Worksheet 5: partial derivatives, tangent planes

1. Let \( f(x, y, z) = x^2 \cos(z)e^{x^2+3y} = f(x, y, z) \)

(a) Find \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \).

\[
\frac{\partial f}{\partial x} = 2x \cos(z)e^{x^2+3y} + x^2 \cos(z) \cdot \frac{2x}{x} (e^{x^2+3y}) = e^{x^2+3y} \cdot 2x \cos(z) + x^2 \cos(z) \cdot e^{x^2+3y} \cdot 2x
\]

(b) Find \( \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial y \partial z} \).

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left( x^2 \cos(z) \cdot e^{x^2+3y} \right) = \frac{d}{dx} \left( \frac{\partial f}{\partial y} \right)
\]

"Second order partial derivative of \( f \) with respect to \( y, z \)."

\[
\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial z} \right) = \frac{d}{dy} \left( \frac{\partial f}{\partial z} \right)
\]

\[
\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial y} \right) = \frac{d}{dz} \left( \frac{\partial f}{\partial y} \right)
\]

See next page for the solution.
Worksheet 5: partial derivatives, tangent planes

1. Let \( f(x, y) = x^2 \cos(x)e^{x^2+3y} \).
   
   (a) Find \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \).

   
   (b) Find \( \frac{\partial^2 f}{\partial y \partial y}, \frac{\partial^2 f}{\partial z \partial y} \) and \( \frac{\partial^2 f}{\partial y \partial z} \).

\[
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( x^2 \cos x \cdot 3e^{x^2+3y} \right) = 3x^2 \cos x \cdot \frac{\partial}{\partial y} (e^{x^2+3y}) \\
= 3x^2 \cos x \cdot e^{x^2+3y} \cdot 3 \\
\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial z} \left( 3x^2 \cos x \cdot e^{x^2+3y} \right) \\
= 3x^2 e^{x^2+3y} \cdot (-\sin z) \\
\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y} \left( -\sin z \cdot x^2 e^{x^2+3y} \right) = -x^2 \sin z \cdot e^{x^2+3y} \cdot 3 \\
\text{The same!}
\]
Note: $f(x,y,z)$ will have 2nd order partial derivatives.

e.g. $\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$

**Claimant Theorem** If all second order partials of $f$ are continuous functions in a neighbourhood of a point in the domain, then the order doesn't matter:

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$

Operator:

Take whatever comes here and differentiate it with respect to $x$.

Note: $\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y}$ is a different thing - it is a product of two first-order partial derivatives, but NOT a second-order partial!
2. Can we find a function $f(x,y)$ such that $f_x(x,y) = x^3y + y^2 - x$, and $f_y(x,y) = x^4 + 2xy$? What if we had $f_x(x,y) = 4x^3y + y^2 - x$ and $f_y(x,y) = x^4 + 2xy$?

Looking for $f(x,y)$: \[
\begin{align*}
    f_x(x,y) &= x^3y + y^2 - x & \text{continuous} \\
    f_y(x,y) &= x^4 + 2xy
\end{align*}
\]

Try: integrate $f_x$ with respect to $x$ (treat $y$ as constant):

get: $f(x,y) = \frac{1}{4}x^4y + \frac{1}{2}xy^2 - \frac{x^2}{2} + c(y)$

Try to see if we can make $f_y(x,y)$ match ($\ast$)

Differentiate my $f(x,y)$ with respect to $y$:

$f_y(x,y) = \frac{1}{4}x^4 + 2xy + c'(y)$

Doesn't match!

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3. Find an equation of the tangent plane to the graph of the function $f(x,y) = x^2 - 3xy$ at the point $(1,2,-5)$.

\[\begin{align*}
    a &= 1 \\
    b &= 2 \\
    f(a,b) &= 1^2 - 3 \cdot 1 \cdot 2 = -5
\end{align*}\]

Linear approximation: need $f_x, f_y$ at $(a,b)$:

\[\begin{align*}
    f_x &= 2x - 3y \\
    f_y &= -3x
\end{align*}\]

\[\begin{align*}
    f_x(1,2) &= 2 - 3 \cdot 2 = -4 \\
    f_y(1,2) &= -3
\end{align*}\]

Answer: $z = -5 - 4(x-1) - 3(y-2)$
Looks like $f(x,y)$ doesn't exist!

Can we prove it?

Suppose such a function existed. Then because $f_x$, $f_y$ are continuous, Clairaut's theorem would apply. So we must have $f_{xy} = f_{yx}$:

Check: $f_x = x^3y + y^2 - x$ then

$$f_{xy} = \frac{\partial}{\partial y} (x^3y + y^2 - x) = x^3 + 2y$$

On the other hand,

$$f_{yx} = \frac{\partial}{\partial x} (f_y) = \frac{\partial}{\partial x} (x^4 + 2xy)$$

$$= 4x^3 + 2y$$

Then such $f$ cannot exist (if Clairaut's theorem is true)

Continued on the next page.
Second part of (2):

now we want: \[
\begin{align*}
    f_x &= 4x^3y + y^2 - x \\
    f_y &= x^4 + 2xy.
\end{align*}
\]

Start with integrating \( f_y \) with respect to \( y \):

\[
f(x,y) = x^4y + x^2y^2 + c(x)
\]

"constant that can depend on \( x \)"

Try to match \( f_x \):

\[
f_x = 4x^3y + y^2 + c'(x)
\]

Want: \( c'(x) = -x \)

So \( c(x) = -\frac{x^2}{2} + C \)

Answer: \[
f(x,y) = x^4y + x^2y^2 - \frac{x^2}{2} + C
\]

Upshot: If I have \( h(x,y) \) and \( g(x,y) \) — two functions

Want: find some \( f(x,y) \) such that

\[
\begin{align*}
    f_x (x,y) &= h(x,y) \\
    f_y (x,y) &= g(x,y)
\end{align*}
\]

then have a chance only if \[
\frac{\partial h}{\partial y} = \frac{\partial g}{\partial x}.
\]
Tangent planes and linear approximations

$f(x_1, \ldots, x_n)$ - function of $n$ variables

(can it be differentiable?)

can approximate it near a given point by a linear function of all these variables.

Recall: 1 variable: $f(x)$

near a point $a$:

$f(x) \approx f(a) + f'(a)(x-a)$

tangent line = graph of the linear approximation of $f(x)$:

$y = f(a) + f'(a)(x-a)$

equation of the tangent line
to the graph of $f(x)$ at $(a, f(a))$
In several variables

\[ f(x_1, \ldots, x_n) \]

at \((a_1, \ldots, a_n)\) — point in the domain

\[ f(x_1, \ldots, x_n) \approx f(a_1, \ldots, a_n) + \frac{\partial f}{\partial x_1}(a_1, \ldots, a_n)(x_1-a_1) + \frac{\partial f}{\partial x_2}(a_1, \ldots, a_n)(x_2-a_2) + \cdots + \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n)(x_n-a_n) \]

linear approximation at \((a_1, \ldots, a_n)\)

For 2 variables:

\[ f(x, y) \] at \((a, b)\):

\[ f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x-a) + \frac{\partial f}{\partial y}(a, b)(y-b) \]

\[ L(x, y) \] (linear function)

(linear approximation of \(f(x, y)\) at \((a, b)\))

\[ z = L(x, y) \] — get a graph of the linear approximation of \(f(x, y)\)

— it is the tangent plane to the graph of \(f(x, y)\) at the point \((a, b, f(a, b))\)

point on the graph of \(f\).