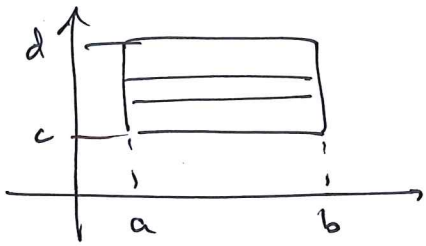
Double integrals

- Review integration
- Last time: integrals over rectangles.

$$\iint_R f(x,y)\,dA = \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy \right)}\,dx$$



$$= \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx \right)}\,dy$$

Fubini's Theorem

(true when $\iint_R |f(x,y)|\,dA$ exists)

true for continuous f. on the closed region R.

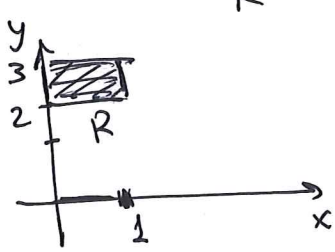
Examples

① $f(x,y) = xy^2$

Find $\iint_R f(x,y)\,dA$, $R = [0,1] \times [2,3]$

$$= \int_0^1 \underbrace{\int_2^3 xy^2\,dy}_{\substack{\text{evaluate this} \\ \text{keep } x \text{ as} \\ \text{a parameter.}}}\,dx \quad \left(= \int_0^1 dx \ \_ \ \_ \right)$$

book.

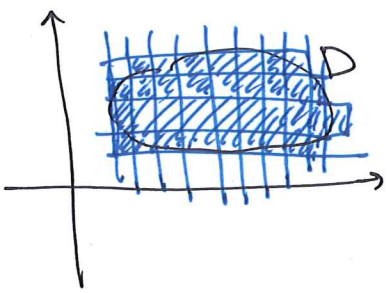


$$= x\int_2^3 y^2\,dy$$

$$= \int_0^1 x \left. \frac{y^3}{3} \right|_2^3 dx = \int_0^1 \underbrace{x \cdot \left(9 - \frac{8}{3}\right)}_{\text{function of } x} dx$$

$$= \int_0^1 \frac{19}{3} x\,dx = \frac{19}{3} \cdot \left. \frac{x^2}{2} \right|_0^1 = \boxed{\frac{19}{6}}.$$

More interesting regions



$$\iint_D f(x,y)\, dA$$

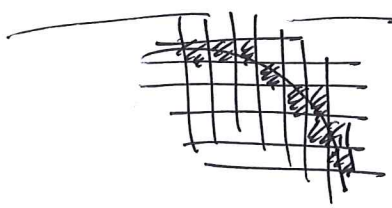
to define: draw a fine grid,
make a Riemann sum as
before using all rectangles
that intersect D

Refine the grid.
Take a limit. ← called $\iint_D f(x,y)\, dA$.

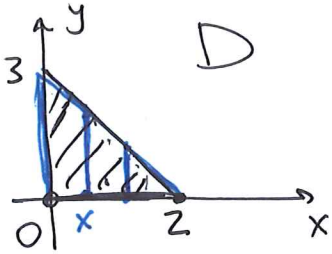
Important point: this is the only way to define what "Area" means for D.

$$\boxed{\text{Area}(D) = \iint_D 1 \cdot dA} = \text{"lim"} \left(\begin{array}{c}\text{sums of areas} \\ \text{of rectangles} \\ \text{covering } D\end{array}\right)$$

$f(x,y) = 1$ for all x, y.



Example



Find $\iint_D e^x \cdot y \, dA$

Need to make it into an iterated integral.

$\boxed{(!) \text{ The limits of our integrals encode the shape of } D.}$ $\rightarrow \int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots}$

1) Make a choice: $\int \left(\int dy \right) dx$

or the other way: $\int \left(\int dx \right) dy$

which variable on the outside?

Let x be on the outside:

given function

$$\int_0^2 \int_0^{\text{the line}} e^x \cdot y \, dy \, dx = \int_0^2 \left(\int_0^{3 - 3/2\, x} e^x y \, dy \right) dx$$

limits encode our triangle.

"the line": need the equation of the line bounding the triangle

$$= \int_0^2 e^x \cdot \left(\int_0^{3 - 3/2\, x} y \, dy \right) dx$$

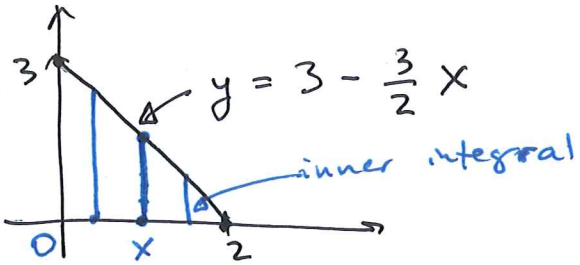
$$= \int_0^2 e^x \cdot \frac{y^2}{2} \Big|_0^{3 - 3/2\, x} dx$$

$$= \frac{1}{2} \int_0^2 \underbrace{e^x \left(3 - \frac{3}{2} x \right)^2}_{\text{should be a function of just } x.} dx = \cdots$$

multiply it out, use integr. by parts to deal with $x^2 e^x$ and $x e^x$.

- The outside integral has to have <u>numbers</u> as limits

(no expressions, no variables - -)

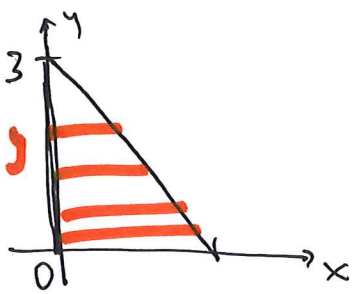


- Once you are <u>inside</u> an integral with respect to x, you can use x in the limits

- <u>In our example, switch the order of integration:</u>

$$\iint_D e^x y \, dA = \int_0^3 \int_0^{\text{the line}} e^x y \, dx \, dy$$

have to rewrite the equation of the line so that it expresses x in terms of y



$y = 3 - \frac{3}{2}x$. solve for x:

$\frac{3}{2}x = 3 - y$, $x = \frac{2}{3}(3-y)$

$$= \int_0^3 \int_0^{2-\frac{2}{3}y} e^x \underline{\underline{y}} \, dx \, dy$$

$$= \int_0^3 y \int_0^{2-\frac{2}{3}y} e^x \, dx \, dy$$

$$= \int_0^3 y \cdot \left(e^{2-2/3\, y} - 1\right) dy$$

would do: $u = 2 - \frac{2}{3}y$

deal with $u e^u$ ← by parts
(quicker than what we had in the other order).

<u>Note</u>: in some examples, integral in one order $dx\,dy$ is impossible, but the other way is fine.

(e.g. e^{x^2}, $\frac{\sin x}{x}$, -- — not integrable in elementrary functions)

Worksheet 13: Double integrals

1. Change the order of integration:

$$\int_0^4 \int_{y/2}^{2+\sqrt{4-y}} f(x,y)\,dx\,dy.$$

: $\frac{y}{2} \le x \le 2+\sqrt{4-y}$

our integral tells us this.

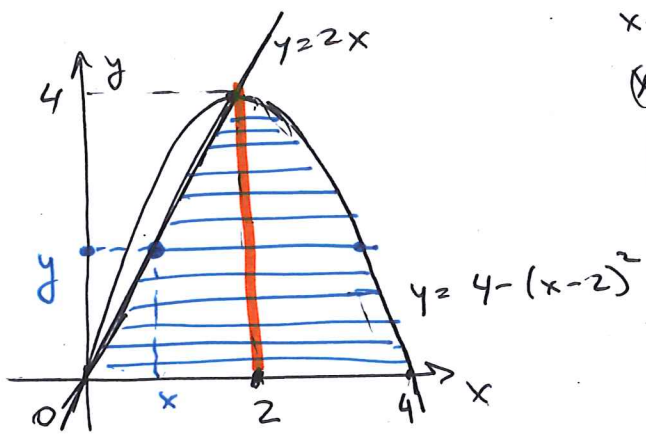
$x = y/2$: $y = 2x$

$x = 2+\sqrt{4-y}$

$x-2 = \sqrt{4-y}$

$(x-2)^2 = 4-y$

$\boxed{y = 4-(x-2)^2}$



$$\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx + \int_2^4 \int_0^{4-(x-2)^2} f(x,y)\,dy\,dx$$

when $0 \le x \le 2$, go up to the line

when $2 \le x \le 4$, go up to the parabola

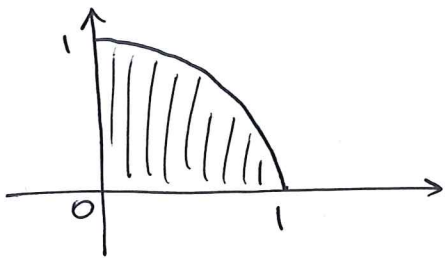
2. Change the integral to polar coordinates:

$$\int_D e^{x^2+y^2}\,dA,$$

where D is the part of the unit disk between the x-axis and the line $x = y$.

Polar coordinates

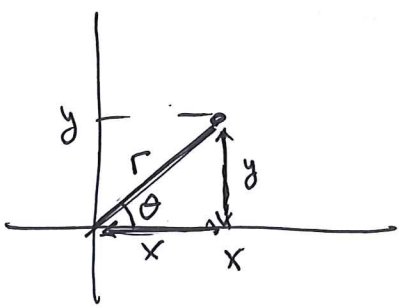
(useful for integrals over pieces of circles centred at (0,0)).
and some other things.



could write:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$$

A lot easier (sometimes) to use polar coordinates



Conversion: $x = r\cos\theta$
$y = r\sin\theta$

$0 \le r$
$0 \le \theta < 2\pi$
(or $-\pi \le \theta \le \pi$)

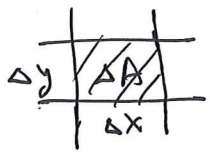
given $f(x,y)$, write $x = r\cos\theta$
$y = r\sin\theta$
plug it into f.

- $$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta$$

"$dx\,dy = \underline{\underline{r}}\,dr\,d\theta$"

dA (like u-substitution)

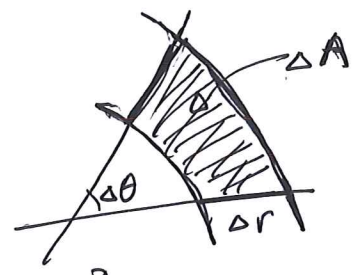
- Where does r come from?



$\Delta A = \Delta x \cdot \Delta y$

"$dA = dx\,dy$"

xy-coordinates



break everything not into rectangles, but into wedges.

$$\Delta A \neq \Delta r\,\Delta\theta$$

in <u>fact</u>: $\boxed{\Delta A \approx r \cdot \Delta r\,\Delta\theta}$