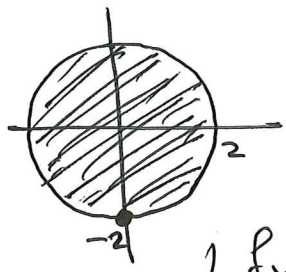
Today: continuing absolute max/min problems.
Lagrange multipliers.

Example: Find max/min of $f(x,y)$ on the disk D: $x^2+y^2 \le 4$

$$f(x,y) = 2x + xy$$



Step 1: Look for critical points inside:

$$\begin{cases} f_x = 2+y \\ f_y = x \end{cases}$$

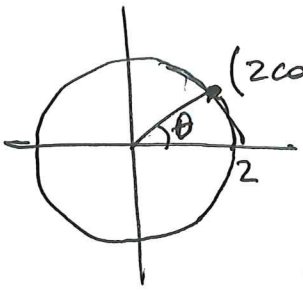
$f_x = f_y = 0$: $\boxed{(0,-2)}$

(on the boundary).

- Step 2 The boundary.

plug in $y = \pm\sqrt{4-x^2}$ – gives an ugly function.

better: • parametrize the circle:
find $x(t), y(t)$ (t – parameter)
such that as t changes, $(x(t), y(t))$ traces the circle.



this is our parametrization!

$t = \theta$:

$(2\cos t, 2\sin t)$ – parametrization of a circle.

Plug this into f:

look for max/min of $f(\overbrace{2\cos t}^{x}, \overbrace{2\sin t}^{y})$, $t \in \mathbb{R}$

(or $\underline{0 \le t \le 2\pi}$).

we had $f(x,y) = 2x + xy$
plug in $x = 2\cos t$, $y = 2\sin t$

Get,

$$g(t) = 2 \cdot 2\cos t + (2\cos t)(2\sin t)$$

$$= 4\cos t + 4\cos t \sin t.$$ ← function of just t

$$g'(t) = 4(-\sin t - \sin^2 t + \cos^2 t)$$

solve: $\underbrace{\cos^2 t}_{= 1 - \sin^2 t} - \sin^2 t - \sin t = 0$

Get: $1 - 2\sin^2 t - \sin t = 0$

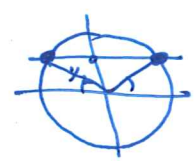
let $w = \sin t$: $1 - 2w^2 - w = 0$

$$w = \frac{1 \pm \sqrt{1+8}}{-4} = -1 \text{ or } \frac{1}{2}$$

Get: $\sin t = -1$ or $\underbrace{\sin t = 1/2}_{t = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}}$

$x = 0$
$y = -2$

get: $(\pm\sqrt{3}, 1)$.



Answer: our points to consider: $\boxed{(\sqrt{3}, 1), (-\sqrt{3}, 1) \quad (0, -2)}$

- <u>Lagrange multipliers</u>

  - method for finding max/min value of $f(x,y)$ or $f(x,y,z)$

  <u>subject to</u> a <u>constraint</u> given by $g(x,y)=0$ or $g(x,y,z)=0$

In our example, we are looking for max/min of $f(x,y) = 2x + xy$

<u>subject to</u> $g(x,y) = x^2+y^2-4 = 0$.

<u>Method of Lagrange multipliers says</u>:

look for points that satisfy the constraint <u>and</u> have the property that $\bar{\nabla} f = \lambda \bar{\nabla} g$

where λ is a scalar (is called Lagrange multiplier).

lambda (Greek)

In practice: $\bar{\nabla} f = \langle 2+y, x \rangle$

(in our example) $\bar{\nabla} g = \langle 2x, 2y \rangle$

$f = 2x + xy$
$g = x^2 + y^2 - 4$

we set up a system of equations:

$$\begin{cases} 2+y = \lambda \cdot 2x \\ x = \lambda \cdot 2y \\ x^2 + y^2 - 4 = 0 \end{cases}$$

$\leftarrow \bar{\nabla} f = \lambda \cdot \bar{\nabla} g$

$\leftarrow g(x,y) = 0$

Solve this system. (do not care to find λ).

here: plug in the second equation into first:

$$\begin{cases} 2+y = 2\lambda \cdot \lambda \cdot 2y \\ 4\lambda^2 y^2 + y^2 - 4 = 0 \end{cases}$$

mult. by y

$$\begin{cases} 4\lambda^2 y = 2+y \\ 4\lambda^2 y^2 + y^2 - 4 = 0 \\ x = 2\lambda y \end{cases}$$

$$\begin{cases} 4\lambda^2 y^2 = 2y + y^2 \\ 4\lambda^2 y^2 + y^2 - 4 = 0 \\ x = 2\lambda y \end{cases}$$

$-2y^2 - 2y + 4 = 0.$

solve for y.

get same answer as before.

Worksheet 12: Lagrange multipliers

1. (a) Minimize the function

$$f(x, y, z) = (x-2)^2 + (y-1)^2 + z^2$$

subject to the constraint $x^2 + y^2 + z^2 = 1$, using the method of Lagrange multipliers.

(b) Give a geometric interpretation of this problem.

Shortcut: write

$f - \lambda \cdot g$ ← a way to remember what to do.

$(x-2)^2 + (y-1)^2 + z^2 - \lambda \cdot (x^2+y^2+z^2-1)$

Now: look for critical point (with respect to all variables including λ).

$$\begin{cases} 2(x-2) - 2\lambda x = 0 \\ 2(y-1) - 2\lambda y = 0 \\ 2z - 2\lambda z = 0 \\ x^2+y^2+z^2 = 1 \end{cases}$$

Solution: 3d equation is:

$$2z - 2\lambda z = 0$$

$$= 2z(1-\lambda) = 0$$

so: $z = 0$ or $\lambda = 1$.

$$\begin{cases} x^2+y^2 = 1 \\ \cancel{2}(x-2) = \cancel{2}\lambda x \\ \cancel{2}(y-1) = \lambda y \end{cases} \iff \begin{cases} x^2+y^2 = 1 \\ x(1-\lambda) = 2 \\ y(1-\lambda) = 1 \end{cases}$$

$$x = \frac{2}{1-\lambda} \qquad (\lambda \neq 1)$$

$$y = \cancel{1/1} - \lambda$$

$$x^2+y^2 = \frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 1.$$

$(1-\lambda)^2 = 5.$ Then $1-\lambda = \pm\sqrt{5}$

Then $x = \pm\frac{2}{\sqrt{5}}$ $\quad y = \pm\frac{1}{\sqrt{5}}$

incorrect!

better: if $1-\lambda = \sqrt{5}$, get

$$\boxed{x = \frac{2}{\sqrt{5}}, \quad y = \frac{1}{\sqrt{5}}}$$

or $1-\lambda = -\sqrt{5}$

$$\boxed{x = -\frac{2}{\sqrt{5}}, \quad y = -\frac{1}{\sqrt{5}}}$$

Case 2 $\lambda = 1$:

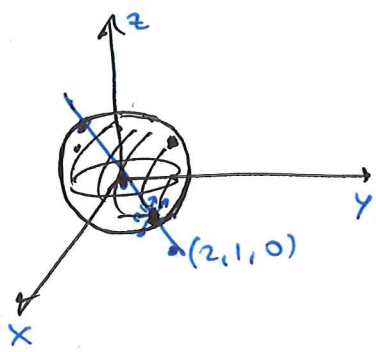
$$\begin{cases} x^2+y^2+z^2=1 \\ 2(y-1)-2y=0 \\ 2(\cancel{x}-2)-2\cancel{x}=0 \end{cases} \longrightarrow \text{impossible!}$$

Now we got: possible points for max or min are:

$$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \leftarrow \boxed{\text{min}}, \quad \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \leftarrow \boxed{\text{max}}$$

Which is which? - plug into f and decide

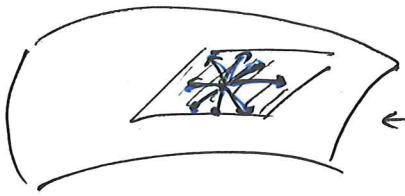
Answer to (b): constraint: we are looking for a point (x,y,z) on the unit sphere.



$f(x,y,z)$ is the square of the distance from (x,y,z) to $(2,1,0)$

So we are looking for a point on the unit sphere that is closest to $(2,1,0)$

Why this works:



← surface $g(x,y,z)=0$

Normally, partial derivatives of $f(x,y,z)$ have to vanish at any max/min point because:

$\bar{\nabla} f \neq 0$ then can move in the direction of $\bar{\nabla} f$ and f will get bigger.

opposite to $\bar{\nabla} f$: f will get smaller.

So: $\bar{\nabla} f \neq 0$, then our point cannot be max/min point.

This was without constraint.

<u>With constraint:</u> we need: $D_{\bar{u}} f = 0$ for all $\bar{u}$ <u>tangent</u> to the constraint.

This means:

$$\underbrace{\bar{\nabla} f}_{} \cdot \bar{u} = 0 \quad \text{for any } \bar{u} \text{ tangent to } g(x,y,z)=0.$$

so $\bar{\nabla} f$ is normal to the same plane as $\bar{\nabla} g$.

$\bar{\nabla} g$ is normal to this tangent plane

Then $\underbrace{\bar{\nabla} f = \lambda \bar{\nabla} g}_{\text{our system of equations.}}$ for some scalar λ.