

Last time: . Vectors

Math 200.
Lecture 2
Thurs Jan 9, 2020

\vec{v} : length, direction
" "
 $\langle a_1, \dots, a_n \rangle$

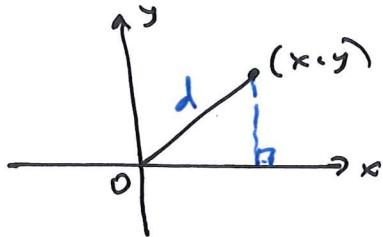
• operations: $\vec{v}_1 + \vec{v}_2$
 $c \cdot \vec{v}_1$ ($c \in \mathbb{R}$ - scalar)

(linear algebra approach).

Today: more geometric - talk about length.

• Distances in \mathbb{R}^3 or 3-dim. space

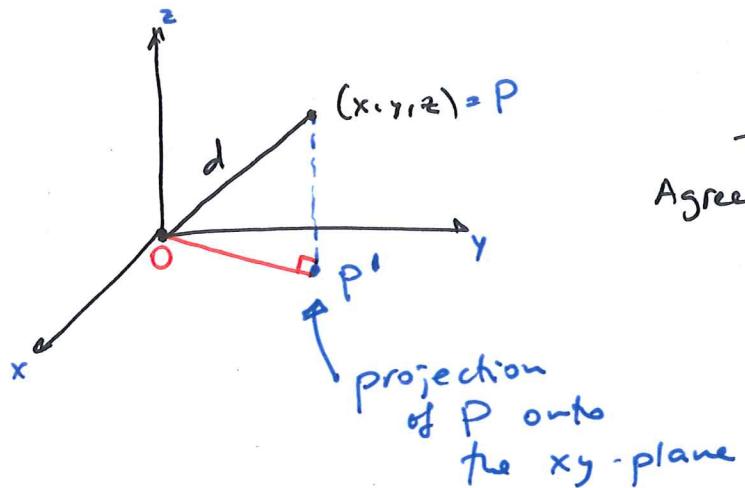
know: in \mathbb{R}^2 :



$$d = \sqrt{x^2 + y^2}$$

Pythagoras' Thm
(can take it as
a def'n of distance).

in \mathbb{R}^3

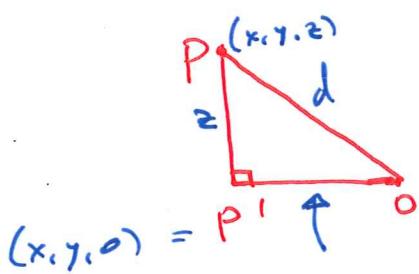


$$d = \sqrt{x^2 + y^2 + z^2}$$

- could define it this way.

Agrees with Pythagoras:

coordinates of $P' = (x, y, 0)$ (the xy -plane in space is where $z=0$)



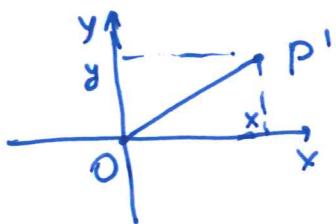
$$\|OP'\| = \sqrt{x^2 + y^2}$$

we have:

$$d^2 = z^2 + \|OP'\|^2$$

$$= z^2 + (x^2 + y^2)$$

so $d = \sqrt{x^2 + y^2 + z^2}$



Distance between two points in space

$$P_1 = (a_1, b_1, c_1)$$

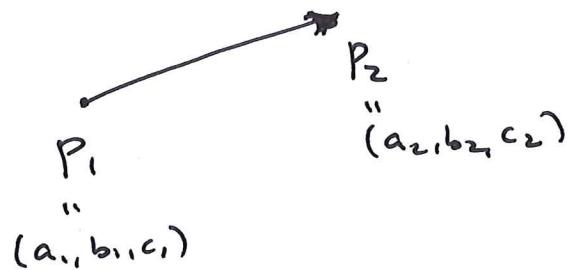
$$P_2 = (a_2, b_2, c_2)$$

$$|P_1 P_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}$$

Def $\vec{v} = \langle a, b, c \rangle$

Then $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$ ← magnitude or length of a vector

Here, we can make a vector



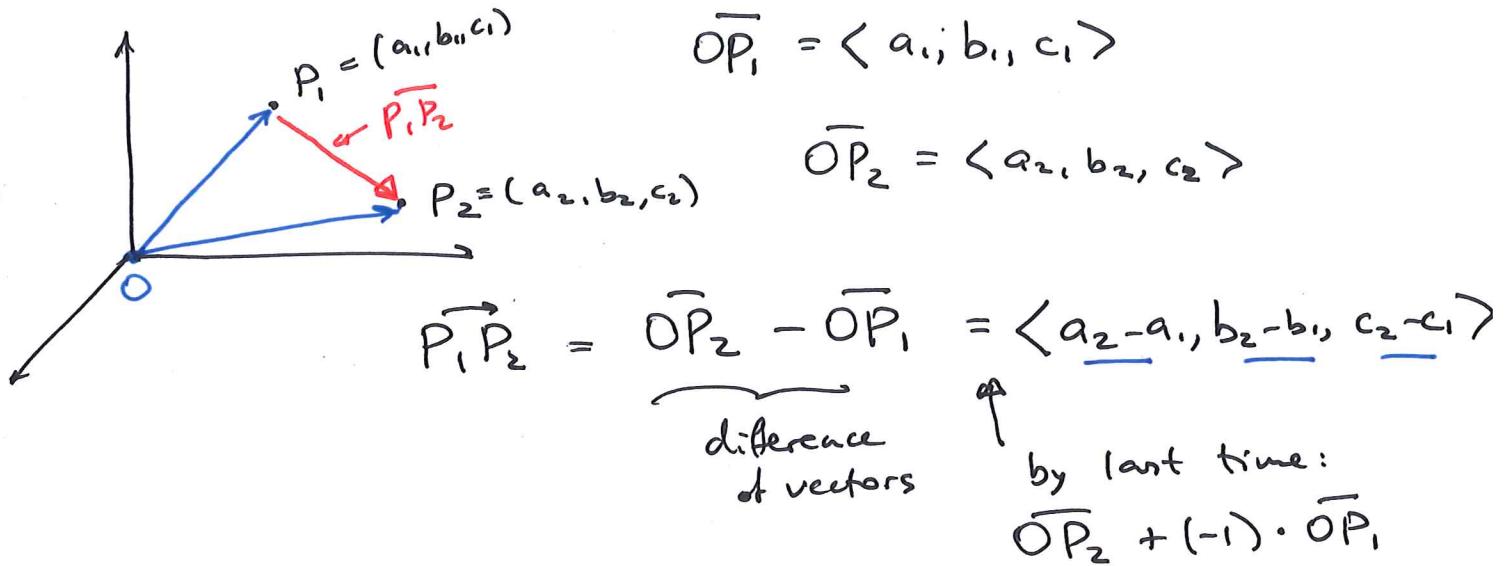
$$\overrightarrow{P_1 P_2} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$$

to make a vector,
subtract coordinates
of its tail from
the coordinates of its
end.

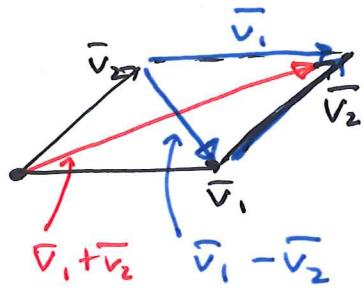
So we have $|\overrightarrow{P_1 P_2}| = \text{distance from } P_1 \text{ to } P_2$

$$= \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}$$

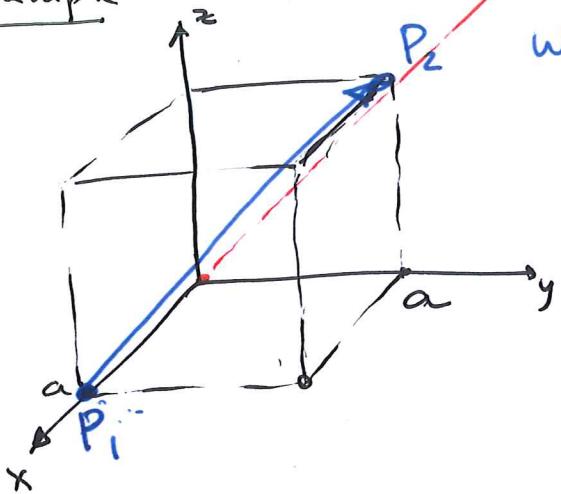
More about the vector $\overrightarrow{P_1 P_2}$:



Parallelogram rule:



Example



What is the blue vector $\overline{P_1P_2}$?
Find its components.
Find its length.

Step 1: coordinates of

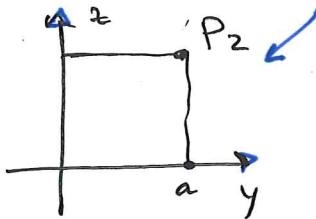
$$P_1 \text{ are: } \begin{cases} x = a \\ y = 0 \\ z = 0 \end{cases}$$

on the
x-axis

$$P_1 = (a, 0, 0)$$

subtract tail from end.

$$P_2 = (0, a, a)$$



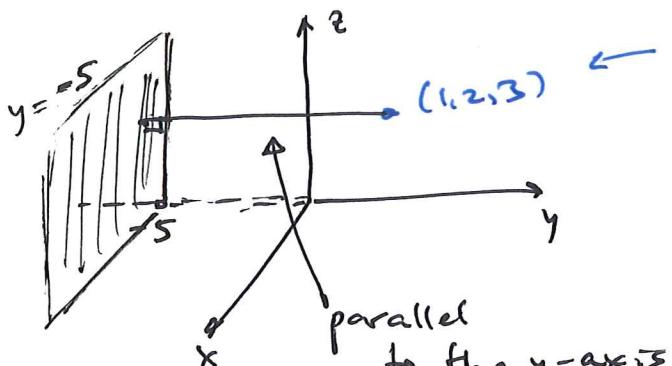
Then

$$\begin{aligned} \overline{P_1P_2} &= \langle 0-a, a-0, a-0 \rangle \\ &= \langle -a, a, a \rangle \end{aligned}$$

$$|\overline{P_1P_2}| = \sqrt{(-a)^2 + a^2 + a^2} = a\sqrt{3}$$

Exercise Let $P = (1, 2, 3)$

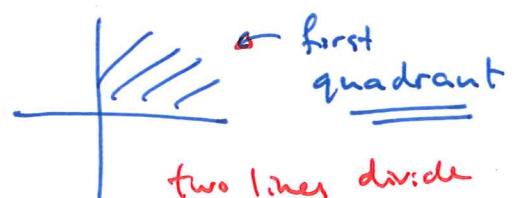
Find the distance from P to the plane $y = -5$.



$$\text{get: } d = |2 - (-5)| \\ = 7$$

↑
how
far we
need to go along
y-axis.

in the first octant ↗ 3 planes
divide \mathbb{R}^3
into 8 regions
 $(x, y, z \geq 0)$



two lines divide
the plane
into
4 regions.

Dot product : (1.3 in the book)

Define a new operation on vectors :

take \bar{v}_1, \bar{v}_2 , make $\bar{v}_1 \cdot \bar{v}_2$ - a number.

Def 1 (algebraic)

$$\bar{v}_1 = \langle a_1, \dots, a_n \rangle, \bar{v}_2 = \langle b_1, \dots, b_n \rangle$$

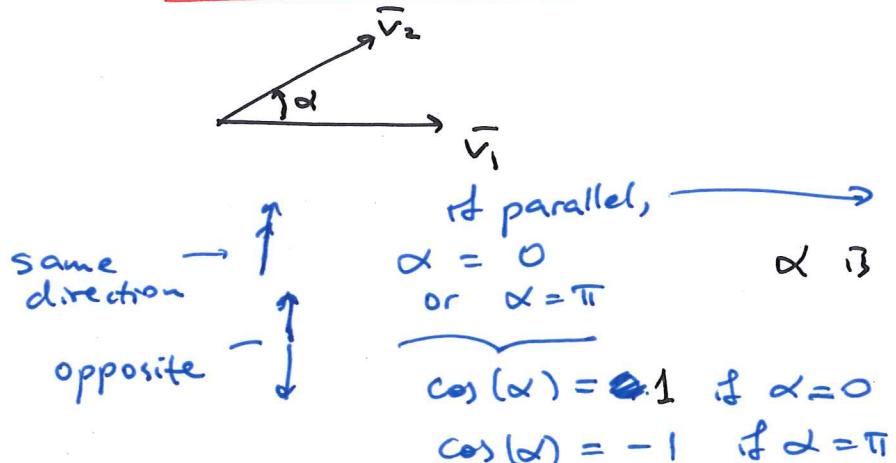
$$\bar{v}_1 \cdot \bar{v}_2 = a_1 \cdot b_1 + a_2 b_2 + \dots + a_n b_n$$

Example $\bar{v}_1 = \langle 1, 2, 3 \rangle \quad \bar{v}_2 = \langle 4, 5, 6 \rangle$

$$\bar{v}_1 \cdot \bar{v}_2 = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

Geometrically

Def 2: $|\bar{v}_1 \cdot \bar{v}_2| = |\bar{v}_1| \cdot |\bar{v}_2| \cdot \cos \alpha$



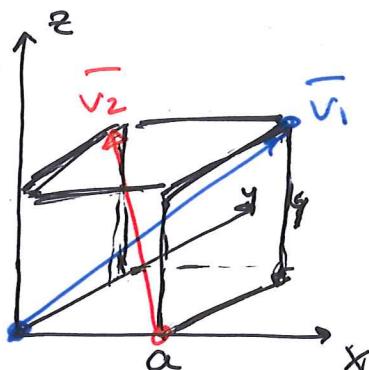
Note: even if \bar{v}_1, \bar{v}_2 live in some n-dimensional space,

they define a plane (unless $\bar{v}_1 \parallel \bar{v}_2$)

α is measured in that plane.

- One has to check these definitions agree ~~on next page~~
- This becomes useful for finding angles:

Example



Find the angle
between \vec{v}_1 and \vec{v}_2

$$\vec{v}_2 = \langle -a, a, a \rangle \quad \text{from the prev. example}$$

$$\vec{v}_1 = \langle a, a, a \rangle$$

$$|\vec{v}_2| = a\sqrt{3} \quad \text{- as before}$$

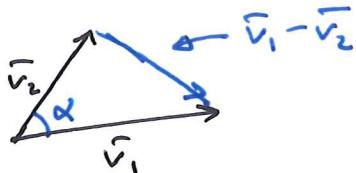
$$|\vec{v}_1| = \sqrt{a^2 + a^2 + a^2} = a\sqrt{3} \quad \text{- same}$$

$$\begin{aligned} \cos \alpha &= \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| |\vec{v}_2|} = \frac{\langle -a, a, a \rangle \cdot \langle a, a, a \rangle}{a^2 \cdot 3} \\ &= \frac{-a^2 + a^2 + a^2}{3a^2} = \frac{1}{3} \end{aligned}$$

Answer $\cos^{-1}\left(\frac{1}{3}\right)$ ← note: always define the angle between lines to be acute
 $(\cos \alpha \geq 0)$

Why $\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| \cdot |\vec{v}_2| \cos \alpha$?

- just in \mathbb{R}^2 , let's try to prove it



- Trick: in the formula $\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| (|\vec{v}_2| \cos \alpha)$
if we take $\vec{v}_1 = \vec{v}_2$, get:

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

compute $|\vec{v}_1 - \vec{v}_2|$ in two ways:

1) use the trick:

$$|\vec{v}_1 - \vec{v}_2|^2 = (\vec{v}_1 - \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2) \xleftarrow{\text{Real properties}} \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2 \cdot \vec{v}_1 - \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_2 \cdot \vec{v}_2$$
$$= |\vec{v}_1|^2 - 2\vec{v}_1 \cdot \vec{v}_2 + |\vec{v}_2|^2$$

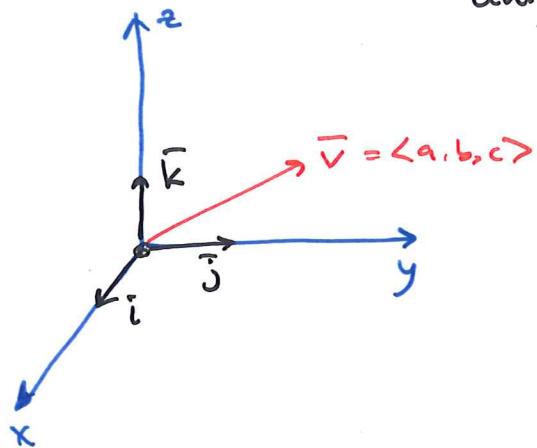
2) use the law of cosines:

$$|\vec{v}_1 - \vec{v}_2|^2 = |\vec{v}_1|^2 + |\vec{v}_2|^2 - 2|\vec{v}_1||\vec{v}_2| \cos \alpha$$

* from the triangle

Compare this, get our formula.

Components of a vector as dot products



Unit vectors of the axes:

they are called $\hat{i}, \hat{j}, \hat{k}$

length 1

\hat{i} along x-axis = $\langle 1, 0, 0 \rangle$

\hat{j} y-axis = $\langle 0, 1, 0 \rangle$

\hat{k} z-axis = $\langle 0, 0, 1 \rangle$

$$\bar{v} = \langle a, b, c \rangle$$

resolving \bar{v} into components

can write:

$$\bar{v} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\text{we also have: } a = \bar{v} \cdot \hat{i}$$

$$b = \bar{v} \cdot \hat{j}$$

$$c = \bar{v} \cdot \hat{k}$$

Read about: vectors, distances, dot products.

CLP-III: 1.1, 1.2, 1.3 ← will finish next.