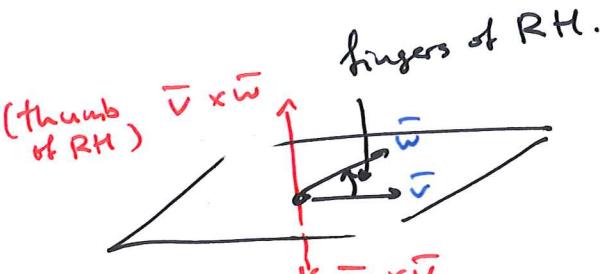


Last time: Started cross product (vector product).

Today: Learn all about it and see applications.

- Recall:  $\bar{v} \times \bar{w}$  - an operation on vectors in  $\mathbb{R}^3$ .



which way it  
goes is determined  
by the right-hand (RH)  
rule:

Fingers point from  $\bar{v}$  to  $\bar{w}$  ← the first vector  
← the second vector

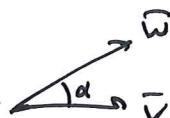
Note:  $\bar{w} \times \bar{v} = -(\bar{v} \times \bar{w})$

By definition

if  $\bar{v} \parallel \bar{w}$ , then  $\bar{v} \times \bar{w} = \bar{w} \times \bar{v} = \bar{0}$ .

- if  $\bar{v} \perp \bar{w}$ , then  $\bar{v}$  and  $\bar{w}$  define a plane in  $\mathbb{R}^3$ .

$\bar{v} \times \bar{w}$  is perpendicular  
to both  $\bar{v}$  and  $\bar{w}$   
(perp. to this plane!)



- length of ~~the~~  $\bar{v} \times \bar{w}$  is  $\|\bar{v} \times \bar{w}\| = \|\bar{v}\| \cdot \|\bar{w}\| \sin \alpha$

## Algebraic definition

first, determinants:  $\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} \stackrel{\text{different notation}}{=} \left| \begin{matrix} a & b \\ c & d \end{matrix} \right| \stackrel{\text{def'n}}{=} ad - bc$

$2 \times 2$ -determinant.

$3 \times 3$ -determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$3 \times 3$   
matrix x

↑ corner left

when cut the row  
and column of  $a_1$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

take it as def'n of  $3 \times 3$  det

signs alternate

cut out  
the column of  $a_2$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ \underline{b_1} & \underline{b_2} & \underline{b_3} \\ c_1 & c_2 & c_3 \end{vmatrix}$$

cut  
out

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

) the determinant  
(a number).

## Algebraic def. of cross product

$$\bar{v} = \langle a_1, b_1, c_1 \rangle$$

$$\bar{w} = \langle a_2, b_2, c_2 \rangle$$

$$\bar{v} \times \bar{w} \stackrel{\text{def}}{=} \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - j \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

way  
to remember,  
not a real determinant

as if  
it were  
a determinant

$$= i(b_1c_2 - c_1b_2) - j(a_1c_2 - c_1a_2) + k(a_1b_2 - a_2b_1)$$

So:  $\boxed{\bar{v} \times \bar{w} = \langle b_1c_2 - c_1b_2, -a_1c_2 + c_1a_2, a_1b_2 - a_2b_1 \rangle}$

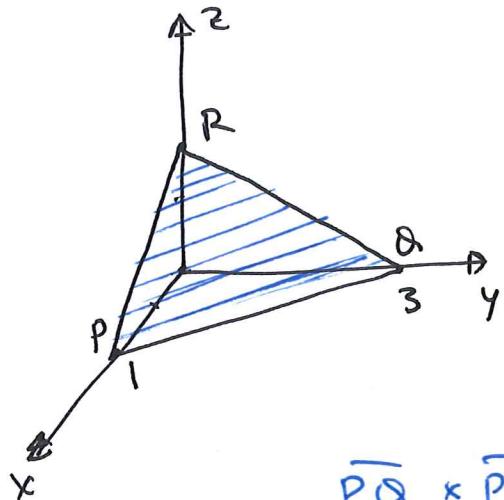
real def'n; use determinants to remember it.

Another way to remember:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = i(b_1c_2 - c_1b_2) + j(c_1a_2 - a_1c_2) + k(a_1b_2 - b_1a_2)$$

Fact: this crazy algebra gives you  
the same vector from the geometric definition!

Example  $P: (1, 0, 0)$ ,  $Q: (0, 3, 0)$ ,  $R: (0, 0, 1)$   
 Find a vector perpendicular to the plane  $PQR$ .



Step 1: make 2 vectors that lie in this plane.

(ex:  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ )

$$\overrightarrow{PQ} = \langle -1, 3, 0 \rangle$$

$$\overrightarrow{PR} = \langle -1, 0, 1 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & 3 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + k \begin{vmatrix} -1 & 3 \\ -1 & 0 \end{vmatrix} = 3i + j + 3k$$

$$\boxed{\vec{v} = \langle 3, 1, +3 \rangle} \quad \text{- perp. to } PQR\text{-plane.}$$

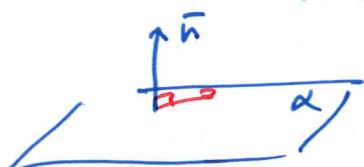
(check: take dot product of the result with  $\overrightarrow{PQ}$ )  
 if don't get 0, then wrong.

Note: we just found a normal vector to a plane.

## Properties of cross product :

- ①.  $\bar{v} \times \bar{w} = -\bar{w} \times \bar{v}$  (order matters.  
NOT commutative)
- ②.  $\bar{v} \times \bar{w} = \bar{0}$  if  $\bar{v} \parallel \bar{w}$
- ③.  $\bar{v}, \bar{w}$  - vectors  $c$  - scalar  
 $(c\bar{v}) \times \bar{w} = \bar{v} \times c\bar{w} = c(\bar{v} \times \bar{w})$
- ④.  $\bar{u} \times (\bar{v} + \bar{w}) = \bar{u} \times \bar{v} + \bar{u} \times \bar{w}$
- ⑤.  $\bar{u} \times (\bar{v} \times \bar{w}) = \underbrace{(\bar{u} \cdot \bar{w})\bar{v}}_{\substack{\text{parallel} \\ \text{to } \bar{u}, \\ \text{normal} \\ \text{to the } \bar{v}, \bar{w} \text{-plane}}} - \underbrace{(\bar{u} \cdot \bar{v})\bar{w}}_{\substack{\text{coefficients}}} \leftarrow \begin{array}{l} \text{linear} \\ \text{combination} \\ \text{of } \bar{v}, \bar{w} \end{array}$   
(it is a vector lying in the plane spanned by  $\bar{v}, \bar{w}$ )  
perp. to  $\bar{u}$  and  $\bar{u}$

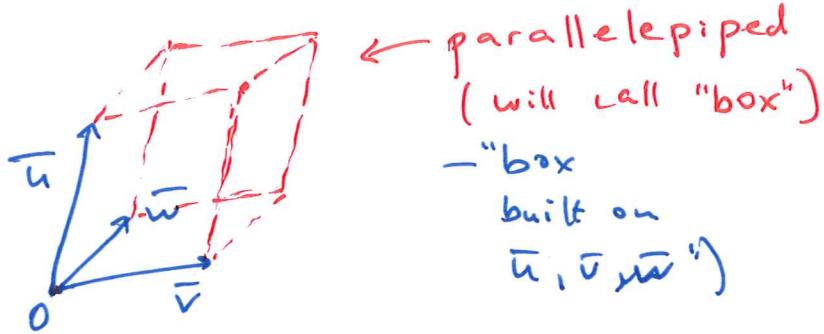
Any vector  $\perp$  (normal vector of a plane  $\alpha$ )  
lies in the plane  $\alpha$



## ⑥ • Scalar triple product or mixed triple product :

$$|\bar{u} \cdot (\bar{v} \times \bar{w})| = \underbrace{\text{vector}}_{\text{number}} \quad \underbrace{\text{volume}}_{\text{of this box!}}$$

absolute value  
of this number



$$= |(\bar{u} \times \bar{v}) \cdot \bar{w}|$$

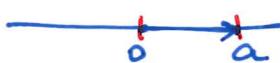
(automatically 0 if they all lie in one plane)

True for any three vectors in  $\mathbb{R}^3$ !

Why this works and how it ties things together.

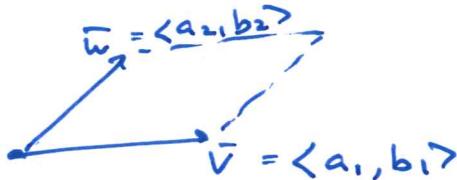
### • determinants and volumes.

on the line  $\mathbb{R}^1$ :



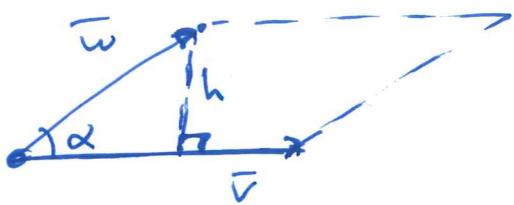
$$\text{length} = |a|$$

on the plane  $\mathbb{R}^2$ :



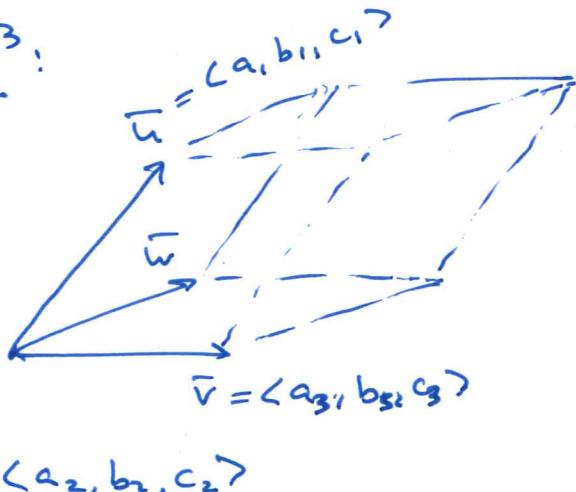
$$\text{Area} = \left| \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right|$$

↑  
absolute value  
of the det



$$\begin{aligned} \text{Area} &= \|\vec{v}\| \cdot h \\ &= \|\vec{v}\| \|\vec{w}\| \sin\alpha \\ &\stackrel{\text{exer}}{=} |a_1 b_2 - b_1 a_2| \end{aligned}$$

in  $\mathbb{R}^3$ :

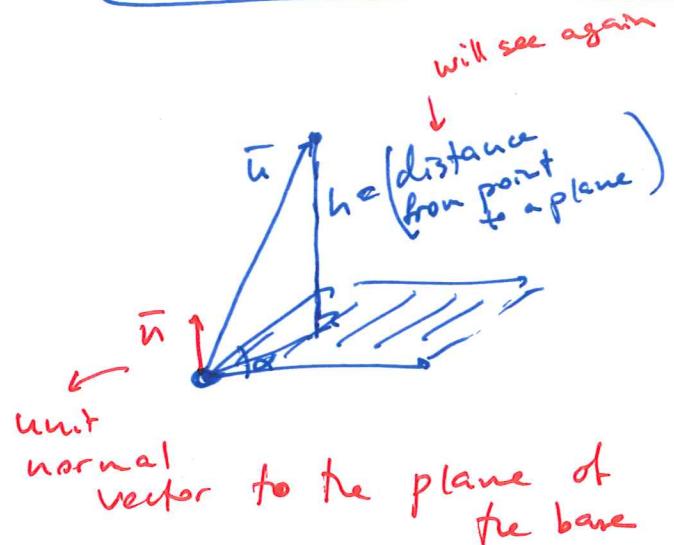


*absolute value*

$$\text{volume} = \left| \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right|$$

determinants  
always give  
volumes of boxes!  
*believe this for now.*

Now can connect the algebraic def with geometric one:



$$\text{volume} = \text{area (base)} \cdot h$$

$$h = \vec{u} \cdot \vec{n}$$

"

$$\|\vec{u}\| \cdot \cos(\text{angle between } \vec{u} \text{ and the normal to the plane})$$

"

length of the projection of  $\vec{u}$  to the direction of  $\vec{n}$

By geometric definition:

$$\vec{n} = \frac{\vec{v} \times \vec{w}}{\|\vec{v} \times \vec{w}\|}$$

$$\text{so volume} = (\text{Area of base}) \cdot h = (\vec{u} \cdot \vec{n}) \cdot \|\vec{v} \times \vec{w}\|$$

$$\|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin\theta = \|\vec{v} \times \vec{w}\|$$

$$= \left( \vec{u} \cdot \frac{\vec{v} \times \vec{w}}{\|\vec{v} \times \vec{w}\|} \right) \cdot \|\vec{v} \times \vec{w}\| = \vec{u} \cdot (\vec{v} \times \vec{w})$$

The point 3:

- using volume =  $(\text{area of base}) \times \text{height}$

get that the geometric def. of  $\vec{v} \times \vec{w}$  is made in such a way that for any vector  $\vec{u}$ ,

$|\vec{u} \cdot (\vec{v} \times \vec{w})|$  gives the volume of this box.

(we just proved)

- Algebraically:

$$u = \langle a_1, b_1, c_1 \rangle$$

$$\vec{v} \times \vec{w} = i \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - j \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & c_3 \end{vmatrix} + k \cdot \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

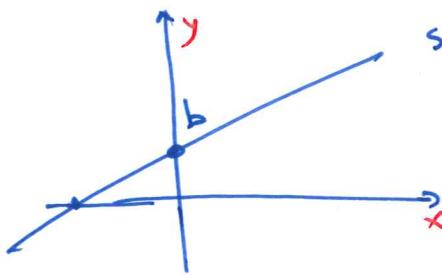
Then  $\vec{u} \cdot (\vec{v} \times \vec{w}) = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \cdot (\text{j-component of } \vec{v} \times \vec{w}) + c_1 \cdot (\text{k-component of } \vec{v} \times \vec{w})$

det of 3x3-det

$$\downarrow = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{matrix} \text{Volume of box.} \\ \uparrow \\ \text{believe} \end{matrix}$$

# Equations of Lines and planes in $\mathbb{R}^3$

## • lines on the plane:



slope-intercept form:  $y = mx + b$

here  $x$  and  $y$  do not have equal rights.

vertical lines ??



$$\underline{x = a}$$

← not the above form.

Different forms:

$$ax + by + c = 0 \quad \leftarrow \text{includes both kinds of lines.}$$

$$\text{If } b \neq 0, \text{ can rewrite it as: } y = -\frac{c + ax}{b}$$

$$= -\frac{a}{b}x - \frac{c}{b}$$

intercept.

By analogy: A single linear equation in  $\mathbb{R}^3$  defines a plane:

$$\underline{ax + by + cz + d = 0} \quad \begin{matrix} \text{- equation of a plane} \\ \text{in } \mathbb{R}^3. \end{matrix}$$

Fact :

• in  $\mathbb{R}^2$  :  $ax+by+c = 0$

the vector  $\langle a, b \rangle$  is perpendicular to this line

• in  $\mathbb{R}^3$ :  $\langle a, b, c \rangle = \bar{n}$  — normal vector to the plane!

Anything to do with planes in  $\mathbb{R}^3$  is easy using normal vectors.

in our example, equation of the plane PQR:

$\langle 3, 1, 3 \rangle$  - normal.

Equation: 
$$3x + y + 3z + d = 0.$$

↑ will find next time