Worksheet II: Absolute Max/Min

Produce the complete list of points where the absolute max or min of \( f(x,y) = x^2 + 3xy - 2y^2 \) on the triangle \( T \) with vertices \((-1,2), (-1,-1) \) and \((2,-1)\) could occur; do not evaluate the function at these points.

NEW:

Office Hours:

The 5-6

Thurs 4:15-5:45

Next test: March 10

Will be extra office hours March 9. Will email.

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Last time: critical points

- how to recognize them on the graph:

local min

saddle point:

(local max in one direction)
(local min in another)

How to recognize them on a contour plot:

local max/min

need to know which way is up to distinguish them

Saddle:

down

may or may not be on the plot

Same level curve!

can get very deformed:

(see homework 7)
Next: Absolute max/min on a closed bounded domain.

$f(x,y)$ (or $f(x,y,z)$ or $f(x_1, -x_{100})$)

What is its absolute max/min value?

(on the whole plane (or a the whole domain) might not exist)

If you bound the domain:

- closed (includes its boundary)

- boundary will be simple (intuitive) for us.

- bounded - fits into a large box around the origin.

Example of a closed bounded domain.
Theorem A continuous function on a closed bounded domain attains its max and min values. (call them absolute max and absolute min)

How to find them?

- Recall for \( f(x) \) - function of 1 variable
  - "closed bounded domain" : \[ a \leq x \leq b \]
  - interval \( [a, b] \) (ends included)

  - to find max/min:
    - Step 1: \( f'(x) \), solve \( f'(x) = 0 \) inside \( (a, b) \)
      [finding critical points inside]
    - Step 2 evaluate \( f(a) \), \( f(b) \) \( \Delta \) end points are on the boundary
      evaluate \( f \) at critical points and at points where \( f'(x) \) does not exist.

Find the max/min of all these values.

\[
\begin{array}{c}
\text{y}
g(x) = f(x) \\
\text{max}
\end{array}
\]

- min, critical pt inside
For functions of 2 variables:

Strategy: 1) Look for critical points inside the domain

2) Analyze the boundary — the more difficult part.

- List of "suspicious points" (where max/min could occur)
- Critical points inside the domain
- Vertices of the triangle (where boundary is not smooth)

- Need to analyze the sides of the triangle.

The picture illustrates this: The highest point could be over one of the edges.
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*Step 1*: critical points.

$f(x,y) = x^2 + 3xy - 2y^2$

$f_x = 2x + 3y$

$f_y = 3x - 4y$

*Solve*: $f_x = f_y = 0$: 

\[
\begin{align*}
2x + 3y &= 0 \\
3x - 4y &= 0
\end{align*}
\]

$\Rightarrow (x,y) = (0,0)$

Check if it is inside $T$.

So it goes on the list.

*Step 2*: Just plug in the equations for the boundary pieces.

This gives a problem in one variable.
1: \( x = -1, \ -1 \leq y \leq 2 \)

Plug it into \( f \):

\[
f(-1, y) = (-1)^2 + 3 \cdot (-1) \cdot y - 2y^2
= 1 - 3y - 2y^2 \quad \text{call it } g_1(y)
\]

Now look for critical points of \( g_1(y) \).

\[
g_1'(y) = -3 - 4y
\]

\[
g_1'(y) = 0: \ -3 - 4y = 0 \quad y = -\frac{3}{4} \quad \text{and satisfies } -1 \leq y \leq 2
\]

Get the point: \((-1, -\frac{3}{4})\) - goes on the list.

2: (blue): \( y = -1, \ -1 \leq x \leq 2 \)

Plug it in:

\[
f(x, -1) = x^2 + 3x \cdot (-1) - 2 \cdot (-1)^2 = x^2 - 3x - 2
\]

Look for critical points: \( g_2'(x) = 2x - 3 \quad x = \frac{3}{2} \)

Get: \( \left( \frac{3}{2}, -1 \right) \) - goes on the list.
Plug in its equation: \[ y = -x + 1. \]

Get:

\[ g_3(x) = f(x, -x+1) = x^2 + 3 \cdot (-x+1) - 2(-x+1)^2 \]
\[ = -4x^2 + 7x - 2 \]

\[ g_3'(x) = -8x + 7 \]

\[ x = \frac{7}{8} \]

Our point: \( \left( \frac{7}{8}, \frac{1}{8} \right) \).

Answer:

\[ (0,0) \leftarrow \text{critical pt of } f \text{ inside } T \]
\[ \left( \frac{7}{8}, \frac{1}{8} \right), (-1, -\frac{3}{4}), \left( \frac{3}{2}, -1 \right) \leftarrow \text{edges of } T \]
\[ (-1, -1), (-1, 2), (2, -1) \leftarrow \text{vertices} \]
What if we had more variables?

\[ f(x, y, z) = x^2 \cos(z) + e^{xy-z} \]

Find max/min on the unit cube.

Steps: 1) Look for critical points

\[ f_x = f_y = f_z = 0 \] inside the cube

2) Have to analyze every face:

   e.g. the face \( z = 0 \)

\[ h_1(x, y) = x^2 + e^{xy} - 1 \]

Plug in \( z = 0 \)

get:

\[ h_1(x, y) = x^2 + e^{xy} - 1 \]

Look for critical points of \( h_1 \).

3) Consider every edge: plug in \( x = 0 \), \( y = 0 \)

get a function of \( z \),

etc.

4) Put vertices on the list.