.



Parametric equation for a line



line through a point $P(x_0, y_0, z_0)$ parallel to $\bar{v} = \langle a, b, c \rangle$

$$\overline{PQ} \parallel \bar{v}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = \langle x - x_0, y - y_0, z - z_0 \rangle$$

Get: $\langle x - x_0, y - y_0, z - z_0 \rangle = t \cdot \bar{v} \quad (t \in \mathbb{R})$

$= t \cdot \langle a, b, c \rangle$

vector-parametric

$$\boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \quad (\text{same } t)}$$

– parametric equation of our line

<u>Remark</u>: if you are trying to write a computer program to plot lines and planes, then:

- parametric equation of a line is very convenient!

  $P_0 = (1,2,2)$ is given.

  take $t = 0.1,\ 0.2,\ 0.3, \ldots$

  compute
  $$x = 1 + t \cdot 3$$
  $$y = 2 + t \cdot 4 \quad \rightarrow \text{plot } (x,y,z)$$
  $$z = 2 + t \cdot 5$$

- compare with planes:

  plane given by $3x + y + 3z = +3$ $(*)$

  how to plot it?

  looks like we have to go throug <u>all</u> (x,y,z)

  check if $(*)$ is true, plot if true.

  in order to plot, <u>need</u> parametric equations.

Problem: it is hard to check if a given point (x, y, z) lies on a given line. (have to solve for t)

Example: let L be the line containing the point $P = (1, 2, 2)$ and parallel to $\vec{v} = \langle 3, 4, 5 \rangle$.

a) Find the parametric equation of L.

b) Does the point $Q = (7, 8, 8)$ belong to L?

(a)
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$
— vector-parametric

$$\langle x, y, z \rangle = \langle 1, 2, 2 \rangle + t \langle 3, 4, 5 \rangle$$

parameter

$$\begin{cases} x = 1 + 3t \\ y = 2 + 4t \\ z = 2 + 5t \end{cases}$$

(as you plug in all possible real numbers for t, you get all the points on our line)

(b): How to decide if Q is on L?

try to solve for t:

$$\begin{cases} 7 = 1 + 3t \\ 8 = 2 + 4t \\ 8 = 2 + 5t \end{cases}$$

→ does there exist a common t?

get $t = 0$

plug into the first: $7 = 1 + 3 \cdot 0$ — not true

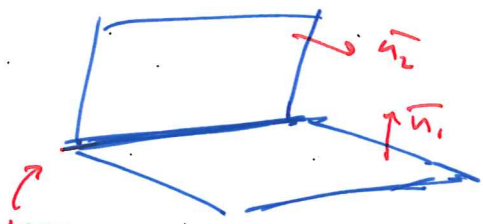
$\boxed{\text{So } Q \text{ is NOT on } L.}$

Worksheet ~~3~~ 2: Distances to lines and planes

(1) (~~From last time~~) Find an equation for the line of intersection of the planes with equations $x - y + 2z = 0$ and $3y + z = 0$.

Hint: 1) this line is perpendicular to both normal vectors of the planes.

2) Need: 1) a point on this line
2) "a direction vector" $\bar{v}$ ← parallel to our line.



lies in both planes, so perp. to both $\bar{n}_1$ and $\bar{n}_2$.

Several solutions:

(unofficial solution)

① we look for common solutions of the two plane equations:

$$\begin{cases} x - y + 2z = 0 \\ 3y + z = 0 \end{cases} \iff \begin{cases} x + \frac{z}{3} + 2z = 0 \\ y = -\frac{z}{3} \end{cases}$$

eliminate variables

$$\iff \begin{cases} x = -\frac{7}{3} z \\ y = -\frac{z}{3} \end{cases}$$

recall: $z = z$!

Now rename: let $z = t$

Get:
$$\begin{cases} x = -\frac{7}{3} t \\ y = -\frac{1}{3} t \\ z = t \end{cases}$$

— looks like a parametric equation of a line! (line containing $(0,0,0)$)

(Note: $x_0 = y_0 = z_0 = 0$ here).

Note: could rename ~~[illegible]~~ $t = 3s$ and get the other answer:

$$\begin{cases} z = 3s \\ y = -s \\ x = -7s \end{cases}$$

② Official solution

- want: a point on this line and a vector parallel to it..

Point: want a common solution to

$$\begin{cases} x - y + 2z = 0 \\ 3y + z = 0 \end{cases}$$

(but do not have to find all common solutions)

can plug in some value of x to make the equations easier.

Plug in $x = 0$, for example ← this will find the intercept of our line with the yz-plane.

Get: $\begin{cases} 2z - y = 0 \\ 3y + z = 0 \end{cases}$

(if after this solution doesn't exists try $z=0$ instead).

Solve, get $y = z = 0$

then try $y=0$ ☺).

Get: $(0,0,0)$ is a common point of both planes.

Next, need the <u>direction vector</u> for our line: $\bar{v}$

Let $\bar{n}_1, \bar{n}_2$ be the normal vectors of our planes.

$\bar{n}_1 = \langle 1, -1, 2 \rangle$ $\qquad$ $\bar{n}_2 = \langle 0, 3, 1 \rangle$ ← read them from equations of the planes.

$\bar{v} \perp \bar{n}_1$ and $\bar{n}_2$

So we can take $\bar{v} = \bar{n}_1 \times \bar{n}_2$

$$\bar{v} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = i \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}$$

$$= -7\bar{i} - \bar{j} + 3\bar{k}$$

<u>Put it together</u>:

$$\begin{cases} x = 0 - 7t \\ y = 0 - t \\ z = 0 + 3t \end{cases}$$

the point we found.