Today: continuing absolute max/min problems.

Lagrange multipliers.

Example: Find max/min of \( f(x, y) \) on the disk \( D: x^2 + y^2 \leq 4 \)

\[ f = 2x + xy \]

**Step 1**: Look for critical points inside:

\[
\begin{align*}
    f_x &= 2 + y \\
    f_y &= x
\end{align*}
\]

\( f_x = f_y = 0 \): \( (0, -2) \) (on the boundary).

**Step 2**: The boundary.

Plug \( x = \pm \sqrt{4-y^2} \) — gives an ugly function.

Better: parametrize the circle:

Find \( x(t), y(t) \) (\( t \) - parameter)

such that as \( t \) changes, \( (x(t), y(t)) \) traces the circle.
\[(2\cos \theta, 2\sin \theta) \rightarrow \text{this is our parametrization!}\]

\[t = \theta:\]

\[(2\cos t, 2\sin t) \rightarrow \text{parametrization of a circle.}\]

Plug this into \(f:\)

Look for maximum of \(f(2\cos t, 2\sin t), t \in \mathbb{R}\)

(or \(0 \leq t \leq 2\pi\)).

We had \(f(x, y) = 2x + xy\)

Plug in \(x = 2\cos t, y = 2\sin t\)

\[g(t) = 2(2\cos t) - 2\cos t + (2\cos t)(2\sin t)\]

\[= 4\cos t + 4\cos t \sin t. \quad \text{← function of just } t\]

\[g'(t) = 4(-\sin t - \sin^2 t + \cos^2 t)\]

Solve:

\[
\frac{\cos^2 t - \sin^2 t - \sin t = 0}{1 - \sin^2 t}
\]

Get:

\[1 - 2\sin^2 t - \sin t = 0\]

Let \(w = \sin t\):

\[1 - 2w^2 - w = 0\]

\[w = \frac{1 \pm \sqrt{1 + 8}}{-2} = -1 \text{ or } \frac{1}{2}\]

Get:

\[\sin t = -1 \text{ or } \sin t = \frac{1}{2}\]

\[x = 0 \quad \text{and} \quad t = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}\]

Get:

\[(\pm \sqrt{3}, 1), (-\sqrt{3}, 1), (0, -2)\]

Answer: our points to consider:

\[(\pm \sqrt{3}, 1), (-\sqrt{3}, 1), (0, -2)\]
Lagrange multipliers

- method for finding max/min value of \( f(x, y) \) or \( f(x, y, z) \)

subject to a constraint given by \( g(x, y) = 0 \) or \( g(x, y, z) = 0 \)

In our example, we are looking for

max/min of \( f(x, y) = 2x + xy \)

subject to \( g(x, y) = x^2 + y^2 - 4 = 0 \).

Method of Lagrange multipliers says:

look for points that satisfy the constraint

and have the property that \( \nabla f = \lambda \nabla g \)

where \( \lambda \) is a scalar (\( \lambda \) called Lagrange multiplier).

\[ \lambda \text{ (Greek)} \]
In practice: \[ \nabla f = \langle 2 + y, x \rangle \quad \text{and} \quad \nabla g = \langle 2x, 2y \rangle \]

(Our example) \[ \nabla f = \lambda \nabla g \quad \text{and} \quad x^2 + y^2 - 4 = 0 \]

we set up a system of equations:

\[
\begin{aligned}
2 + y &= \lambda \cdot 2x \\
\lambda &= \lambda \cdot 2y \\
x^2 + y^2 - 4 &= 0
\end{aligned}
\]

Solve this system. (Do not care to find \( \lambda \)).

Here: plug in the second equation into first:

\[
\begin{aligned}
2 + y &= 2 \lambda \cdot \lambda \cdot 2y \\
4x^2y^2 + y^2 - 4 &= 0
\end{aligned}
\]

\[
\begin{aligned}
4x^2y & = 2 + y \\
4x^2y^2 + y^2 - 4 &= 0
\end{aligned} \quad \text{mult. by } y
\]

\[
\begin{aligned}
-4x^2y^2 &= 2y + y^2 \\
4x^2y^2 + y^2 - 4 &= 0
\end{aligned}
\]

\[x = 2 \lambda y\]

\[-2y^2 - 2y + 4 = 0\]

Solve for \( y \).

get same answer as before.
Worksheet 12: Lagrange multipliers

1. (a) Minimize the function

\[ f(x, y, z) = (x - 2)^2 + (y - 1)^2 + z^2 \]

subject to the constraint \( x^2 + y^2 + z^2 = 1 \), using the method of Lagrange multipliers.

(b) Give a geometric interpretation of this problem.

\[
\text{Shortcut: write } \quad f - \lambda \cdot g \\
\text{\( (x-2)^2 + (y-1)^2 + z^2 - \lambda \cdot (x^2 + y^2 + z^2 - 1) \)}
\]

Now: look for critical point \( \text{ (with respect to all variables, including } x \text{) } \)

\[
\begin{align*}
2(x-2) - 2\lambda x &= 0 \\
2(y-1) - 2\lambda y &= 0 \\
2z - 2\lambda z &= 0 \\
x^2 + y^2 + z^2 &= 1
\end{align*}
\]
Solution: 3rd equation is:

\[ 2z - 2\lambda z = 0 \]
\[ = 2z(1-\lambda) = 0 \]
so: \( z = 0 \) or \( \lambda = 1 \).

\[
\begin{align*}
\begin{cases}
X^2 + y^2 = 1 \\
\lambda(x-2) = \lambda\lambda x \\
\lambda(y-1) = \lambda y
\end{cases}
\iff
\begin{cases}
X^2 + y^2 = 1 \\
X(1-\lambda) = 2 \\
y(1-\lambda) = 1
\end{cases}
\end{align*}
\]

\[ X = \frac{2}{1-\lambda} \quad (\lambda \neq 1) \]

\[ y = \frac{1}{1-\lambda} \]

\[ x^2 + y^2 = \frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 1. \]

\[ (1-\lambda)^2 = 5. \quad \text{Then} \quad 1-\lambda = \pm\sqrt{5} \]

\[ \text{Then} \quad x = \pm \frac{2}{\sqrt{5}} \quad y = \pm \frac{1}{\sqrt{5}} \]

\[ \text{incorrect!} \]

Better: if \( 1-\lambda = \sqrt{5} \),

\[
\begin{align*}
\left\{ \begin{array}{l}
X = \frac{2}{\sqrt{5}} \\
y = \frac{1}{\sqrt{5}}
\end{array} \right.
\end{align*}
\]

or \( 1-\lambda = -\sqrt{5} \)

\[
\begin{align*}
\left\{ \begin{array}{l}
X = -\frac{2}{\sqrt{5}} \\
y = -\frac{1}{\sqrt{5}}
\end{array} \right.
\end{align*}
\]
Case 2 \( x = 1 \):

\[
\begin{align*}
    x^2 + y^2 + z^2 &= 1 \\
    2(y - 1) - 2y &= 0 \\
    2(x - 2) - 2x &= 0
\end{align*}
\]

\( \rightarrow \) impossible!

Now we get: possible points for \( \max \) or \( \min \) are:

\[
(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}), \quad (-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})
\]

Which is which? - plug into \( f \) and decide

Answer to (6): constraint: we are looking for a point \( (x, y, z) \) on the unit sphere.

\( f(x, y, z) \) is the square of the distance from \( (x, y, z) \) to \( (2, 1, 0) \)

So we are looking for a point on the unit sphere that is closest to \( (2, 1, 0) \)
Why this works:

\[ \text{surface } g(x,y,z) = 0 \]

Normally, partial derivatives of \( f(x,y,z) \) have to vanish at any max/min point because:

- \( \nabla f \neq 0 \), then can move in the direction of \( \nabla f \) and \( f \) will get bigger.
- opposite to \( \nabla f \): \( f \) will get smaller.

So: \( \nabla f \neq 0 \), then our point cannot be max/min point.

This was without constraint.

With constraint: we need: \( \nabla f \cdot \bar{u} = 0 \) for all \( \bar{u} \) tangent to the constraint.

This means:

\[ \nabla f \cdot \bar{u} = 0 \text{ for any } \bar{u} \text{ tangent} \]

so \( \nabla f \) is normal to the same plane as \( \nabla g \).

Then \( \nabla f = \lambda \nabla g \) for some scalar \( \lambda \).

\( \nabla g \) is normal to this tangent plane.

Our system of equations.