1 A brief summary of sheaf cohomology

1.1 How does one make a (sheaf) cohomology

- We like functors applied to exact sequences.
- Functors not always exact, cohomology tries to fix this.

One defines cohomology groups as

- Derived functors in a category with enough injectives.
- Cocycles modulo coboundaries in cochain complexes.

In Sheaf cohomology:

- The functor of interest is Γ, the global section functor.
- There are enough injectives.
- The cochain complex given by the Cech complex computes the cohomology groups for smooth schemes.

1.2 What can a sheaf cohomology (not) do?

It can:

– compute Euler characteristics

– Study vector bundles of $X(\mathbb{C})$.

But it cannot:

– compute Betti numbers as $\dim_{\mathbb{Q}} H^k(X, \mathbb{Q})$.

– Study fibre bundles

Example 1.1. The $(\mathbb{Z}/5)$-bundle defined by

$$0 \longrightarrow \frac{\mathbb{Z}}{5} \longrightarrow E \longrightarrow F \longrightarrow 0$$

2 Étale morphisms

2.1 Étale morphisms of varieties

Moral: maps of schemes satisfying the inverse mapping property.

Example 2.1.

$$\begin{aligned} f^{\#} \colon \mathbb{A}^1_{\mathbb{C}} &\to \mathbb{A}^1_{\mathbb{C}} \\ \lambda &\mapsto \lambda^2 \end{aligned}$$

for any $\lambda \neq 0$.

Definition 2.2 (Tangent cone)**.** Let $X := \operatorname{Spec} k[x_1, \ldots, x_n]/I$ be an affine variety over an algebraically closed field. The *tangent cone* at $p \in X(k)$ is defined as

$$\operatorname{Spec} k[x_1, \ldots, x_n]/I_*, \quad I_* = \langle f_* : f \in I \rangle$$

where f_* denotes the homogeneous part of f (with respect to $\mathfrak{m}_p$) of lowest degree.

Example 2.3. Let $f(x, y) = xy + x^3 + y^3$ and let

$$E \colon Z(f) \subseteq \mathbb{A}^2_k.$$

The tangent cone of E at $P := (0, 0)$ is

$$C_P(E) \colon Z(xy) \subseteq \mathbb{A}^2_k$$

Proposition 2.4. *Let X be a variety over an algebraically closed field. Then the tangent cone to $p \in X(k)$ is given by*

$$C_P(X) = \operatorname{Spec} \operatorname{gr}(\mathcal{O}_{X,p}) := \operatorname{Spec} \bigoplus_n \mathfrak{m}_p^n / \mathfrak{m}_p^{n+1}$$

with $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{X,p}$.

Example 2.5. Let $f(x,y) = xy + x^3 + y^3$ and let p and E be as before. Then

$$\begin{aligned}
\mathfrak{m}_p &= \langle x, y \rangle \\
\mathfrak{m}_p^2 &= \langle x^2, xy, y^2 \rangle \\
\mathfrak{m}_p^3 &= \langle x^3, x^2y, xy^2, y^3 \rangle = \langle x^3, x^2y, xy^2, y^3, xy \rangle \\
\mathfrak{m}_p^4 &= \langle x^4, \ldots, y^4, xy \cdot x, xy \cdot y \rangle
\end{aligned}$$

We see that

$$\operatorname{gr} \mathcal{O}_{E,p} = k[x,y]/\langle xy \rangle.$$

Corollary 2.6. *Let $f \colon Y \to X$ be a morphism. Then there is an induced covariant morphism of tangent cones.*

Definition 2.7 (étale)**.** A morphism of k-varieties $f \colon Y \to X$ is *étale* if the induced morphism of tangent cones at every point is an isomorphism.

Example 2.8.

$$f \colon k[x] \xrightarrow{x \mapsto t^2} k[t]$$

$$f(\mathfrak{m}_{\lambda^2}) = \begin{cases} \mathfrak{m}_\lambda & \text{if } \lambda \neq 0 \\ \mathfrak{m}_\lambda^2 & \text{otherwise} \end{cases}$$

It turns out f is étale everywhere except 0.

2.2 A review of some commutative algebra

Definition 2.9 (unramified)**.** A morphism of local rings $f \colon A \to B$ is *unramified* if

(a) $f(\mathfrak{m}_A)B = \mathfrak{m}_B$ and

(b) the field $B/\mathfrak{m}_B$ is a finite and separable extension of $A/\mathfrak{m}_A$.

A morphism of schemes $f \colon Y \to X$ is unramified if it is of finite type and if each morphism of local rings $f^* \colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is unramified.

Definition 2.10 (flat)**.** A morphism of rings $f \colon A \to B$ is *flat* if the functor

$$M \mapsto M \otimes_A B$$

is (left) exact.

If this functor is also faithful (injective on morphisms) we say that f is faithfully flat.

A morphism of schemes $f \colon Y \to X$ is *flat* if each morphism of local rings $f^* \colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat. The morphism f is *faithfully flat* if it is surjective.

Remark 2.11. For modules, faithfully flat is equivalent to saying

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow L \otimes_A B \longrightarrow M \otimes_A B \longrightarrow N \otimes_A B \longrightarrow 0$$

is exact.

2.3 Étale morphisms of schemes

Definition 2.12 (étale)**.** A morphism of schemes $f \colon Y \to X$ is *étale* if it is flat and unramified.

Example 2.13. Let k be a number field and L/k a finite extension. Then

$$f \colon \operatorname{Spec} L \to \operatorname{Spec} k$$

is étale, even if L/k is ramified as a field extension. However,

$$f \colon \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_k$$

is not necessarily étale, but still fppf.

Proposition 2.14 (Hartshorne, ex III.10.3)**.** *Let $f\colon Y \to X$ be a morphism of non-singular schemes. Then the following are equivalent:*

(a) *f is étale.*

(b) *f is smooth of relative dimension 0.*

(c) *f is flat and $\Omega_{Y/X} = 0$.*

Proof. (b) $\Leftrightarrow$ (c) is immediate from definitions and the fact that $\Omega_{Y/X}$ is locally free of rank 0 on Y if it is zero. For (b/c) $\Rightarrow$ (a), note that

$$f^*\Omega_X \longrightarrow \Omega_Y \longrightarrow \Omega_{Y/X} \longrightarrow 0$$

By the hypothesis on $\Omega_{Y/X}$, We have $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2$. In particular, $f^*(\mathfrak{m}_{f(p)})$ contains the generators for $\mathfrak{m}_p$. That f is finite follows from the hypothesis of (b).
For $(a) \Rightarrow (b/c)$, we have an isomorphism of tangent spaces at each point, so looking at

$$f^*\Omega_X \longrightarrow \Omega_Y \longrightarrow \Omega_{Y/X} \longrightarrow 0$$

we see that the cokernel must be trivial. Furthermore, flatness of f is automatic so we have verified (c). □

Example 2.15 (A non-etale cover of curves)**.** Let

$$E\colon y^2 = x(x-1)(x-\lambda)$$

and define

$$\begin{aligned} \pi\colon E &\to \mathbb{P}^1 \\ (x,y) &\mapsto x. \end{aligned}$$

We see that

$$\begin{array}{ccccccc} \pi^*\Omega_{\mathbb{P}^1} & \longrightarrow & \Omega_E & \longrightarrow & \Omega_{E/\mathbb{P}^1} & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ \mathcal{O}_E(-\operatorname{div} y) & \longrightarrow & \mathcal{O}_E & \longrightarrow & \mathcal{O}_{(0,0)} \oplus \ldots \oplus \mathcal{O}_\infty^{\otimes(-3)} & \longrightarrow & 0 \end{array}$$

Note that the points where π is not étale are recorded by $\Omega_{E/\mathbb{P}^1}$.

Corollary 2.16. *Let $f\colon Y \to X$ be an étale morphism of varieties. Then it is also an étale morphism of schemes.*

2.4 Properties of an étale morphism

Proposition 2.17. *(a) Every open immersion is étale.*

(b) The composition of two étale morphisms is étale.

(c) Every base change of an étale morphism is étale.

(d) If $\varphi \circ \psi$ and φ are étale, then so is ψ.

Proposition 2.18. *Let $\varphi\colon Y \to X$ be étale. Then*

(a) For all $y \in Y, x = \varphi(y)$, we have $\mathcal{O}_{Y,y}, \mathcal{O}_{X,x}$ have the same Krull dimension.

(b) φ is quasi-finite.

(c) φ is open.

(d) If X is reduced, so is Y.

(e) If X is normal, so is Y.

(f) If X is regular, so is Y.

Example 2.19 (Standard étale morphism)**.** Let A be a ring, let $f(T) \in A[T]$ be a monic polynomial, and let $b \in A[T]/f$ be an element such that

$$f'(T) \in ((A[T]/f)_b)^\times .$$

As f is monic, $A[T]/f$ is a free, and hence flat, A module. By the choice of b, the morphism

$$\operatorname{Spec}(A[T]/f)_b \to \operatorname{Spec} A$$

is unramified and thus étale. Any morphism of this type is a *standard étale morphism*. Locally, every étale morphism is a standard étale morphism.

2.5 Things for next time

Proposition 2.20 (2.15)**.** *Let $\varphi\colon Y \to X$ be an étale morphism of varieties. If X is connected, then every section to φ is an isomorphism of X onto a connected component of Y.*

Definition 2.21 (étale neighbourhood)**.** An *étale neighbourhood* of $x \in X$ is an étale morphism $i\colon U \to X$ together with a point $u \in U$ such that $i(u) = x$. A morphism of étale neighbourhoods is a morphism of schemes over X respecting the marked point.

Corollary 2.22 (2.16)**.** *Let $\varphi\colon (V, v) \to (U, u)$ be a morphism of étale neighbourhoods of $x \in X$. Then φ is unique.*