Today: • Last bits about gradients, tangent planes, implicit differentiation

• Critical points. Next week: optimization: max/min of functions of several variables.

(These notes refer to worksheet 8 – see the last page of notes from the previous class)

Relevance of Problem 1: ← from the worksheet, see #8, last class

- 2 ways of finding the tangent plane to the graph of $f(x,y)$: ← fn of 2 variables

① $z = f(x,y)$ – graph. Let $(a,b, f(a,b))$

Then write linearization of f at (a,b):

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$z = L(x,y)$ ← graph of a linear function (a plane!)

– the tangent plane.

② Make $F(x,y,z) = z - f(x,y)$ ← fn of 3 variables

Our graph is a <u>level surface</u> of F

(defined by $F(x,y,z) = 0$)

<u>Example</u>: $f(x,y) = \sqrt{x^2+y^2}$

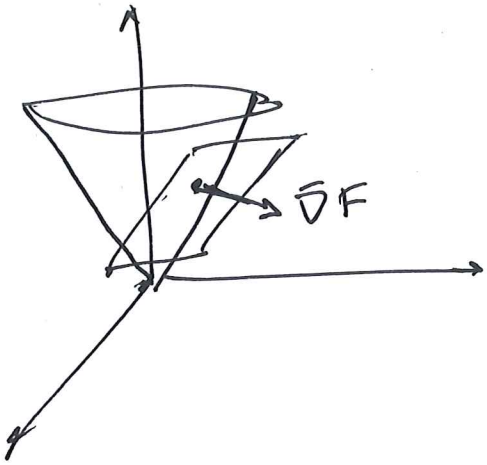
$z = \sqrt{x^2+y^2}$ – can use linearization to get tangent plane.

or: $F(x,y,z) = z - \sqrt{x^2+y^2}$ – fn of 3 variables.

∇F is normal to the level surface $z - \sqrt{x^2-y^2} = 0$.

Gives the same normal vector and the same plane!

"normal to level surface" = "normal to the <u>tangent plane</u> of the level surface"



Application to implicit differentiation:

Let z be defined as an implicit function of x, y by: $F(x, y, z) = 0$.

We used to solve for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Simpler formula: 1) Forget all about implicit functions, just find F_x, F_y, F_z.

2) $$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$ – explained in a post online

Reason: compare 2 ways of writing the tangent plane.

Example: Surface defined by

$$x^3 y - y^3 z + z^2 x = xy$$

Find the equation of the tangent plane at $(1, 0, 0)$ and find $\frac{\partial z}{\partial x}$ at this point.

Hint: $F(x, y, z) = x^3 y - y^3 z + z^2 x - xy$.

(in this example we'll find out that at this point, z is NOT a function of x, y and the tangent plane is undefined)

$F_x = 3x^2y + z^2 - y$

$F_y = x^3 - 3y^2z - x$

$F_z = -y^3 + 2zx$

at $(1,0,0)$: $F_x = 0$ $F_y = \overset{0}{\cancel{1}}$, $\boxed{F_z = 0.}$

at $(1,0,0)$, $\bar{\nabla} F = \langle 0, \overset{0}{\cancel{1}}, 0 \rangle$ ← critical point.

Then the equation of the tangent plane is: $0 = 0$?

$\cancel{0\cdot(x-1) + (-1)\cdot(y-0) + 0\cdot(z-0) = 0}$

$\Leftrightarrow \quad -y = 0$

So ~~xz-plane~~ is tangent to this surface

at ~~$(1,0,0)$~~.

Want to use our formulas for $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$:

Get: have to divide by 0!

So actually they do not exist!

In this situation (if $F_z = 0$) cannot think of z as an implicit function of x, y.

(This happens at some points on pretty much any surface)

Worksheet 8: gradients

1. Let $f(x, y)$ be a function of two variables. Find the gradient of the function $F(x, y, z) = z - f(x, y)$.

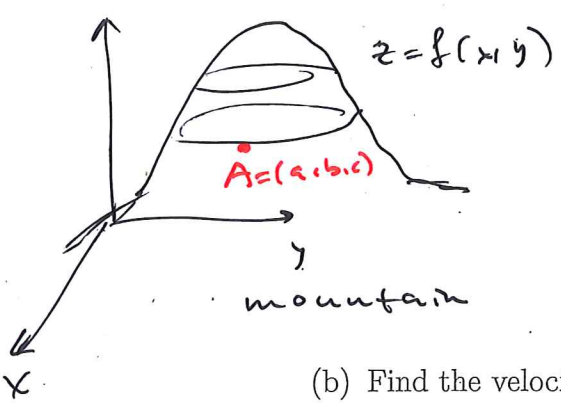
some function.

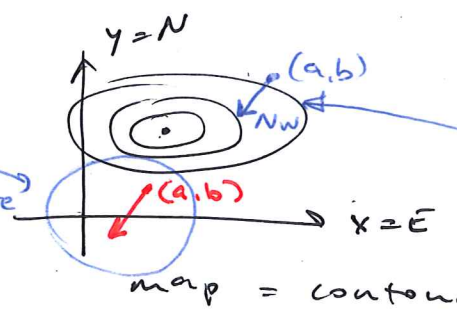
$$\bar{\nabla} F = \langle -f_x, -f_y, 1 \rangle$$

See previous page for remarks about why we are looking at this function

2. A hiker is walking up the trail on the mountain following the direction of the steepest ascent. The hiker's speed is $3km/hr$. When the hiker is at point $A = (a, b, c)$ on the mountain, his compass is telling him that he is walking directly Northwest; the slope of the trail is 30°.

   (a) If the mountain is thought of the graph of the altitude function $z = f(x, y)$, find $|\nabla f|$ at the point (a, b).





actually this picture makes no sense as the gradient must point up. So, (a, b) has to be here if the gradient points NW.

map = contour plot of f(x, y)

   (b) Find the velocity vector of the hiker at the point A.

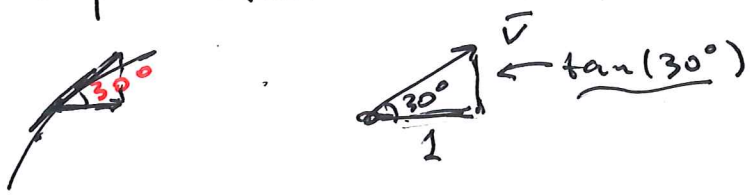
see next few pages for the solution.

What are we given about $\bar{\nabla} f$ at (a,b) ?

- Given: ① the steepest ascent is NW

$$\Rightarrow \quad \bar{\nabla} f \parallel \underbrace{\langle -1, 1 \rangle}_{NW}$$

② slope in this direction is $30°$



Given: if $\bar{u}$ is a unit vector pointing NW then $D_{\bar{u}} f = \tan(30°) = \frac{1}{\sqrt{3}}$.

Also know: $D_{\bar{u}} f = \bar{\nabla} f \cdot \bar{u}$, so

$D_{\bar{u}} f = |\bar{\nabla} f|$ when $\bar{u} \uparrow\uparrow \bar{\nabla} f$.

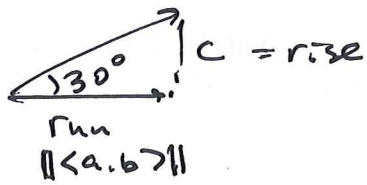
Answer: $\boxed{|\bar{\nabla} f| = \frac{1}{\sqrt{3}}}$.

2(b): velocity of the hiker $\bar{v}$:

$$\bar{v} = \langle a, b, c \rangle$$

Given: 1) $\langle a, b \rangle \uparrow\uparrow \langle -1, 1 \rangle$

2) $\text{slope} = \dfrac{c}{\sqrt{a^2+b^2}} = \tan 30^\circ$



3) $\|\bar{v}\| = 3 \text{ km/hr}$

So we have: 1) $\bar{v} = \langle -a, a, c \rangle$

2) $a^2 + a^2 + c^2 = 3$

3) $\dfrac{c}{a\sqrt{2}} = \dfrac{1}{\sqrt{3}}$

Compute $\rightsquigarrow$ get a, c. (see online). (separate link to the solutions.)

Critical points:

Def: $f(x_1, \ldots, x_n)$. A critical point is a point in the domain of f at which all partial derivatives of f are 0.
(or f is not differentiable)

(this means, $\vec{\nabla} f = 0$ at a critical point (if f is differentiable)).

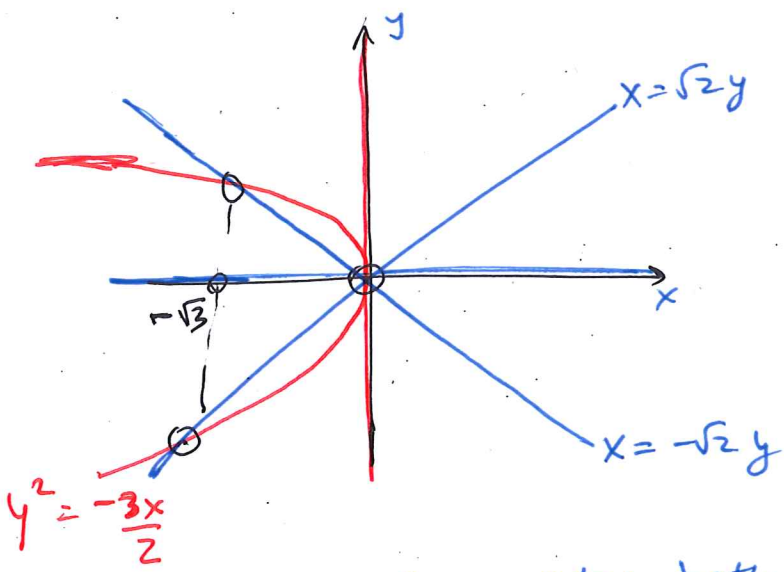
Worksheet 10: critical points

1. Find and classify all the critical points of the function

$$f(x,y) = x^3 + x^2y^2 - y^4.$$

$$f_x = 3x^2 + 2xy^2 = x(3x + 2y^2)$$
$$f_y = 2yx^2 - 4y^3 = y(2x^2 - 4y^2)$$

Solve: $\begin{cases} x(3x+2y^2) = 0 \\ y(2x^2-4y^2) = 0 \end{cases} \iff \begin{cases} \left[\begin{array}{l} x = 0 \\ 3x+2y^2 = 0 \end{array}\right. & \text{— red} \\ \left[\begin{array}{l} y = 0 \\ x^2 = 2y^2 \end{array}\right. & \text{— blue} \end{cases}$



$3x + 2y^2 = 0$

$y^2 = -\frac{3x}{2}$

$x^2 = 2y^2$

$x = \pm\sqrt{2}\cdot y$

Critical points: both blue and red.

Get: $(0,0)$ and: $\begin{cases} y = \sqrt{-\frac{3x}{2}} \\ y = -\frac{x}{\sqrt{2}} \end{cases}$

and the one will be same x, negate the y.

$y = -\frac{x}{\sqrt{2}} = \sqrt{\frac{-3x}{2}}$ square: $\frac{x^2}{2} = \frac{-3x}{2}$

$x = -\sqrt{3}$.

Our points: $\boxed{(0,0),\ \left(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}\right),\ \left(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}\right)}$.

2nd derivative Test for functions of 2 variables.

Comes from approximating $f(x,y)$ by a degree 2 polynomial:

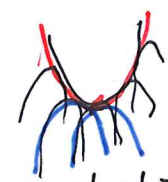
we talked about linear approximation.

Better: a quadratic approximation.

graph: $z = (\text{quadratic of } x, y)$

Prototypes: $z = a^2x^2 + b^2y^2$ or $z = a^2x^2 - b^2y^2$

- elliptic paraboloid



hyperbolic paraboloid.

- faces down:

$z = -(a^2x^2 + b^2y^2)$

How to tell what best approximates the graph of f?

$$D = \underbrace{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}}_{\text{Hessian matrix}} = f_{xx} f_{yy} - (f_{xy})^2$$

$f_{xy} = f_{yx}$

← evaluate it at your point.

- if $D > 0$ then: if $f_{xx} > 0$, then our point is local min 

  if $f_{xx} < 0$, local max 

- if $D < 0$ then our point is <u>saddle</u> point

- if $D = 0$ — undetermined.

(<u>exercise</u>: let $f(x,y) = a^2x^2 + b^2y^2$.
compute D. see $D > 0$.
Then compute D for $f = a^2y^2 - b^2y^2$.
see $D < 0$.)