Double integrals

- Review integration

- Last time: integrals over rectangles.

\[
\iint_R f(x,y) \, dA = \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx
\]

Fubini's Theorem
(true when \( \iiint |f(x,y)| \, dA \) exists)

true for continuous \( f \)

on the closed region \( R \).

Examples

\( f(x,y) = xy^2 \)

Find \( \iiint f(x,y) \, dA \), \( R = [0,1] \times [2,3] \)

\[
\int_0^1 \int_2^3 xy^2 \, dy \, dx = \int_0^1 \left( \int_2^3 xy^2 \, dy \right) \, dx
\]

evaluate this

11 keep \( x \) as a parameter.

\[
= \int_0^1 \left( \frac{y^3}{3} \right) \bigg|_2^3 \, dx = \int_0^1 \frac{1}{3} x \left( 9 - \frac{8}{3} \right) \, dx
\]

\[
= \int_0^1 \frac{1}{3} x^2 \left( \frac{19}{3} \right) \, dx = \frac{19}{3} \cdot \frac{x^3}{3} \bigg|_0^1 = \frac{19}{3}
\]
More interesting regions

\[ \iint_D f(x,y) \, dA \]

to define: draw a fine grid,
make a Riemann sum as
before using all rectangles
that intersect \( D \)

Refine the grid.
Take a limit. – called \( \int \int_D f(x,y) \, dA \).

**Important point:** This is the only way to
define what "area" means for \( D \).

\[ \text{Area}(D) = \iiint_D 1 \cdot dA = \lim_{\text{sum of areas of rectangles covering } D} \]

\[ f(x,y) = 1 \text{ for all } x,y. \]
Example

Find \( \iint_D e^{x+y} \, dA \)

Need to make it into an iterated integral.

The limits of our integrals encode the shape of \( D \).

1) Make a choice: \( \int (\int dy) \, dx \)

or the other way: \( \int (\int dx) \, dy \)

Which variable on the outside?

Let \( x \) be on the outside:

\[
\int_0^2 \int_0^{3 - \frac{3}{2}x} e^{x+y} \, dy \, dx
\]

"The line" need the equation of the line bounding the triangle

\[
= \int_0^2 e^x \cdot \left( \int_0^{3 - \frac{3}{2}x} y \, dy \right) \, dx
\]

\[
= \int_0^2 e^x \cdot \frac{y^2}{2} \bigg|_0^{3 - \frac{3}{2}x} \, dx
\]

\[
= \frac{1}{2} \int_0^2 e^x \left( 3 - \frac{3}{2}x \right)^2 \, dx = \ldots
\]

should be a function of just \( x \).
The outside integral has to have numbers as limits (no expressions, no variables).

Once you are inside an integral with respect to \(x\), you can use \(x\) in the limits.

In our example, switch the order of integration:

\[
\iint_D e^x y \, dA = \int_0^3 \int_0^{3 - \frac{3}{2}x} e^x \, dy \, dx
\]

Have to rewrite the equation of the line so that it expresses \(x\) in terms of \(y\):

\[
y = 3 - \frac{3}{2}x
\]

Solve for \(x\):

\[
\frac{3}{2}x = 3 - y, \quad x = \frac{2}{3}(3 - y)
\]
\[
\int_0^3 \int_0^{2-\frac{2}{3}y} e^x \, dx \, dy = \\
= \int_0^3 y \int_0^{2-\frac{2}{3}y} e^x \, dx \, dy = \\
= \int_0^3 y \cdot \left( e^{2-\frac{2}{3}y} - 1 \right) \, dy
\]

\[\text{Would do: } u = 2 - \frac{2}{3}y\]

\[\text{Deal with } u e^u \text{ by parts (quicker than what we had in the other order).}\]

**Note:** In some examples, integral is one order dx dy is impossible, but the other way is fine.

(e.g. \(e^{x^2}\), \(\frac{\sin x}{x}\), . . . not integrable in elementary functions)
Worksheet 13: Double integrals

1. Change the order of integration:

   \[ \int_{y/2}^{4} \int_{y/2}^{2+\sqrt{4-y}} f(x, y) \, dx \, dy. \]

   Our integral tells us this.

   \[ \frac{y}{2} \leq x \leq 2 + \sqrt{4-y} \]

   \[ x = 2 + \sqrt{4-y} \]

   \[ x - 2 = \sqrt{4-y} - 2 \]

   \[ (x-2)^2 = 4-y \]

   \[ y = 4 - (x-2)^2 \]

   \[ \int_{0}^{2} \int_{0}^{2x} f(x, y) \, dy \, dx \]

   \[ + \int_{2}^{4} \int_{0}^{4 - (x-2)^2} f(x, y) \, dy \, dx \]

   When \( 0 \leq x \leq 2 \),

   go up to the line

   when \( 2 \leq x \leq 4 \),

   go up to the parabola

2. Change the integral to polar coordinates:

   \[ \int_{D} e^{x^2+y^2} \, dA, \]

   where \( D \) is the part of the unit disk between the \( x \)-axis and the line \( x = y \).
Polar coordinates

(Useful for integrals over pieces of circles centred at (0,0), and some other things.)

Could write:
\[
\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y) \, dy \, dx
\]

A lot easier (sometimes) to use polar coordinates

Conversion:

\[
\begin{align*}
X &= r \cos \theta \\
Y &= r \sin \theta
\end{align*}
\]

\[0 \leq r \leq 2\pi \]

(or \(-\pi \leq \theta \leq \pi\))

Given \(f(x,y)\), write \(X = r \cos \theta\)
\(Y = r \sin \theta\)
plug it into \(f\).
\[ \iint_D f(x, y) \, dA = \iint_D f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \]

- Where does \( r \) come from?

\[ \Delta A = \Delta x \cdot \Delta y \]

"\( dA = dx \, dy \)"

xy-coordinates

\[ \delta A \neq dA \, d\theta \]

in fact: \[ \delta A \approx r \cdot \Delta r \, \Delta \theta \]

break everything not into rectangles, but into wedges.