Today's topics: • higher-order partial derivatives,
• linear approximations and tangent planes

Thursday Feb. 6 Lecture 10

Worksheet 5: partial derivatives, tangent planes

1. Let $f(x,y,z) = x^2\cos(z)e^{x^2+3y}$. $= f(x,y,z)$

(a) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$.

product rule

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2\cos z \cdot e^{x^2+3y}\right) = e^{x^2+3y}\cdot\frac{\partial}{\partial x}(x^2\cos z) + x^2\cos z\,\frac{\partial}{\partial x}(e^{x^2+3y})$$

treat y, z as constants.

$$= e^{x^2+3y}\cdot 2x\cos z + x^2\cos z\cdot e^{x^2+3y}\cdot 2x$$

chain rule, x, z are "constant".

$$\boxed{\frac{\partial f}{\partial y} = x^2\cos z\, e^{x^2+3y}\cdot 3}$$

$$\frac{\partial f}{\partial z} = \boxed{x^2\cdot(-\sin z)\cdot e^{x^2+3y}}$$

(b) Find $\frac{\partial^2 f}{\partial^2 y}$, $\frac{\partial^2 f}{\partial z\partial y}$ and $\frac{\partial^2 f}{\partial y\partial z}$.

$$\frac{\partial^2 f}{\partial^2 y} \overset{\text{def}}{=} f_{yy} \overset{\text{def}}{=} \frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial f}{\partial y}}_{\text{function of } x,y,z}\Big)$$

$$\| $$

how many differentiations total

$$\frac{\partial^2 f}{\partial z\partial y} \overset{\text{def}}{=} f_{yz} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)$$

"second order partial derivative of f with respect to y, z".

See next page for the solution.

Worksheet 5: partial derivatives, tangent planes

1. Let $f(x,y) = x^2\cos(z)e^{x^2+3y}$.

   (a) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$.

   (b) Find $\frac{\partial^2 f}{\partial^2 y}$, $\frac{\partial^2 f}{\partial z \partial y}$ and $\frac{\partial^2 f}{\partial y \partial z}$.

$$\frac{\partial^2 f}{\partial^2 y} = \frac{\partial}{\partial y}\Big(\underbrace{x^2\cos z \cdot 3e^{x^2+3y}}_{\text{from (a)}}\Big) = 3x^2\cos z \cdot \frac{\partial}{\partial y}\left(e^{x^2+3y}\right)$$

$$= 3x^2\cos z \cdot e^{x^2+3y} \cdot 3$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\Big(\underbrace{3x^2\cos z \cdot e^{x^2+3y}}_{\text{from (a) as above}}\Big)$$

$$= \underbrace{3x^2 e^{x^2+3y} \cdot (-\sin z)}$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\Big(\underbrace{-\sin z\, x^2 e^{x^2+3y}}_{\text{from (a)}}\Big) = \underbrace{-x^2\sin z \cdot e^{x^2+3y} \cdot 3}_{\text{The same!}}$$

Note: $f(x,y,z)$ will have $\underline{9}$ 2nd order partial derivatives

e.g. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ - - -

Clairaut Theorem If all second order partials of f are continuous functions in a neighbourhood of a point the domain, then the order doesn't matter:

$$\underbrace{\frac{\partial}{\partial x}}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

operator:
take whatever comes here and differentiate it with respect to x

[Note: $\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y}$ is a different thing - it is a product of two first-order partial derivatives, but NOT a second-order partial!]

2. Can we find a function $f(x,y)$ such that $f_x(x,y) = x^3y + y^2 - x$, and $f_y(x,y) = x^4 + 2xy$? What if we had $f_x(x,y) = 4x^3y + y^2 - x$ and $f_y(x,y) = x^4 + 2xy$?

. Looking for $f(x,y)$: $\begin{cases} f_x(x,y) = x^3y + y^2 - x \\ f_y(x,y) = x^4 + 2xy \quad (*) \end{cases}$ ← continuous

Try: integrate f_x with respect to x (treat y as constant):

get: $f(x,y) = \frac{1}{4}x^4y + xy^2 - \frac{x^2}{2} + c(y)$

Try to see if we can make $f_y(x,y)$ match $(*)$

Differentiate my $f(x,y)$ with respect to y:

$f_y(x,y) = \frac{1}{4}x^4 + 2xy + c'(y)$

can it match?

Doesn't match!

→ continued on the next page

3. Find an equation of the tangent plane to the graph of the function $f(x,y) = x^2 - 3xy$ at the point $(1, 2, -5)$.

$a = 1$

$b = 2$

$f(a,b) = 1^2 - 3 \cdot 1 \cdot 2 = -5$ ← our point is on the graph.

Linear approximation: need f_x, f_y at (a,b):

$f_x = 2x - 3y$ $\qquad f_x(1,2) = 2 - 3 \cdot 2 = -4$

$f_y = -3x$ $\qquad f_y(1,2) = -3$

Answer: $z = -5 - 4(x-1) - 3(y-2)$

(2) continued:

Looks like $f(x,y)$ doesn't exist!

Can we prove it?

Suppose such a function existed.
Then because f_x, f_y are continuous, Clairaut's theorem would apply.

So we must have $f_{xy} = f_{yx}$:

Check: $f_x = x^3y + y^2 - x$ then

$$f_{xy} = \frac{\partial}{\partial y}(x^3y + y^2 - x) = x^3 + 2y$$

On the other hand,

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(x^4 + 2xy)$$

$$= 4x^3 + 2y$$

Then such f cannot exist
(if Clairaut's Theorem is true)

continued on the next page.

Second part of (2):

now we want: $\begin{cases} f_x = 4x^3y + y^2 - x \\ f_y = x^4 + 2xy. \end{cases}$

Start with integrating f_y with respect to y:

$$f(x,y) = \underbrace{x^4 \cdot y + xy^2}_{} + c(x)$$

"constant that can depend on x"

Try to match f_x:

$$f_x = 4x^3y + y^2 + c'(x)$$

Want: $c'(x) = -x$

So $c(x) = -\frac{x^2}{2} + C$ ← constant.

Answer: $f(x,y) = \underbrace{x^4 y + yx^2}_{} - \frac{x^2}{2} + C$

Upshot: If I have $h(x,y)$ and $g(x,y)$ – two continuous functions

Want: find some $f(x,y)$ such that

$$f_x(x,y) = h(x,y)$$
$$f_y(x,y) = g(x,y)$$

Then have a chance only if $\boxed{\frac{\partial h}{\partial y} = \frac{\partial g}{\partial x}.}$

Tangent planes and linear approximations

$f(x_1, \dots, x_n)$ - function of $\underline{n}$ variables

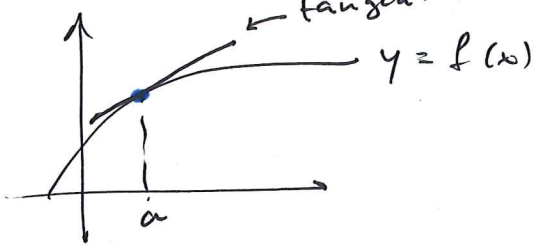
(if it is differentiable)

can approximate it near a given point by a linear function of all these variables.

Recall: 1 variable: $f(x)$

near a point a:

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}$$



tangent line = graph of the linear approximation of $f(x)$:

$$\underbrace{y = f(a) + f'(a)(x-a)}_{\text{equation of the tangent line to the graph of } f(x) \text{ at } (a, f(a))}$$

- In several variables

$f(x_1, \dots x_n)$

at $(a_1, \dots, a_n)$ – point in the domain

$$f(x_1, \dots x_n) \approx \underbrace{f(a_1, \dots, a_n) + \left.\frac{\partial f}{\partial x_1}\right|_{(a_1, \dots, a_n)} (x_1 - a_1) + \left.\frac{\partial f}{\partial x_2}\right|_{(a_1, \dots, a_n)} (x_2 - a_2) + \dots + \left.\frac{\partial f}{\partial x_n}\right|_{(a_1, \dots, a_n)} (x_n - a_n)}_{\text{linear approximation at } (a_1, \dots, a_n)}$$

For 2 variables:

$f(x,y)$ at (a,b):

$$f(x,y) \approx \underbrace{f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)} (y-b)}_{L(x,y)}$$

$L(x,y)$ = linear function (linear approximation of $f(x,y)$ at (a,b))

$z = L(x,y)$ – get a graph of the linear approximation of $f(x,y)$

– it is the tangent plane to the graph of $f(x,y)$ at the point $(\underbrace{a, b, f(a,b)}_{\text{point on the graph of } f.})$