Recall: Last time: linearization
(linear approximation) \( f(x,y) \)
- our main tool!

Today:
- "differentials" (error estimates)
  - Chain rule for functions of several variables.

Recall: \( f(x,y) \) - a function of \( x,y \) (continuously differentiable)
we write:
\[
L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)
\]
- linearization of \( f(x,y) \) at the point \((a,b)\).

The graph of \( L(x,y) \) is a plane (since \( L(x,y) \) is a linear function), and this plane is the tangent plane to the graph of \( f(x,y) \) at \((a,b)\).
Today, think of this as:

\[ \Delta x \text{ - small change in } x \quad \text{ (} x \text{ is close to the value } a ) \]

\[ \Delta x = x - a \]

\[ \Delta y \text{ - small change in } y \quad \text{ (} y \text{ is close to the value } b ) \]

What is the corresponding change in the value of \( f \): 

\[ \Delta f \approx L(x,y) - L(a,b) = f_x(a,b) \Delta x + f_y(a,b) \Delta y \]

we use the linearization to approximate \( f \).
Definition: \[ df = f_x(a,b) \cdot dx + f_y(a,b) \cdot dy \]

"the differential of \( f \) at \( (a,b) \)"

Think of this a way to remember how to compute a small change of \( f(x,y) \).

\[ \frac{df}{dx} \quad \frac{df}{dy} \]

Example: This works in any number of variables.

Take \( f = V(l, w, h) = l \cdot w \cdot h \) - function of 3 variables.

\[ l, w, h \]

Suppose we measured:

\[ l = 10 \text{ cm} \]
\[ w = 15 \text{ cm} \]
\[ h = 12 \text{ cm} \]

Error of each measurement \( \pm 1 \text{ mm} \).

(used a ruler), or measuring tape.

How big (approximately) is the resulting error of the volume?

(we get: \( V(10,15,12) = 10 \cdot 15 \cdot 12 \), but how precise is this?)
\[ \Delta V \approx 0.10 \text{ cm}^3 \]
\[ 1 \text{ cm}^3 \]
\[ 10 \text{ cm}^3 \]
\[ 100 \text{ cm}^3 \]
\[ 0.01 \text{ cm}^3 \]

<table>
<thead>
<tr>
<th>Volume</th>
<th>Votes</th>
</tr>
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<tbody>
<tr>
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<td>3</td>
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<tr>
<td>100</td>
<td>2</td>
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<tr>
<td>0.01</td>
<td>1</td>
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</tbody>
</table>

\[ \text{true answer: } 45 \text{ cm}^3. \]

\[ 1 \text{ mm} = 0.1 \text{ cm} \] — our error for \( l, w, h \).

\[ \Delta V \approx \frac{\partial V}{\partial l} \bigg|_{(l=10, w=15, h=12)} \Delta l \approx 0.1 \text{ cm} \]

\[ \Delta W \approx 0.1 \text{ cm} \]

\[ \Delta l \approx 0.1 \text{ cm} \]

\[ \Delta W \approx 0.1 \text{ cm} \]

\[ \Delta h \approx 0.1 \text{ cm} \]

\[ \Delta V \approx \frac{\partial V}{\partial w} \bigg|_{(l=10, w=15, h=12)} \Delta w \approx 0.1 \text{ cm} \]

\[ \Delta V \approx \frac{\partial V}{\partial h} \bigg|_{(l=10, w=15, h=12)} \Delta h \approx 0.1 \text{ cm} \]

\[ \Delta V \approx \frac{\partial V}{\partial l} \cdot \Delta l + \frac{\partial V}{\partial w} \cdot \Delta w + \frac{\partial V}{\partial h} \cdot \Delta h \]

\[ \Delta V = \frac{\partial V}{\partial l} \cdot \Delta l + \frac{\partial V}{\partial w} \cdot \Delta w + \frac{\partial V}{\partial h} \cdot \Delta h \]
\[ \frac{\partial V}{\partial l} = w \cdot h \]
\[ \frac{\partial V}{\partial w} = l \cdot h \]
\[ \frac{\partial V}{\partial h} = w \cdot l \]

Evaluate these derivatives at \((10, 15, 12)\):

\[ \frac{\partial V}{\partial l} \bigg|_{(10, 15, 12)} = 15.12 \text{ cm}^2 \]
\[ \frac{\partial V}{\partial w} \bigg|_{(10, 15, 12)} = 10.12 \text{ cm}^2 \]
\[ \frac{\partial V}{\partial h} \bigg|_{(10, 15, 12)} = 10.15 \text{ cm}^2 \]

Get: \( \Delta V \propto 0.1 \left( 15.12 + 10.12 + 10.15 \right) \text{ cm}^3 \)

\( \Delta V = 15.12 + 12 + 15 = 45 \text{ cm}^3 \)

\( \Delta A = l \cdot \Delta w + w \cdot \Delta l \)

Don't take this into account when using differentials.
Remark: For a function of 2 variables, $L(x,y)$ gives an equation of the tangent plane to the graph, but for 3 or more variables, we are not thinking about the graph!

For 3 variables, linearization itself is a function of 3 variables; both graph of $f(x,y,z)$ and graph of $L(x,y,z)$ live in $\mathbb{R}^4$; so the graph of $L(x,y,z)$ is "tangent 3-space" to the graph of $f$.

When we write $z = f(x,y)$, get a graph of $f(x,y)$ (for a function of 2 variables), but if we write $f(x,y,z)$, we are dealing with a function of 3 variables; here the role of $z$ is the same as the role of $x,y$.

Favourite examples: ① $f(x,y) =$ altitude above sea level at a point with geographic coordinates $x,y$.  

graph of $f(x,y) =$ mountain.

② Temperature (in this room) $T(x,y,z)$ — at every point there's some temperature!
Implicit differentiation

Recall from calc-1: implicit functions:

\[ f(x,y) = c \] fixed constant

This makes \( y \) an 'implicit function' of \( x \).

Example: \( y \) is given as an implicit function of \( x \) by:

\[ * \]  \( xy^2 + y^3 - x^2 = 10 \)

Find \( \frac{dy}{dx} \), \( x^2 - 4 \) \( \uparrow \) relation on \( x,y \).

Which makes \( y \) a 'function' of \( x \) (maybe doesn't satisfy vertical line test)

"Implicit" because we don't have a formula for it, so it's not explicit.

Still, we can find \( \frac{dy}{dx} \).

Differentiate both sides, but every time we see \( y \), treat it as a function of \( x \):

Get:

\[ 1 \cdot y^2 + x \cdot 2y \cdot y' + 3y^2 \cdot y' - 2xy - x^2 \cdot y' = 0 \]

Solve for \( \frac{dy}{dx} \):

\[ \frac{dy}{dx} = \frac{2xy - y^2}{3y^2 + 2xy - x^2} \]
If \((a, b)\) satisfies \((\phi)\), then \(\frac{dy}{dt}\) gives us the slope of the tangent line to this curve at this point.
Worksheet 6: implicit differentiation, chain rule

1. Let $z$ be an implicit function of $x, y$ defined by:
\[ x^3 + 3y^2 + 5z^2 = 58 \]

Differentiate both sides.

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 2, 3)$.

Note: Later we'll get a better way.

Do the partials as usual, but treat $z$ as a function of $x, y$.

(i.e. when doing $\frac{\partial z}{\partial x}$, treat $2$ as function of $y$ as constant).

Then solve for $\frac{\partial z}{\partial x}$.

\[
\frac{\partial}{\partial x}: \quad 2x + 10z \cdot \frac{\partial z}{\partial x} = 0
\]

\[
\frac{\partial}{\partial y}: \quad 6y + 10z \cdot \frac{\partial z}{\partial y} = 0
\]

Evaluate at $(1, 2, 3)$:

\[
\frac{\partial z}{\partial x} \bigg|_{(1,2,3)} = -\frac{2}{10} = -\frac{1}{5}
\]

\[
\frac{\partial z}{\partial y} \bigg|_{(1,2,3)} = -\frac{6}{10} = -\frac{3}{5}
\]

2. Let $F(x, y, z) = \cos(x)e^{3y} + z^2$. Let $x(t) = 5t^2, y(t) = 2t, z(t) = \sin(t)$.

Find:

\[
\frac{d}{dt} F(x(t), y(t), z(t)).
\]

\[
F_x = -\sin x \cdot e^{3y}
\]
\[
F_y = 3e^{3y} \cos x
\]
\[
F_z = 2z
\]

\[
\frac{dx}{dt} = 10t
\]
\[
\frac{dy}{dt} = 2
\]
\[
\frac{dz}{dt} = \cos t
\]

\[
\frac{dF}{dt} = (-\sin x \cdot e^{3y}) \cdot 10t + 3e^{3y} \cos x \cdot 2 + 2z \cdot \cos t
\]

Note: In what I am doing.
Upshot & Question!

F(x, y, z) - function of 3 variables.

If we set \( F(x, y, z) = c \)

(\( F(x, y, z) = 58 \) in our example)

get a level surface \( F(x, y, z) \)

(\( F(x, y, z) = 58 \) in our example, it is an ellipsoid)

If you want to stay on this ellipsoid, you can choose \( x, y \) so that determines \( z \)

(or could choose \( x, z \) so determines \( y \))

Choose one variable so it is an implicit function of the other two

(in our example, could solve for \( z \)

and make it explicit: \( z = \pm \sqrt{58 - x^2 - 3y^2} \))

In more complicated examples, cannot solve!

When we finish chain rule, we'll get a better way to do implicit differentiation!
Chain rule:
(recall: \( f(x) \) — function of one variable \( x \);

suddenly we make \( x \) a function of \( t \):

\[ X = x(t) \]

get \( f(x(t)) \)

\[
\frac{d}{dt} f(x(t)) = f'(x(t)) \cdot x'(t)
\]

Now: \( F(x, y, z) \) — a function of 3 (or any number)
of variables.

Let \( x = x(t) \)
\( y = y(t) \)
\( z = z(t) \)

become functions of a single variable \( t \).

Then \( F(x(t), y(t), z(t)) \) — function of \( t \)

How to find \( \frac{dF}{dt} \)?

(What does it mean?)

imagine that \( F \) is temperature at \((x, y, z)\).

Suppose there's a fly, at the time \( t \)
it is at \((x(t), y(t), z(t))\)

Then \( \frac{dF}{dt} \) = rate of change of the temperature that the fly
is experiencing.
\[
\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial t} \frac{dt}{dt}
\]

Chain rule (in the situation of 1 variable \( t \))

"Proof" and more general cases of chain rule - next class.

Please read CLP-\( \Delta \) !