Recall: Last time: linearization (linear approximation) of a function $f(x,y)$

- our main tool!

Today: • "differentials" (error estimates)

• Chain rule for functions of several variables.

Recall: $f(x,y)$ - a function of x,y (continuously differentiable)

we write:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- linearization of $f(x,y)$ at the point (a,b).

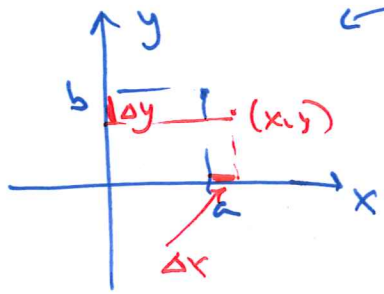
(the graph of $L(x,y)$ is a plane (since $L(x,y)$ is a linear function), and this plane is the tangent plane to the graph of $f(x,y)$ at (a,b)).

Today, think of this as:

Δx - small change in x (x is close to the value a)

$$\Delta x := x - a$$

Δy - small change in y (y is close to the value b)



← picture in the domain of $f(x,y)$

What is the corresponding change in the value of f:

$$\underbrace{f(x,y) - f(a,b)}_{\Delta f} \approx L(x,y) - \underset{\substack{\shortparallel \\ f(a,b)}}{L(a,b)} = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

use the linearization to approximate f.

Def: $df = f_x(a,b)\cdot \underbrace{\Delta x}_{\text{"}dx\text{"}} + f_y(a,b)\cdot \underbrace{\Delta y}_{\text{"}dy\text{"}}$

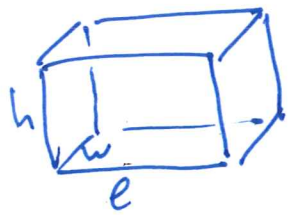
"the differential of f at (a,b)"

think of this a way to remember how to compute a small change of $f(x,y)$.

Example: This works in any number of variables.

Take $f = V(\ell, w, h) = \ell \cdot w \cdot h$ – Function of 3 variables.

sides of the box



Suppose we measured:

$\ell = 10$ cm

$w = 15$ cm

$h = 12$ cm

error of each measurement ≈ 1 mm.

(used a ruler).
or measuring tape

How big (approximately) is the resulting error of the volume?

(we get: $V(10,15,12) = 10\cdot 15\cdot 12$, but how precise is this?)

Vote: ΔV = the error!

$\Delta V \approx$

	votes
$0.1\ cm^3$	3
$1\ cm^3$	4
$10\ cm^3$	10
$100\ cm^3$	2
$0.01\ cm^3$	1

true answer: $45\ cm^3$.

$\boxed{1\ mm = 0.1\ cm}$ — our error for l, w, h.

$$\Delta V \approx \left.\frac{\partial V}{\partial l}\right|_{(10,15,12)} \overbrace{(l-10)}^{\Delta l \approx 0.1cm} + \left.\frac{\partial V}{\partial w}\right|_{(10,15,12)} \overbrace{(w-15)}^{\Delta w \approx 0.1cm}$$

$$+ \left.\frac{\partial V}{\partial h}\right|_{(10,15,12)} \cdot \overbrace{(h-12)}^{\Delta h \approx 0.1cm}$$

$$\boxed{\text{"}\ dV = \frac{\partial V}{\partial l} \cdot dl + \frac{\partial V}{\partial w} \cdot dw + \frac{\partial V}{\partial h} \cdot dh\ \text{"}}$$

$$\frac{\partial V}{\partial \ell} = w \cdot h$$

$$\frac{\partial V}{\partial w} = \ell \cdot h$$

$$\frac{\partial V}{\partial h} = w \cdot \ell$$

Evaluate these derivatives at $(\overset{\ell}{10}, \overset{w}{15}, \overset{h}{12})$:

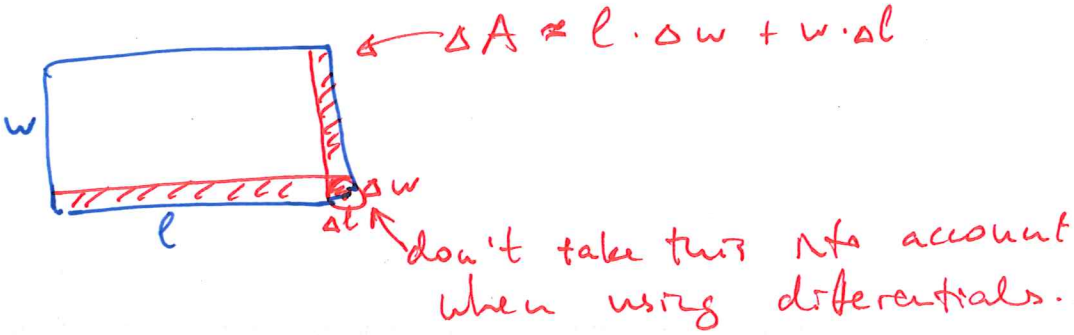
$$\left.\frac{\partial V}{\partial \ell}\right|_{(10,15,12)} = 15 \cdot 12 \text{ cm}^2$$

$$\left.\frac{\partial V}{\partial w}\right|_{(\ \ )} = 10 \cdot 12 \text{ cm}^2$$

$$\left.\frac{\partial V}{\partial h}\right|_{(\ \ )} = 10 \cdot 15 \text{ cm}^2$$

<u>Get</u>: $\Delta V \approx \underbrace{0.1}_{\substack{\Delta\ell = \Delta w = \Delta h \\ \text{(in cm)}}} (15 \cdot 12 + 10 \cdot 12 + 10 \cdot 15) \text{ cm}^3$

$$= 1.5 \cdot 12 + 12 + 15 = 18 + 12 + 15 = \boxed{45 \text{ cm}^3}$$



Remark: For a function of 2 variables, $L(x,y)$ gives an equation of the tangent plane to the graph;
linearization

but for 3 or more variables, we are not thinking about the graph!

(For 3 variables, linearization itself is a function of 3 variables; both graph of $f(x,y,z)$ and graph of $L(x,y,z)$ live in $\mathbb{R}^4$;

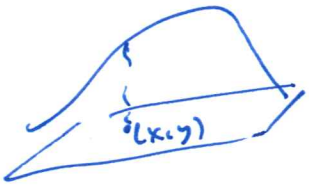
so the graph of $L(x,y,z)$ is "tangent 3-space" to the graph of f in $\mathbb{R}^4$")

When we write $z = f(x,y)$ ← get a graph of $f(x,y)$ (for a function of 2 variables)

here z is special (not a variable)

but if we write $f(x,y,z)$, we are dealing with a function of 3 variables; here the role of z is the same as the role of x, y.

Favourite examples: ① $f(x,y)$ = altitude above sea level at the point with geographic coordinates x, y.



graph of $f(x,y)$ = mountain.

② Temperature (in this room) $T(x,y,z)$ — at every point there's some temperature!

Implicit differentiation

Recall from Calc-1: implicit functions:

$$f(x,y) = c \leftarrow \text{fixed constant}$$

this makes y an 'implicit function' of x.

Example: y is given as an implicit function of x by:

(*) $xy^2 + y^3 - x^2y = 10$ ← Note: this is a level curve for $f(x,y) = xy^2+y^3-x^2y$.

Find $\frac{dy}{dx}$,

relation on x, y. Which makes y a 'function' of x (maybe doesn't satisfy vertical line test)

"implicit" because we don't have a formula for it, so it's not explicit.

Still, we can find $\frac{dy}{dx}$.

Differentiate both sides, but every time we see y, treat it as a function of x:

Get:

$$1\cdot y^2 + x\cdot 2y\cdot \underbrace{y'}_{\frac{dy}{dx}} + 3y^2\cdot y' - 2xy - x^2\cdot y' = 0$$

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{2xy - y^2}{3y^2 + 2xy - x^2}$

If (a,b) satisfies $(*)$, then $\frac{dy}{dx}$ gives us the slope of the tangent line to this curve at this point.

has to match our point: (1,2,3) lies on the level surface $F(x,y,z) = 58$

Worksheet 6: implicit differentiation, chain rule

1. Let z be an implicit function of x, y defined by:

$$x^2 + 3y^2 + 5z^2 = 58$$

differentiate both sides.

. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 2, 3)$. Note: later we'll get a better way :)

Do the partials as usual, but treat z as a function of x, y.
(ie. when doing $\frac{\partial z}{\partial x}$, treat z as function but y as constant).

Then solve for $\frac{\partial z}{\partial x}$.

$\frac{\partial}{\partial x}$: $\quad 2x + 10z \cdot \frac{\partial z}{\partial x} = 0$

$\frac{\partial}{\partial y}$: $\quad 6y + 10z \frac{\partial z}{\partial y} = 0$

note on what I am doing.

$$\boxed{\frac{\partial z}{\partial x} = -\frac{2x}{10z} \qquad \frac{\partial z}{\partial y} = -\frac{6y}{10z}}$$

evaluate at $(1,2,3)$:

$\left.\frac{\partial z}{\partial x}\right|_{(1,2,3)} = -\frac{2}{30}$

$\frac{\partial z}{\partial y} = \frac{-12}{30}$

2. Let $F(x, y, z) = \cos(x)e^{3y} + z^2$. Let $x(t) = 5t^2$, $y(t) = 2t$, $z(t) = \sin(t)$. Find

$$\frac{d}{dt}F(x(t), y(t), z(t)).$$

$F_x = -\sin x \cdot e^{3y}$

$F_y = 3e^{3y}\cos x$

$F_z = 2z$

$\frac{dx}{dt} = 10t$

$\frac{dy}{dt} = 2$

$\frac{dz}{dt} = \cos t$

$$\frac{dF}{dt} = \left(-\sin x \cdot e^{3y}\right) \cdot 10t + 3e^{3y}\cos x \cdot 2 + 2z \cdot \cos t$$

Upshot of Question 1:

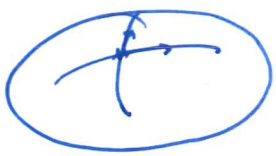
$F(x,y,z)$ – function of 3 variables.

If we set $F(x,y,z) = c$

(in our example, $F(x,y,z) = 58$)

get a level surface of $F(x,y,z)$

(in our example, it is an ellipsoid)



If you want to stay on this ellipsoid, you can choose $x, y \rightsquigarrow$ that determines z

(or could choose $x, z \rightsquigarrow$ determines y)

Choose one variable $\rightsquigarrow$ it is an implicit function of the other two

(in our example, could solve for z and make it explicit: $z = \pm\sqrt{\frac{58 - x^2 - 3y^2}{5}}$)

In more complicated examples, cannot solve!

When we finish chain rule, we'll get a better way to do implicit differentiation!

Chain rule:

(recall: $f(x)$ – function of one variable x;
suddenly we ~~[illegible]~~ make x a function of t:

$$x = x(t)$$

get $f(x(t))$

$$\frac{d}{dt} f(x(t)) = f'(x(t)) \cdot x'(t)$$
$$= \underline{\frac{df}{dx} \cdot \frac{dx}{dt}}$$

Now: $F(x, y, z)$ – a function of 3 (or any number) of variables.

Let $x = x(t)$
$y = y(t)$
$z = z(t)$ become functions of a single variable t.

Then $F(x(t), y(t), z(t))$ – <u>function of t</u>

How to find $\frac{dF}{dt}$?

(What does it mean?
imagine that F is temperature at (x, y, z).
~~[illegible]~~ Suppose there's a fly, at the time t
it is at $(x(t), y(t), z(t))$

Then $\frac{dF}{dt}$ = rate of change of the temperature that the fly is experiencing.

Answer:

$$\underbrace{\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt}}_{\text{chain rule (in the situation of 1 variable } t)}$$

"Proof" and more general cases of chain rule – next class.
(explanation)

Please read CLP-VI !