Today: Last bits about gradients, tangent planes, implicit differentiation.

- Critical points. Next week: optimization, max/min of functions of several variables.

(These notes refer to Worksheet 8 – see the last page of notes from the previous class.)
Relevance of Problem 1: from the worksheet #8, see last class

2 ways of finding the tangent plane to the graph of \( f(x,y) = \text{fn of 2 variables} \):

1. \( z = f(x,y) \) - graph. Let \((a,b, f(a,b))\)

   Then write linearization of \( f \) at \((a,b)\):

   \[
   L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)
   \]

   \( z = L(x,y) \) - graph of a linear function (a plane!)

   - the tangent plane.

2. Make \( F(x,y,z) = z - f(x,y) \) - fn of 3 variables

   Our graph is a level surface of \( F \) (defined by \( F(x,y,z) = 0 \))

   **Example:** \( f(x,y) = \sqrt{x^2+y^2} \)

   \( z = \sqrt{x^2+y^2} \) - can use linearization to get tangent plane.

   or: \( F(x,y,z) = z - \sqrt{x^2+y^2} \) - fn of 3 variables.

   \( \nabla F \) is normal to the level surface: \( z = \sqrt{x^2+y^2} = 0 \).

   Given the same normal vector and the same plane!

   "normal to level surface" = "normal to the tangent plane of the level surface"
Application to implicit differentiation:

Let \( z \) be defined as an implicit function of \( x, y \) by: \( F(x, y, z) = 0 \).

We used to solve for \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \).

Simpler formula:

1) Forget all about implicit functions, just find \( F_x, F_y, F_z \).

\[ \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \]

2) Explained in a point online.

Reason: compare 2 ways of writing the tangent plane.

Example: Surface defined by

\[ x^3y - y^3z + z^2x = xy \]

Find the equation of the tangent plane at \((1, 0, 0)\) and find \( \frac{\partial z}{\partial x} \) at this point.

Hint: \( F(x, y, z) = x^3y - y^3z + z^2x - xy \).

(in this example we'll find out that at this point, \( z \) is not a function of \( x, y \) and the tangent plane is undefined)
\[ F_x = 3x^2y + x^2 - y \]
\[ F_y = x^3 - 3y^2z - x \]
\[ F_z = -y^3 + 2xz \]

at \((1,0,0)\):  \( F_x = 0 \), \( F_y = 2 \), \( F_z = 0 \)

at \((1,0,0)\), \( \nabla F = \langle 0, 2, 0 \rangle \) \text{ is a critical point.}

Then the equation of the tangent plane is: \( 0 = 0 \)?

\[ 0 \cdot (x-1) + 2 \cdot (y-0) + 0 \cdot (z-0) = 0 \]

\( \implies -y = 0 \)

\( \text{So } xz \text{-plane is tangent to this surface at } (1,0,0). \)

Want to use our formulas for \( \frac{\partial z}{\partial x} \) \text{ and } \( \frac{\partial z}{\partial y} \): \text{ get: have to divide by } 0! \text{ so actually they do not exist!}

In this situation (if \( F_z = 0 \)) \text{ cannot think of } z \text{ as an implicit function of } x, y. \text{ (This happens at some points on pretty much any surface.)}
Worksheet 8: gradients

1. Let \( f(x, y) \) be a function of two variables. Find the gradient of the function \( F(x, y, z) = z - f(x, y) \).

\[
\nabla F = \langle -f_x, -f_y, 1 \rangle
\]

See previous page for remarks about why we are looking at this function.

2. A hiker is walking up the trail on the mountain following the direction of the steepest ascent. The hiker's speed is 3\( km/hr \). When the hiker is at point \( A = (a, b, c) \) on the mountain, his compass is telling him that he is walking directly Northwest; the slope of the trail is 30°.

(a) If the mountain is thought of the graph of the altitude function \( z = f(x, y) \), find \( |\nabla f| \) at the point \( (a, b) \).

(b) Find the velocity vector of the hiker at the point \( A \).
What are we given about $\nabla f$ at $(a,b)$?

- Given: 1. the steepest ascent is NW
  $\implies \hat{f} \parallel <-1, 1> \parallel \text{NW}$

2. slope in this direction is $30^\circ$

  $\tan(30^\circ) = \frac{\sqrt{3}}{3}$

Given: if $\hat{u}$ is a unit vector pointing NW
then $D_{\hat{u}} f = \tan(30^\circ) = \frac{1}{\sqrt{3}}$.

Also know: $D_{\hat{u}} f = \nabla f \cdot \hat{u}$, so
$D_{\hat{u}} f = |\nabla f|$ when $\hat{u} \parallel \nabla f$.

Answer: $|\nabla f| = \frac{1}{\sqrt{3}}$. 
2(b): velocity of the hiker \( \overrightarrow{v} \):

\[ \overrightarrow{v} = \langle a, b, c \rangle \]

**Given:**

1) \( \langle a, b \rangle \uparrow \uparrow \langle -1, 1 \rangle \)

2) slope = \( \frac{c}{\sqrt{a^2 + b^2}} = \tan 30^\circ \)

\[ \triangle \]

3) \( \| \overrightarrow{v} \| = 3 \text{ km/hr} \)

So we have:

1) \( \overrightarrow{v} = \langle -a, a, c \rangle \)

2) \( a^2 + a^2 + c^2 = 3 \)

3) \( \frac{c}{a\sqrt{2}} = \frac{1}{\sqrt{3}} \)

Compute \( a, c \). (see online). (separate link to the solutions.)
Critical points:

3) Def: \( f(x_1, \ldots, x_n) \). A critical point is a point in the domain of \( f \) at which all partial derivatives of \( f \) are 0.

(or \( f \) is not differentiable)

(this means, \( \nabla f = 0 \) at a critical point

(if \( f \) is differentiable)\).
Worksheet 10: critical points

1. Find and classify all the critical points of the function

\[ f(x, y) = x^3 + x^2y^2 - y^4. \]

\[ f_x = 3x^2 + 2xy^2 = x(3x + 2y^2) \]
\[ f_y = 2xy^2 - 4y^3 = y(2x^2 - 4y^2) \]

\[ \text{Solve: } \begin{cases} x(3x + 2y^2) = 0 \\ y(2x^2 - 4y^2) = 0 \end{cases} \]

\[ \Rightarrow \begin{cases} x = 0 \\ 3x + 2y^2 = 0 \\ y = 0 \\ x^2 = 2y^2 \end{cases} \]

\[ 3x + 2y^2 = 0 \]
\[ y^2 = -\frac{3x}{2} \]
\[ x^2 = 2y^2 \]
\[ x = \pm \sqrt{2} \cdot y \]

Critical points: both blue and red.

Get: \((0, 0)\) and: \( \begin{cases} y = \sqrt{-\frac{3x}{2}} \\ y = -\frac{x}{\sqrt{2}} \end{cases} \)

and the one will be same \(x\), negate the \(y\).
\[ y = -\frac{x}{\sqrt{2}} = \sqrt{-\frac{3x^2}{2}} \]

square: \[ \frac{x^2}{2} = -\frac{3x}{2} \]

\[ x = -\sqrt{3} . \]

Our points: \((0, 0), (\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}})\).

\[ 2\text{nd derivative Test for functions of 2 variables} \]

comes from approximating \( f(x,y) \) by a degree 2 polynomial:

we talked about linear approximation.

Better: a **quadratic** approximation.

graph: \( z = \text{quadratic of } x,y \)

Prototypes:

\[ z = a^2 x^2 + b^2 y^2 \] or \[ z = a^2 x^2 - b^2 y^2 \]

- elliptic paraboloid
- face down:
  \[ z = -(a^2 x^2 + b^2 y^2) \]

hyperbolic paraboloid
How to tell what best approximates the graph of $f$?

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$ — evaluate it at your point.

- $D > 0$ then:
  - if $f_{xx} > 0$, then the point is local min
  - if $f_{xx} < 0$, then local max

- $D < 0$ then the point is saddle point

- $D = 0$ — undetermined.

(exercise: Let $f(x, y) = ax^2 + by^2$. Compute $D$, see $D > 0$. Then compute $D$ for $f = a^2x^2 - b^2y^2$. See $D < 0$.)