

Today: • Chain rule

1) Explanation & the rule:

$$F(t) = f(x(t), y(t), z(t))$$

$$\frac{dF}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

2) New application: directional derivatives!
(The gradient vector)

3) Chain rule in general:

$$F(u, v) := f(x(u, v), y(u, v), z(u, v))$$

$$\frac{\partial F}{\partial u} = ? \quad \frac{\partial F}{\partial v} = ?$$

Explanation for chain rule:

i) in single variable:

$$f(x)$$

Plug in $x = x(t)$. Suppose at $t = \underline{a}$ we have $x(a) = \underline{b}$.
Want to find $\frac{df}{dt} \Big|_{t=a}$.

Write the linear approximation:

$$\Delta f \approx f'(b) \underbrace{\Delta x}_{\approx x'(a) \Delta t} \quad \text{when } x \text{ is near } b.$$

$$\Delta f \approx f'(b) \cdot x'(a) \Delta t$$

$$\text{so } \frac{\Delta f}{\Delta t} \xrightarrow{\Delta t \rightarrow 0} f'(b) x'(a) \quad - \text{get chain rule!}$$

Now suppose we have $f(x, y)$

let $x = x(t)$ at $t = t_0$ - fixed value of t , $x(t_0) = a$
 $y = y(t)$ at $t = t_0$ - fixed value of t , $y(t_0) = b$.

We know: $\Delta f \approx \frac{\partial f}{\partial x} \Big|_{(a,b)} \underbrace{\Delta x}_{\substack{\text{near} \\ (a,b)}} + \frac{\partial f}{\partial y} \Big|_{(a,b)} \underbrace{\Delta y}_{y'(t_0) \Delta t}$

$$\Delta f \approx \frac{\partial f}{\partial x} \Big|_{(a,b)} \cdot x'(t_0) \Delta t + \frac{\partial f}{\partial y} \Big|_{(a,b)} \cdot y'(t_0) \Delta t.$$

$$\left[\frac{\Delta f}{\Delta t} \xrightarrow{\Delta t \rightarrow 0} \frac{\partial f}{\partial x} \Big|_{(a,b)} \cdot \frac{dx}{dt} \Big|_{t=t_0} + \frac{\partial f}{\partial y} \Big|_{(a,b)} \cdot \frac{dy}{dt} \Big|_{t=t_0} \right] \text{ - chain rule.}$$

partial derivatives of f

x and y are functions of one variable t , so we write $\frac{dx}{dt}$.

Way to remember chain rule:
(but this NOT mathematically correct)

They do NOT really cancel!

In every term you should get " $\frac{df}{dt}$ " if you pretend to cancel the notations

Geometrically, what does this $\frac{d}{dt}(f(x(t), y(t)))$ mean?

or $\frac{d}{dt} f(x(t), y(t), z(t))$

(will talk more
about geometric
meaning next class)

(True but not helpful: if you graph this function of t ,
then ~~the~~ its derivative is the slope of the tangent line
to that graph; but this is not related to
any geometry of the graph of $f(x, y)$
or $f(x, y, z) \leftarrow$ doesn't have a reasonable graph →)

It has the better meaning:

pretend that f is some function of the environment
(e.g. temperature)

And $(x(t), y(t))$ or $(x(t), y(t), z(t))$ are
coordinates of some creature travelling on the
plane or in space respectively.

Then $\frac{df}{dt}$ measures the rate of change of
 f ("temperature") as that the creature
experiences.

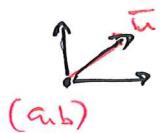
Directional derivatives :

For now, on \mathbb{R}^2 :

$$f(x, y)$$

We already defined $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Reasonable to define a derivative of f in any direction given by a unit vector \bar{u}



$$\underbrace{D_{\bar{u}} f}_{(a,b)} \stackrel{\text{def}}{=} \lim_{h \rightarrow 0}$$

$$\frac{f(a+h \cdot u_1, b+h \cdot u_2) - f(a, b)}{h}$$

directional derivative of f
in the direction
of \bar{u}
at (a, b)

$$\bar{u} = \langle u_1, u_2 \rangle$$

$$\begin{matrix} h \\ (a+h u_1, b+h u_2) \\ (a, b) \end{matrix}$$

To a point
at distance h
from (a, b)
in the direction \bar{u}

Meaning: ① imagine that the function f measures temperature of the plane at a given point.
 A bug is crawling on the plane with ~~speed~~ a velocity

but speed = 1
 $\|\vec{u}\|$

Then $D_{\vec{u}} f |_{(a,b)} =$ rate of change of temperature that the bug experiences

\rightarrow
 Only true if it crawls at speed = 1.

(Note: this agrees with the def. of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$
 if we take $\vec{u} = \langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$, respectively:

$$D_i f = \frac{\partial f}{\partial x} \quad D_j f = \frac{\partial f}{\partial y} \quad).$$

How to compute: we have chain rule.

Let the coordinates of the bug be linear functions:

$$x(t) = a + u_1 t, \quad y(t) = b + u_2 t \quad \leftarrow \begin{array}{l} \text{bug going} \\ \text{on a straight} \\ \text{line with} \\ \text{constant} \\ \text{velocity } \vec{u}. \end{array}$$

We know:

$$\frac{d}{dt} f(\underbrace{a + u_1 t, b + u_2 t}_{x'(t)=\vec{u}}) = \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2 \quad (x)$$

Upshot: to find $D_{\vec{u}} f$ for any \vec{u} ,
you just need to make sure \vec{u} is
a unit vector, and use this formula

(if you know $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, you know all
directional derivatives!)

(If the function is smooth enough).

Def Let $f(x, y)$ be a function of 2 variables.

$\nabla f = \left\langle \frac{\partial f}{\partial x} \Big|_{(a,b)}, \frac{\partial f}{\partial y} \Big|_{(a,b)} \right\rangle$ - the gradient vector
(depends on the point)
"nabla" ↑
at (a, b)

(∇f is a 2-vector
if f has 2 variables)

If f has 3 variables, get

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \text{ - a vector in } \mathbb{R}^3$$

If f has 50 variables, get

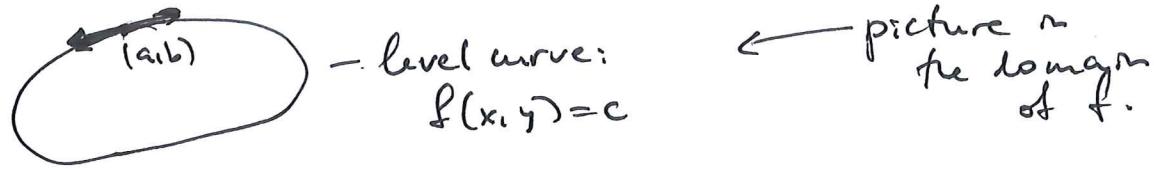
$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{50}} \right\rangle \text{ - vector in } \mathbb{R}^{50}.$$

Now can write (*) simply:

$$D_{\bar{u}} f = \nabla f \cdot \bar{u}$$
 dot product. remember this!

rate of change of f
in the direction of \bar{u}

Consequence: ∇f for $f(x,y)$



if \bar{u} is tangent to the level curve of f at (a,b)
then $D_{\bar{u}} f = 0$ (level curve means: f is NOT changing)

get: $\bar{\nabla} f \cdot \bar{u} = 0$

so: $\bar{\nabla} f$ is perpendicular to the tangent line
at (a,b) to the level curve.

Similarly, in 3-space:

$f(x,y,z)$ has level surfaces: $f(x,y,z) = c$

$\bar{\nabla} f$ is normal to the tangent plane at a given point

Use the example

From last time:

Worksheet 6: implicit differentiation, chain rule

1. Let z be an implicit function of x, y defined by:

$$x^2 + 3y^2 + 5z^2 = 58$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 2, 3)$.

here we answer a different question:
find the tangent plane
to this surface
at the point
 $(1, 2, 3)$

New way of finding the tangent plane:

let $F(x, y, z) = x^2 + 3y^2 + 5z^2$.

Then our surface is the level surface of F .

At $(1, 2, 3)$, the gradient of F is a normal vector
of the tangent plane.

$$\nabla F: \quad \frac{\partial F}{\partial x} = 2x \quad \frac{\partial F}{\partial z} = 10z \quad \text{At } (1, 2, 3) \quad \frac{\partial F}{\partial (1, 2, 3)} = \langle 2, 12, 30 \rangle$$

$\frac{\partial F}{\partial y} = 6y$

Area

Equation of the tangent plane:

$$2(x-1) + 12(y-2) + 30(z-3) = 0.$$

2. Let $F(x, y, z) = \cos(x)e^{3y} + z^2$. Let $x(t) = 5t^2$, $y(t) = 2t$, $z(t) = \sin(t)$.

Find

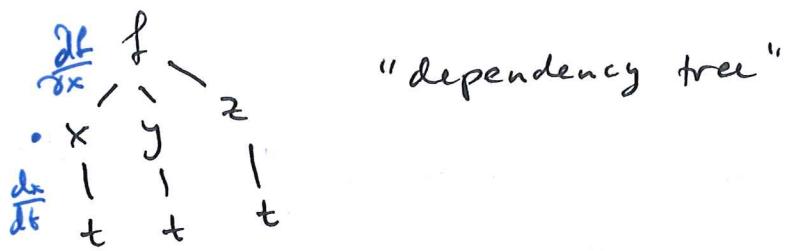
$$\frac{d}{dt} F(x(t), y(t), z(t)).$$

Can compare
it with last time.

See a note
on tangent planes
(separate link).

General chain rule:

We have so far:



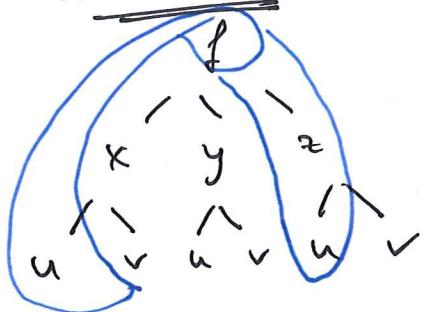
"dependency tree"

We have a rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

What is:

$$F(u, v) := f(x(u, v), y(u, v), z(u, v))$$



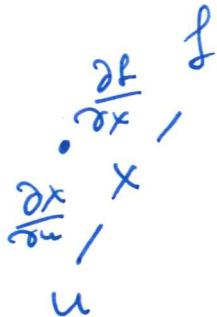
$$\frac{\partial F}{\partial u} = ?$$

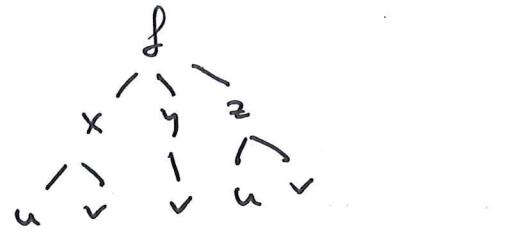
$$\frac{\partial F}{\partial v} = ?$$

Get: $\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \cdot \underline{\frac{\partial x}{\partial u}} + \frac{\partial f}{\partial y} \cdot \underline{\frac{\partial y}{\partial u}} + \frac{\partial f}{\partial z} \cdot \underline{\frac{\partial z}{\partial u}}$

To remember: look for paths from f to u.

add paths on each path, multiply the partials.





Worksheet 7: more on chain rule

1. Let $f(x, y, z) = x^3 + \ln(yz)$, let $x = u^2 + v^2$, $y = 5v$, $z = uv$. Find $\frac{\partial f}{\partial u}$ and evaluate it at $(u, v) = (1, 2)$.

$$\frac{\partial f}{\partial x} = 3x^2 \quad \frac{\partial x}{\partial u} = 2u \quad \frac{\partial y}{\partial u} = 0 \quad \frac{\partial z}{\partial u} = v$$

$$\frac{\partial f}{\partial y} = \frac{1}{y}$$

$$\frac{\partial f}{\partial u} = 3x^2 \cdot 2u + \left(\frac{1}{y} \cdot 0\right) + \frac{1}{z} \cdot v$$

$$\frac{\partial f}{\partial z} = \frac{1}{z}$$

only need
 $\frac{\partial x}{\partial u}$

$\frac{\partial x}{\partial u}$

Evaluate: $\frac{\partial x}{\partial u}|_{(1,2)} = 2 \quad \frac{\partial y}{\partial u}|_{(1,2)} = 0 \quad \frac{\partial z}{\partial u}|_{(1,2)} = 2$

Where to evaluate the partials of f ? see next page.

2. The wave equation in one space dimension is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where c is a constant (for electromagnetic waves, it is the speed of light).

Prove that any function of the form $u(x, t) = f(x - ct) + g(x + ct)$, where f and g are functions of a single variable, satisfies this equation.

We are plugging in $x(u,v)$, $y(u,v)$, $z(u,v)$ into f .
cannot evaluate f or $\frac{\partial f}{\partial x}$ at (u,v) directly!

when $u=1, v=2$

we compute x, y, z :

$$x = u^2 + v^2 = 5$$

$$y = 5v = 10$$

$$z = u \cdot v = 1 \cdot 2 = 2$$

So when $(u,v) = (1,2)$, $(x,y,z) = \boxed{(5, 10, 2)}$
we need to evaluate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ at this point.