Equations of lines and planes in $\mathbb{R}^3$

- Recall: last time we discussed how to make a normal vector \( \vec{n} \) to a plane.

- How to write an equation of a plane in $\mathbb{R}^3$
  - use normal vector (a vector perpendicular to your plane)
  
  \[ \vec{n} = \langle a, b, c \rangle \text{ - normal vector} \]
  
  \[ P = (x_0, y_0, z_0) \text{ - a point in the plane} \]
  
  "X-not" means some fixed constant. (Fixed \( x_0 \)).
  
  Easy to write the equation of the plane:

  \[
  a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.
  \]

  - plane normal to \( \langle a, b, c \rangle \) containing \( P \).
  
  Components of the normal vector.
Why this works:

\[ \mathbf{\overline{PA}} \text{ is a vector in our plane, so } \mathbf{n} \perp \mathbf{\overline{PA}} \]

\[ \mathbf{n} \cdot \mathbf{\overline{PA}} = 0 \]

\[ \mathbf{n} \cdot \mathbf{\overline{PA}} = (a, b, c) \cdot (x-x_0, y-y_0, z-z_0) = 0 \]

\[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \quad (\star) \]

(The converse also works)

\[ \mathbf{Q} = (x, y, z) \text{ lies in the plane if and only if } (\star) \text{ is satisfied.} \]
Example: from last class, we had $P, Q, R$

$P = (1, 0, 0) \quad Q = (0, 3, 0) \quad R = (0, 0, 1)$

$\vec{n} = \langle 3, 1, 3 \rangle$ - normal to the plane $PQR$. (last time)

$3(x-1) + 1(y-0) + 3(z-0) = 0$ - equation of $PQR$

Recall: we expected an equation of the form $3x + 4y + 3z + d = 0$

where $d = 3(-1) + 0 + 0 = -3$

comes from coordinates of some $P$ in the plane.

Way to check: what if we use $Q$?

$3 \cdot 0 + 1 \cdot (-3) + 3 \cdot 0 = -3$ - the same!

(so our $\vec{n}$ is likely correct)
1) Find the equation of the plane perpendicular to $\mathbf{n} = \langle 1, 2, 3 \rangle$
and containing the point $(4, 5, 6)$.

Answer: $1 \cdot (x - 4) + 2 \cdot (y - 5) + 3 \cdot (z - 6) = 0$

Note: automatically set that our point satisfies the equation.
Lines in $\mathbb{R}^3$

- Cannot expect to have a single equation like for the plane: in $\mathbb{R}^3$, need to have 2 constraints to specify a line.

**Parametric equation for a line**

$\mathbf{Q}(x_1, y_1, z_1) = \mathbf{P}$

Line through a point $\mathbf{P}(x_0, y_0, z_0)$ parallel to $\mathbf{v} = \langle a, b, c \rangle$

$\overrightarrow{PQ} \parallel \mathbf{v}$

$\overrightarrow{OQ} - \overrightarrow{OP} = \langle x - x_0, y - y_0, z - z_0 \rangle$

Get, $\langle x - x_0, y - y_0, z - z_0 \rangle = t \cdot \mathbf{v}$ $\ (t \in \mathbb{R})$

\[
\begin{align*}
    x &= x_0 + ta \\
    y &= y_0 + tb \\
    z &= z_0 + tc \\
\end{align*}
\]

- parametric equation of our line
Remark: if you are trying to write a computer program to plot lines and planes, then:

- parametric equation of a line is very convenient!
  - \( P_0 = (1, 2, 2) \) is given.
  - take \( t = 0.1, 0.2, 0.3, \ldots \)
  - compute \( x = 1 + t \cdot 3 \)
  - \( y = 2 + t \cdot 4 \) → plot \((x, y, z)\)
  - \( z = 2 + t \cdot 5 \)

- compare with planes:
  - plane given by \( 3x + y + 3z = 3 \) (\( \star \))
  - how to plot it?
  - looks like we have to go through all \((x, y, z)\)
  - check if (\( \star \)) is true, plot if true.

  In order to plot, need parametric equations.
Problem: It is hard to check if a given point \((x, y, z)\) lies on a given line. (have to solve for \(t\))

Example: let \(L\) be the line containing the point \(P = (1, 2, 2)\) and parallel to \(D = \langle 3, 4, 5 \rangle\).

a) Find the parametric equation of \(L\).

b) Does the point \(Q = (7, 8, 8)\) belong to \(L\)?

(a) \[
\begin{bmatrix}
\frac{x}{2} \\
\frac{y}{2} \\
\frac{z}{2}
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
2
\end{bmatrix} + t \begin{bmatrix}
3 \\
4 \\
5
\end{bmatrix}
\]

\[
\langle x, y, z \rangle = \langle 1, 2, 2 \rangle + t \langle 3, 4, 5 \rangle
\]

parameter (as you plug in all possible real numbers for \(t\), you get all the points on our line)

\[
\begin{align*}
x &= 1 + 3t \\
y &= 2 + 4t \\
z &= 2 + 5t
\end{align*}
\]

(b): how to decide if \(Q\) is on \(L\)?

try to solve for \(t\):

\[
\begin{align*}
7 &= 1 + 3t \quad \text{→ does there exist a common } t? \\
8 &= 2 + 4t \\
8 &= 2 + 5t
\end{align*}
\]

get \(t = 0\)

plug into the first: \(7 = 1 + 3 \cdot 0\)

So \(Q\) is NOT on \(L\).
Worksheet 3: Distances to lines and planes

(1) (From last time) Find an equation for the line of intersection of the planes with equations $x - y + 2z = 0$ and $3y + z = 0$.

Hint: 1) this line is perpendicular to both normal vectors of the planes.

2) Need: 1) a point on this line
2) "a direction vector" $\vec{d}$ parallel to our line.

C lies in both planes, so perp to both $\vec{u}_1$ and $\vec{u}_2$. 
Several solutions:

(Official solution)

1. We look for common solutions of the two plane equations:

\[
\begin{align*}
\begin{cases}
 x - y + 2z &= 0 \\ 3y + z &= 0
\end{cases}
\iff
\begin{cases}
 y &= -\frac{2}{3} \\
 z &= \frac{7}{3}
\end{cases}
\]

(\text{eliminate variables})

(=)

\[
\begin{cases}
 x &= -\frac{7}{3} z \\
 y &= -\frac{2}{3} z
\end{cases}
\]

(recall: \( z = z \))

Now rename: let \( z = t \)

Get:

\[
\begin{cases}
 x &= -\frac{7}{3} t \\
 y &= -\frac{1}{3} t \\
 z &= t
\end{cases}
\]

Looks like a parametric equation of a line!

\[
\text{(Note: } x_0 = y_0 = z_0 = 0 \text{ here}.)
\]

Note: could rename \( z = 3s \) and get the other answer:

\[
\begin{cases}
 z &= 3s \\
 y &= -s \\
 x &= -7s
\end{cases}
\]
2. Official Solution

- want a point on this line and a vector parallel to it.

Point: want a common solution to
\[
\begin{align*}
\begin{cases}
x - y + 2z &= 0 \\
3y + 2z &= 0
\end{cases}
\end{align*}
\]

(but do not have to find all common solutions)

Can plug in some value of \(x\) to make the equations easier.

Plug in \(x = 0\), for example — this will find the intercept of our line with the \(yz\)-plane.

Get,
\[
\begin{align*}
\begin{cases}
2z - y &= 0 \\
3y + 2z &= 0
\end{cases}
\end{align*}
\]

Solve, get
\[
y = 2z = 0
\]

(Get: \((0,0,0)\) is a common point of both planes.

(If after this solution doesn't exist, try \(z = 0\) instead).

Then try \(y = 0\).)
Next, need the direction vector for our line: \( \mathbf{v} \)

Let \( \mathbf{n}_1, \mathbf{n}_2 \) be the normal vectors of our planes.

\[ \mathbf{n}_1 = \langle 1, -1, 2 \rangle \quad \mathbf{n}_2 = \langle 0, 3, 1 \rangle \]

Read them from equations of the planes.

\( \mathbf{v} \perp \mathbf{n}_1 \) and \( \mathbf{n}_2 \)

So we can take \( \mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 \)

\[ \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} \]

\[ \mathbf{v} = -7\mathbf{i} + 3\mathbf{k} = -\mathbf{j} + 3\mathbf{k} \]

Put it together:

\[ \begin{cases} x = 0 + 7t \\ y = 0 - t \\ z = 0 + 3t \end{cases} \]

The point we found.