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For any $k \geq 1$, we study the distribution of the difference between the number of integers $n \leq x$ with $\omega(n) = k$ or $\Omega(n) = k$ in two different arithmetic progressions, where $\omega(n)$ is the number of distinct prime factors of n and $\Omega(n)$ is the number of prime factors of n counted with multiplicity. Under some reasonable assumptions, we show that, if k is odd, the integers with $\Omega(n) = k$ have preference for quadratic nonresidue classes; and if k is even, such integers have preference for quadratic residue classes. This result confirms a conjecture of Richard Hudson. However, the integers with $\omega(n) = k$ always have preference for quadratic residue classes. Moreover, as k increases, the biases become smaller and smaller for both of the two cases.

1. Introduction and statement of results

First, we consider products of k primes in arithmetic progressions. Let

$$\pi_k(x; q, a) = |\{n \leq x : \omega(n) = k,\ n \equiv a \bmod q\}|,$$

and

$$N_k(x; q, a) = |\{n \leq x : \Omega(n) = k,\ n \equiv a \bmod q\}|,$$

where $\omega(n)$ is the number of distinct prime divisors of n, and $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. For example, when $k = 1$, $N_1(x; q, a)$ is the number of primes $\pi(x; q, a)$ in the arithmetic progression $a \bmod q$; and $\pi_1(x; q, a)$ counts the number of prime powers $p^l \leq x$ for all $l \geq 1$ in the arithmetic progression $a \bmod q$.

Dirichlet [1837] showed that, for any a and q with $(a, q) = 1$, there are infinitely many primes in the arithmetic progression $a \bmod q$. Moreover, for any $(a, q) = 1$,

$$\pi(x; q, a) \sim \frac{x}{\phi(q) \log x},$$

where ϕ is Euler's totient function [Davenport 2000]. Analogous asymptotic formulas are available for products of k primes. Landau [1909] showed that, for each fixed integer $k \geq 1$,

$$N_k(x) := |\{n \leq x : \Omega(n) = k\}| \sim \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

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The same asymptotic is also true for the function $\pi_k(x) := |\{n \le x : \omega(n) = k\}|$. For more precise formulas, see [Tenenbaum 1995, II.6, Theorems 4 and 5]. Using similar methods as in [Tenenbaum 1995; Davenport 2000], one can show that, for any fixed residue class $a \bmod q$ with $(a, q) = 1$,

$$N_k(x; q, a) \sim \pi_k(x; q, a) \sim \frac{1}{\phi(q)} \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

For the case of counting primes ($\Omega(n) = 1$), Chebyshev [1853] observed that there seem to be more primes in the progression $3 \bmod 4$ than in the progression $1 \bmod 4$. That is, it appears that $\pi(x; 4, 3) \ge \pi(x; 4, 1)$. In general, for any $a \not\equiv b \bmod q$ and $(a, q) = (b, q) = 1$, one can study the behavior of the functions

$$\Delta_{\omega_k}(x; q, a, b) := \pi_k(x; q, a) - \pi_k(x; q, b),$$
$$\Delta_{\Omega_k}(x; q, a, b) := N_k(x; q, a) - N_k(x; q, b).$$

Denote $\Delta(x; q, a, b) := \Delta_{\Omega_1}(x; q, a, b)$. Littlewood [1914] proved that $\Delta(x; 4, 3, 1)$ changes sign infinitely often. Actually, $\Delta(x; 4, 3, 1)$ is negative for the first time at $x = 26,861$ [Leech 1957]. Knapowski and Turán published a series of papers starting with [1962] about the sign changes and extreme values of the functions $\Delta(x; q, a, b)$. And such problems are colloquially known today as "prime race problems". Irregularities in the distribution, that is, a tendency for $\Delta(x; q, a, b)$ to be of one sign is known as "Chebyshev's bias". For a nice survey of such works, see [Ford and Konyagin 2002; Granville and Martin 2006].

Chebyshev's bias can be well understood in the sense of *logarithmic density*. We say a set S of positive integers has *logarithmic density*, if the following limit exists:

$$\delta(S) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{n \le x \\ n \in S}} \frac{1}{n}.$$

Let $\delta_{f_k}(q; a, b) = \delta(P_{f_k}(q; a, b))$, where $P_{f_k}(q; a, b)$ is the set of integers with $\Delta_{f_k}(n; q, a, b) > 0$, and $f = \Omega$ or ω. In order to study the Chebyshev's bias and the existence of the logarithmic density, we need the following assumptions:

(1) The *extended Riemann hypothesis* ($\mathrm{ERH_q}$) for Dirichlet L-functions modulo q.

(2) The *linear independence conjecture* ($\mathrm{LI_q}$), the imaginary parts of the zeros of all Dirichlet L-functions modulo q are linearly independent over $\mathbb{Q}$.

Under these two assumptions, Rubinstein and Sarnak [1994] showed that, for Chebyshev's bias for primes ($\Omega(n) = 1$), the logarithmic density $\delta_{\Omega_1}(q; a, b)$ exists, and in particular, $\delta_{\Omega_1}(4; 3, 1) \approx 0.996$ which indicates a strong bias for primes in the arithmetic progression $3 \bmod 4$. Recently, using the same assumptions, Ford and Sneed [2010] studied the Chebyshev's bias for products of two primes with $\Omega(n) = 2$ by transforming this problem into manipulations of some double integrals. They connected $\Delta_{\Omega_2}(x; q, a, b)$ with $\Delta(x; q, a, b)$, and showed that $\delta_{\Omega_2}(q; a, b)$ exists and the bias is in the opposite direction to the case of primes, in particular, $\delta_{\Omega_2}(4; 3, 1) \approx 0.10572$ which indicates a strong bias for the arithmetic progression $1 \bmod 4$.

By orthogonality of Dirichlet characters, we have

$$\Delta_{\Omega_k}(x; q, a, b) = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \bmod q} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{n \leq x \\ \Omega(n)=k}} \chi(n), \tag{1-1}$$

and

$$\Delta_{\omega_k}(x; q, a, b) = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \bmod q} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{n \leq x \\ \omega(n)=k}} \chi(n). \tag{1-2}$$

The inner sums over n are usually analyzed using analytic methods. Neither the method of Rubinstein and Sarnak [1994] nor the method of Ford and Sneed [2010] readily generalizes to handle the cases of more prime factors ($k \geq 3$). From the point of view of L-functions, the most natural sum to consider is

$$\sum_{\substack{n_1 \cdots n_k \leq x \\ n_1 \cdots n_k \equiv a \bmod q}} \Lambda(n_1) \cdots \Lambda(n_k). \tag{1-3}$$

However, estimates for $\Delta_{\Omega_k}(x; q, a, b)$ or $\Delta_{\omega_k}(x; q, a, b)$ cannot be readily recovered from such an analogue by partial summation. Ford and Sneed [2010] overcome this obstacle in the case $k = 2$ by means of the 2-dimensional integral

$$\int_0^\infty \int_0^\infty \sum_{p_1 p_2 \leq x} \frac{\chi(p_1 p_2) \log p_1 \log p_2}{p_1^{u_1} p_2^{u_2}} \, du_1 \, du_2.$$

Analysis of an analogous k-dimensional integral leads to an explosion of cases, depending on the relative sizes of the variables u_j, and becomes increasingly messy as k increases.

We take an entirely different approach, working directly with the unweighted sums. We express the associated Dirichlet series in terms of products of the logarithms of Dirichlet L-functions, then apply Perron's formula, and use Hankel contours to avoid the zeros of $L(s, \chi)$ and the point $s = \frac{1}{2}$. Using the same assumptions (1) and (2), we show that, for any $k \geq 1$, both

$$\delta_{\Omega_k}(q; a, b) \quad \text{and} \quad \delta_{\omega_k}(q; a, b)$$

exist. Moreover, we show that, as k increases, if a is a quadratic nonresidue and b is a quadratic residue, the bias oscillates with respect to the parity of k for the case $\Omega(n) = k$, but $\delta_{\omega_k}(q; a, b)$ increases from below $\frac{1}{2}$ monotonically.

For some of our results, we need only a much weaker substitute for condition $\mathrm{LI_q}$, which we call the *simplicity hypothesis* ($\mathrm{SH_q}$): $\forall \chi \neq \chi_0 \bmod q$, $L\left(\frac{1}{2}, \chi\right) \neq 0$ and the zeros of $L(s, \chi)$ are simple. Let

$$N(q, a) := \#\{u \bmod q : u^2 \equiv a \bmod q\}.$$

Then, using the weaker assumptions $\mathrm{SH_q}$ and $\mathrm{ERH_q}$, we prove the following theorems.

Theorem 1. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k \geq 1$, *and fixed large* T_0,

$$\Delta_{\Omega_k}(x; q, a, b) = \frac{1}{(k-1)!} \frac{\sqrt{x}(\log\log x)^{k-1}}{\log x} \left\{ \frac{(-1)^k}{\phi(q)} \sum_{\chi \neq \chi_0} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{|\gamma_\chi| \leq T_0 \\ L\left(\frac{1}{2}+i\gamma_\chi, \chi\right)=0}} \frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi} + \frac{(-1)^k}{2^{k-1}} \frac{N(q,a) - N(q,b)}{\phi(q)} + \Sigma_k(x; q, a, b, T_0) \right\},$$

where

$$\limsup_{Y\to\infty} \frac{1}{Y} \int_1^Y |\Sigma_k(e^y; q, a, b, T_0)|^2 \, dy \ll \frac{\log^2 T_0}{T_0}.$$

Since $\Delta_{\Omega_1}(x; q, a, b) = \Delta(x; q, a, b)$, we get the following corollary.

Corollary 1.1. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k \geq 2$,

$$\frac{\Delta_{\Omega_k}(x; q, a, b) \log x}{\sqrt{x}(\log\log x)^{k-1}} = \frac{(-1)^{k+1}}{(k-1)!} \left(1 - \frac{1}{2^{k-1}}\right) \frac{N(q,a) - N(q,b)}{\phi(q)} + \frac{(-1)^{k+1}}{(k-1)!} \frac{\Delta(x; q, a, b) \log x}{\sqrt{x}} + \Sigma'_k(x; q, a, b),$$

where, as $Y \to \infty$,

$$\frac{1}{Y} \int_1^Y |\Sigma'_k(e^y; q, a, b)|^2 \, dy = o(1).$$

In the above theorem, the constant $(-1)^k/(2^{k-1}) \cdot (N(q,a) - N(q,b))/(\phi(q))$ represents the bias in the distribution of products of k primes counted with multiplicity. Richard Hudson conjectured that, as k increases, the bias would change directions according to the parity of k. Our result above confirms his conjecture (under $\mathrm{ERH_q}$ and $\mathrm{SH_q}$). Figures 1.1 and 1.2 show the graphs corresponding to $(q, a, b) = (4, 3, 1)$ for $2\log x/(\sqrt{x}(\log\log x)^2) \cdot \Delta_{\Omega_3}(x; 4, 3, 1)$ and $6\log x/(\sqrt{x}(\log\log x)^3) \cdot \Delta_{\Omega_4}(x; 4, 3, 1)$, plotted on a logarithmic scale from $x = 10^3$ to $x = 10^8$. In these graphs, the functions do not appear to be oscillating around $\frac{1}{4}$ and $-\frac{1}{8}$ respectively as predicted in our theorem. This is caused by some terms of order $1/\log\log x$ and even lower order terms, and $\log\log 10^8 \approx 2.91347$ and $1/\log\log 10^8 \approx 0.343233$. However, we can still observe the expected direction of the bias through these graphs.

For the distribution of products of k primes counted without multiplicity, we have the following theorem. In this case, the bias will be determined by the constant $(N(q,a) - N(q,b))/(2^{k-1}\phi(q))$ in the theorem below.

Theorem 2. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k \geq 1$, *and fixed large* T_0,

$$\Delta_{\omega_k}(x; q, a, b) = \frac{1}{(k-1)!} \frac{\sqrt{x}(\log\log x)^{k-1}}{\log x} \left\{ \frac{(-1)^k}{\phi(q)} \sum_{\chi \neq \chi_0} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{|\gamma_\chi| \leq T_0 \\ L\left(\frac{1}{2}+i\gamma_\chi, \chi\right)=0}} \frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi} + \frac{N(q,a) - N(q,b)}{2^{k-1}\phi(q)} + \widetilde{\Sigma}_k(x; q, a, b, T_0) \right\},$$

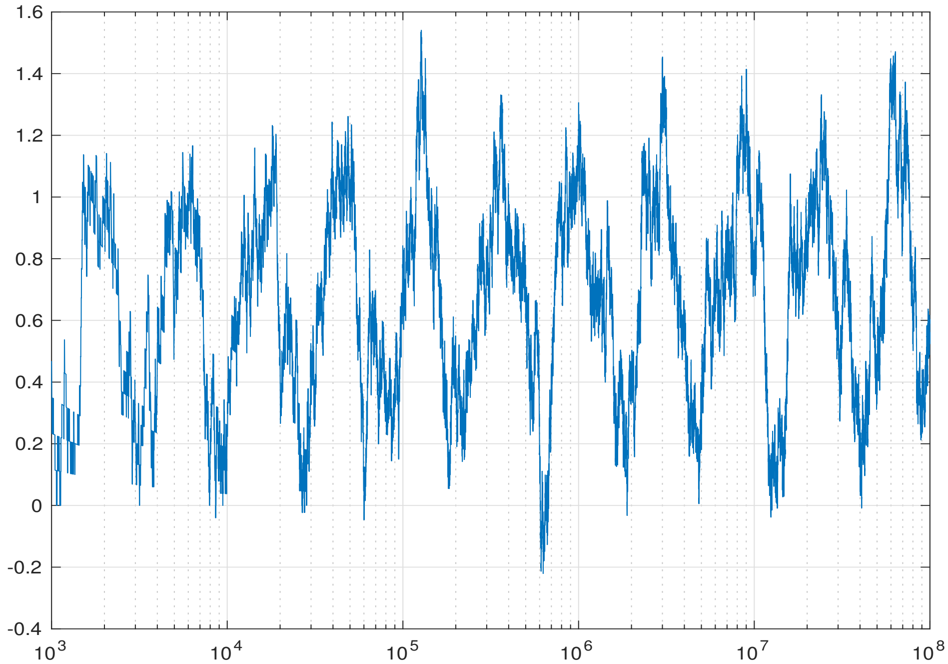

Figure 1.1. $\dfrac{2\log x}{\sqrt{x}(\log\log x)^2}\Delta_{\Omega_3}(x;4,3,1)$

where

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y |\widetilde{\Sigma}_k(e^y;q,a,b,T_0)|^2\,dy \ll \frac{\log^2 T_0}{T_0}.$$

Corollary 2.1. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k\ge 1$,

$$\frac{\Delta_{\omega_k}(x;q,a,b)\log x}{\sqrt{x}(\log\log x)^{k-1}} = \left(\frac{1}{2^{k-1}}+(-1)^{k+1}\right)\frac{N(q,a)-N(q,b)}{(k-1)!\phi(q)}+\frac{(-1)^{k+1}}{(k-1)!}\frac{\Delta(x;q,a,b)\log x}{\sqrt{x}}+\widetilde{\Sigma}'_k(x;q,a,b),$$

where, as $Y\to\infty$,

$$\frac{1}{Y}\int_1^Y |\widetilde{\Sigma}'_k(e^y;q,a,b)|^2\,dy = o(1).$$

For the distribution of $\Delta(x;q,a,b)$, Rubinstein and Sarnak [1994] showed the following theorem. This is the version from [Ford and Sneed 2010].

Theorem RS. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{LI_q}$. *For any* $a\not\equiv b \bmod q$ *and* $(a,q)=(b,q)=1$, *the function*

$$\frac{u\Delta(e^u;q,a,b)}{e^{u/2}}$$

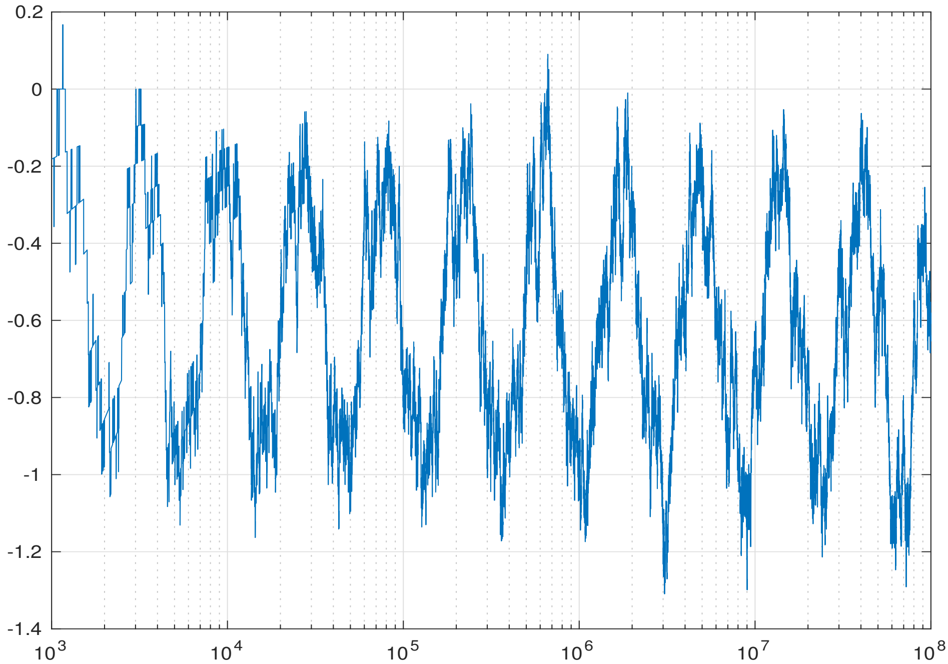

Figure 1.2. $\dfrac{6\log x}{\sqrt{x}(\log\log x)^3}\Delta_{\Omega_4}(x; 4, 3, 1)$

has a probabilistic distribution. This distribution has mean $(N(q, b) - N(q, a))/\phi(q)$, *is symmetric with respect to its mean*, *and has a continuous density function.*

Corollaries 1.1, 2.1, and Theorem RS imply the following result.

Theorem 3. *Let* $a \not\equiv b \bmod q$ *and* $(a, q) = (b, q) = 1$. *Assuming* $\mathrm{ERH_q}$ *and* $\mathrm{LI_q}$, *for any* $k \geq 1$, $\delta_{\Omega_k}(q; a, b)$ *and* $\delta_{\omega_k}(q; a, b)$ *exist. More precisely, if a and b are both quadratic residues or both quadratic nonresidues, then* $\delta_{\Omega_k}(q; a, b) = \delta_{\omega_k}(q; a, b) = \frac{1}{2}$. *Moreover, if a is a quadratic nonresidue and b is a quadratic residue, then, for any* $k \geq 1$,

$$1 - \delta_{\Omega_{2k-1}}(q; a, b) < \delta_{\Omega_{2k}}(q; a, b) < \tfrac{1}{2} < \delta_{\Omega_{2k+1}}(q; a, b) < 1 - \delta_{\Omega_{2k}}(q; a, b),$$
$$\delta_{\omega_k}(q; a, b) < \delta_{\omega_{k+1}}(q; a, b) < \tfrac{1}{2},$$
$$\delta_{\Omega_{2k}}(q; a, b) = \delta_{\omega_{2k}}(q; a, b), \quad \textit{and} \quad \delta_{\Omega_{2k-1}}(q; a, b) + \delta_{\omega_{2k-1}}(q; a, b) = 1.$$

Remark. The above results confirm a conjecture of Richard Hudson proposed years ago in his communications with Ford. Borrowing the methods from [Rubinstein and Sarnak 1994, Section 4], we are able to calculate $\delta_{\Omega_k}(q; a, b)$ and $\delta_{\omega_k}(q; a, b)$ precisely for special values of q, a, and b. In particular, we record in Table 1.1 the logarithmic densities up to products of 10 primes for two cases: $q = 3$, $a = 2$, $b = 1$, and $q = 4$, $a = 3$, $b = 1$.

k	$\delta_{\Omega_k}(3;2,1)$	$\delta_{\omega_k}(3;2,1)$	$\delta_{\Omega_k}(4;3,1)$	$\delta_{\omega_k}(4;3,1)$
1	0.99906†	0.00094	0.9959†	0.0041
2	0.069629	0.069629	0.10572‡	0.10572
3	0.766925	0.233075	0.730311	0.269689
4	0.35829	0.35829	0.380029	0.380029
5	0.571953	0.428047	0.380029	0.380029
6	0.463884	0.463884	0.469616	0.469616
7	0.518075	0.481925	0.515202	0.484798
8	0.49096	0.49096	0.492398	0.492398
9	0.50452	0.49548	0.503801	0.496199
10	0.49774	0.49774	0.498099	0.498099

Table 1.1. For $q=3$, $a=2$ and $b=1$ (left-hand side) and $q=4$, $a=3$ and $b=1$ (right-hand side). † [Rubinstein and Sarnak 1994] ‡ [Ford and Sneed 2010]

For fixed q and large k, we give asymptotic formulas for $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$.

Theorem 4. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{LI_q}$. *Let* $A(q)$ *be the number of real characters* mod q. *Let a be a quadratic nonresidue and b be a quadratic residue, and* $(a,q)=(b,q)=1$. *Then, for any nonnegative integer* K, *and any* $\epsilon>0$,

$$\delta_{\Omega_k}(q;a,b)=\frac{1}{2}+\frac{(-1)^{k-1}}{2\pi}\sum_{j=0}^{K}\left(\frac{1}{2^{k-1}}\right)^{2j+1}\frac{(-1)^jA(q)^{2j+1}C_j(q;a,b)}{(2j+1)!}+O_{q,K,\epsilon}\left(\frac{1}{(2^{k-1})^{2K+3-\epsilon}}\right), \tag{1-4}$$

$$\delta_{\omega_k}(q;a,b)=\frac{1}{2}-\frac{1}{2\pi}\sum_{j=0}^{K}\left(\frac{1}{2^{k-1}}\right)^{2j+1}\frac{(-1)^jA(q)^{2j+1}C_j(q;a,b)}{(2j+1)!}+O_{q,K,\epsilon}\left(\frac{1}{(2^{k-1})^{2K+3-\epsilon}}\right), \tag{1-5}$$

where $C_j(q;a,b)$ *is some constant depending on* $j,q,a,$ *and* b. *In particular, for* $K=0$,

$$\delta_{\Omega_k}(q;a,b)=\frac{1}{2}+(-1)^{k-1}\frac{A(q)C_0(q;a,b)}{2^k\pi}+O_{q,\epsilon}\left(\frac{1}{(2^k)^{3-\epsilon}}\right),$$
$$\delta_{\omega_k}(q;a,b)=\frac{1}{2}-\frac{A(q)C_0(q;a,b)}{2^k\pi}+O_{q,\epsilon}\left(\frac{1}{(2^k)^{3-\epsilon}}\right).$$

Remark. We have a formula for $C_j(q;a,b)$,

$$C_j(q;a,b)=\int_{-\infty}^{\infty}x^{2j}\Phi_{q;a,b}(x)\,dx,$$

where

$$\Phi_{q;a,b}(z)=\prod_{\chi\neq\chi_0}\prod_{\substack{\gamma_\chi>0\\ L\left(\frac{1}{2}+i\gamma_\chi\right)=0}}J_0\left(\frac{2|\chi(a)-\chi(b)|z}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right),$$

and $J_0(z)$ is the Bessel function,

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{(m!)^2}.$$

Numerically, $C_0(3; 2, 1) \approx 3.66043$ and $C_0(4; 3, 1) \approx 3.08214$. When q is large, using the method in [Fiorilli and Martin 2013, Section 2], we can find asymptotic formulas for $C_j(q; a, b)$,

$$C_j(q; a, b) = \frac{(2j-1)!!\sqrt{2\pi}}{V(q; a, b)^{j+1/2}} + O_j\left(\frac{1}{V(q; a, b)^{j+3/2}}\right),$$

where $(2j-1)!! = (2j-1)(2j-3)\cdots 3 \cdot 1$, $(-1)!! = 1$, and

$$V(q; a, b) = \sum_{\chi \bmod q} |\chi(b) - \chi(a)|^2 \sum_{\substack{\gamma_\chi \in \mathbb{R} \\ L\left(\frac{1}{2}+i\gamma_\chi, \chi\right)=0}} \frac{1}{\frac{1}{4} + \gamma_\chi^2}.$$

By Proposition 3.6 in [Fiorilli and Martin 2013], under $\mathrm{ERH_q}$, $V(q; a, b) \sim 2\phi(q) \log q$.

2. Formulas for the associated Dirichlet series and origin of the bias

Let χ be a nonprincipal Dirichlet character, and denote

$$F_{f_k}(s, \chi) := \sum_{f(n)=k} \frac{\chi(n)}{n^s},$$

where $f = \Omega$ or ω. The formulas for $F_{f_k}(s, \chi)$ are needed to analyze the character sums in (1-1) and (1-2). The purpose of this section is to express $F_{f_k}(s, \chi)$ in terms of Dirichlet L-functions, and to explain the source of the biases in the functions $\Delta_{\Omega_k}(x; q, a, b)$ and $\Delta_{\omega_k}(x; q, a, b)$.

Throughout the paper, the notation $\log z$ will always denote the *principal branch* of the logarithm of a complex number z.

2A. ***Symmetric functions.*** Let $x_1, x_2, \ldots$ be an infinite collection of indeterminates. We say a formal power series $P(x_1, x_2, \ldots)$ with bounded degree is a *symmetric function* if it is invariant under all finite permutations of the variables $x_1, x_2, \ldots$.

The n-th *elementary symmetric function* $e_n = e_n(x_1, x_2, \ldots)$ is defined by the generating function $\sum_{n=0}^{\infty} e_n z^n = \prod_{i=1}^{\infty}(1 + x_i z)$. Thus, e_n is the sum of all square-free monomials of degree n. Similarly, the n-th *homogeneous symmetric function* $h_n = h_n(x_1, x_2, \ldots)$ is defined by the generating function $\sum_{n=0}^{\infty} h_n z^n = \prod_{i=1}^{\infty} 1/(1 - x_i z)$. We see that, h_n is the sum of all possible monomials of degree n. And the n-th *power symmetric function* $p_n = p_n(x_1, x_2, \ldots)$ is defined to be $p_n = x_1^n + x_2^n + \cdots$.

The following result is due to Newton or Girard (see [Macdonald 1995, Chapter 1, (2.11) and (2.11'), page 23] or [Mendes and Remmel 2015, Chapter 2, Theorems 2.8 and 2.9]).

Lemma 5. *For any integer $k \geq 1$,*

$$kh_k = \sum_{n=1}^{k} h_{k-n} p_n, \tag{2-1}$$

$$ke_k = \sum_{n=1}^{k} (-1)^{n-1} e_{k-n} p_n. \tag{2-2}$$

2B. *Formula for $F_{\Omega_k}(s, \chi)$.* For $\Re(s) > 1$, we define

$$F(s, \chi) := \sum_p \frac{\chi(p)}{p^s},$$

the sum being over all primes p. Since

$$\log L(s, \chi) = \sum_{m=1}^{\infty} \sum_p \frac{\chi(p^m)}{m p^{ms}}, \tag{2-3}$$

we then have

$$F(s, \chi) = \log L(s, \chi) - \tfrac{1}{2} \log L(2s, \chi^2) + G(s), \tag{2-4}$$

where $G(s)$ is absolutely convergent for $\Re(s) \geq \sigma_0$ for any fixed $\sigma_0 > \frac{1}{3}$. Henceforth, σ_0 will be a fixed abscissa $> \frac{1}{3}$, say $\sigma_0 = 0.34$. Because $L(s, \chi)$ is an entire function for nonprincipal characters χ, formula (2-4) provides an analytic continuation of $F(s, \chi)$ to any simply connected domain within the half-plane $\{s : \Re(s) \geq \sigma_0\}$ which avoids the zeros of $L(s, \chi)$ and the zeros and possible pole of $L(2s, \chi^2)$.

For any complex number s with $\Re(s) \geq \sigma_0 > \frac{1}{3}$, let $x_p = \chi(p)/p^s$ if p is a prime, 0 otherwise. Then, by (2-1) in Lemma 5, we have the following relation

$$kF_{\Omega_k}(s, \chi) = \sum_{n=1}^{k} F_{\Omega_{k-n}}(s, \chi) F(ns, \chi^n). \tag{2-5}$$

For example, for $k = 1$, $F_{\Omega_1}(s, \chi) = F(s, \chi)$. For $k = 2$,

$$2F_{\Omega_2}(s, \chi) = F^2(s, \chi) + F(2s, \chi^2).$$

For $k = 3$,

$$\begin{aligned} 3!F_{\Omega_3}(s, \chi) &= 2F_{\Omega_2}(s, \chi)F(s, \chi) + 2F(s, \chi)F(2s, \chi^2) + 2F(3s, \chi^3) \\ &= F^3(s, \chi) + 3F(s, \chi)F(2s, \chi^2) + 2F(3s, \chi^3). \end{aligned}$$

For $k = 4$,

$$\begin{aligned} 4!F_{\Omega_4}(s, \chi) &= 3!F_{\Omega_3}(s, \chi)F(s, \chi) + 3!F_{\Omega_2}(s, \chi)F(2s, \chi^2) + 3!F(s, \chi)F(3s, \chi^2) + 3!F(4s, \chi^4) \\ &= F^4(s, \chi) + 6F^2(s, \chi)F(2s, \chi^2) + 8F(s, \chi)F(3s, \chi^3) + 6F(4s, \chi^4) + 3F^2(2s, \chi^2). \end{aligned}$$

For any integer $l \geq 1$, we define the set

$$S_{m,l}^{(k)} := \{(n_1, \cdots, n_l) \mid n_1 + \cdots + n_l = k - m, 2 \leq n_1 \leq n_2 \leq \cdots \leq n_l, n_j \in \mathbb{N}(1 \leq j \leq l)\}$$

Let $S_m^{(k)} = \bigcup_{l\ge 1} S_{m,l}^{(k)}$. Thus any element of $S_m^{(k)}$ is a partition of $k-m$ with each part ≥ 2. For any $\boldsymbol{n} = (n_1, n_2, \cdots, n_l) \in S_m^{(k)}$, denote

$$F(\boldsymbol{n}s, \chi) := \prod_{j=1}^{l} F(n_j s, \chi^{n_j}).$$

Hence, by (2-5) and induction on k, we deduce the following result.

Lemma 6. *For $k=1$, $F_{\Omega_1}(s,\chi) = F(s,\chi)$. For any $k \ge 2$, we have*

$$k! F_{\Omega_k}(s,\chi) = F^k(s,\chi) + \sum_{m=0}^{k-2} F^m(s,\chi) F_{\boldsymbol{n_m}}(s,\chi), \tag{2-6}$$

where $F_{\boldsymbol{n_m}}(s,\chi) = \sum_{\boldsymbol{n}\in S_m^{(k)}} a_m^{(k)}(\boldsymbol{n}) F(\boldsymbol{n}s,\chi)$ for some $a_m^{(k)}(\boldsymbol{n}) \in \mathbb{N}$.

2C. ***Formula for $F_{\omega_k}(s,\chi)$.*** By definition, we have

$$F_{\omega_k}(s,\chi) = \sum_{p_1<p_2<\cdots<p_k} \prod_{n=1}^{k} \biggl(\sum_{j=1}^{\infty} \frac{\chi(p_n^j)}{p_n^j} \biggr).$$

Denote

$$\tilde{F}(s,\chi) := \sum_p \biggl(\frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \biggr),$$

and for any $u \in \mathbb{N}^+$,

$$\tilde{F}(s,\chi;u) := \sum_p \biggl(\frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \biggr)^u = \sum_p \sum_{j=u}^{\infty} \biggl(D_u(j) \frac{\chi(p^j)}{p^{js}} \biggr),$$

where $D_u(j) = \binom{j-1}{u-1}$ is the number of ways of writing j as sum of u ordered positive integers.

By (2-3), we have

$$\tilde{F}(s,\chi) = \tilde{F}(s,\chi;1) = \sum_p \sum_{j=1}^{\infty} \frac{\chi(p^j)}{p^{js}} = \log L(s,\chi) + \tfrac{1}{2}\log L(2s,\chi^2) + \tilde{G}_1(s) \tag{2-7}$$

and

$$\tilde{F}(s,\chi;2) = \sum_p \sum_{j=2}^{\infty} (j-1) \frac{\chi(p^j)}{p^{js}} = \log L(2s,\chi^2) + \tilde{G}_2(s), \tag{2-8}$$

where $\tilde{G}_1(s)$ and $\tilde{G}_2(s)$ are absolutely convergent for $\Re(s) \ge \sigma_0$. Formula (2-7) provides an analytic continuation of $\tilde{F}(s,\chi)$ to any simply connected domain within the half-plane $\{s : \Re(s) \ge \sigma_0\}$ which avoids the zeros of $L(s,\chi)$ and the zeros and possible pole of $L(2s,\chi^2)$. Moreover, for any fixed $u \ge 3$, $\tilde{F}(s,\chi;u)$ is absolutely convergent for $\Re(s) \ge \sigma_0$.

For any complex number s with $\Re(s) \geq \sigma_0$, take $x_p = \sum_{j=1}^{\infty} \frac{\chi(p^j)}{p^{js}}$ if p is a prime, 0 otherwise. Then by (2-2) in Lemma 5, we get the following formula,

$$kF_{\omega_k}(s, \chi) = F_{\omega_{k-1}}(s, \chi)\tilde{F}(s, \chi) - \sum_{n=2}^{k}(-1)^n F_{\omega_{k-n}}(s, \chi)\tilde{F}(s, \chi; n). \tag{2-9}$$

For example, for $k = 1$, $F_{\omega_1}(s, \chi) = \tilde{F}(s, \chi)$. For $k = 2$,

$$2F_{\omega_2}(s, \chi) = \tilde{F}^2(s, \chi) - \tilde{F}(s, \chi; 2).$$

For $k = 3$,

$$\begin{aligned} 3!F_{\omega_3}(s, \chi) &= 2F_{\omega_2}(s, \chi)\tilde{F}(s, \chi) - 2F_{\omega_1}(s, \chi)\tilde{F}(s, \chi; 2) + 2\tilde{F}(s, \chi; 3) \\ &= \tilde{F}^3(s, \chi) - 3\tilde{F}(s, \chi)\tilde{F}(s, \chi; 2) + 2\tilde{F}(s, \chi; 3). \end{aligned}$$

For $k = 4$,

$$\begin{aligned} 4!F_{\omega_4}(s, \chi) &= 3!F_{\omega_3}(s, \chi)\tilde{F}(s, \chi) - 3!F_{\omega_2}(s, \chi)\tilde{F}(s, \chi; 2) + 3!\tilde{F}(s, \chi)\tilde{F}(s, \chi; 3) - 3!\tilde{F}(s, \chi; 4) \\ &= \tilde{F}^4(s, \chi) - 6\tilde{F}^2(s, \chi)\tilde{F}(s, \chi; 2) + 8\tilde{F}(s, \chi)\tilde{F}(s, \chi; 3) - 6\tilde{F}(s, \chi; 4) + 3\tilde{F}^2(s, \chi; 2). \end{aligned}$$

Hence, by (2-9) and induction on k, we get the following result.

Lemma 7. *For $k = 1$, $F_{\omega_1}(s, \chi) = \tilde{F}(s, \chi)$. For any $k \geq 2$, we have*

$$k!F_{\omega_k}(s, \chi) = \tilde{F}^k(s, \chi) + \sum_{m=0}^{k-2} \tilde{F}^m(s, \chi)\tilde{F}_{\boldsymbol{n}_{\boldsymbol{m}}}(s, \chi), \tag{2-10}$$

where $\tilde{F}_{\boldsymbol{n}_{\boldsymbol{m}}}(s, \chi) = \sum_{\boldsymbol{n} \in S_m^{(k)}} b_m^{(k)}(\boldsymbol{n})\tilde{F}(\boldsymbol{n}s, \chi)$ *for some* $b_m^{(k)}(\boldsymbol{n}) \in \mathbb{Z}$, *and for any* $\boldsymbol{n} = (n_1, \ldots, n_l) \in S_m^{(k)}$, $\tilde{F}(\boldsymbol{n}s, \chi) := \prod_{j=1}^{l} \tilde{F}(s, \chi; n_j)$.

2D. *Origin of the bias.* In this section, we heuristically explain the origin of the bias in our theorems.

(1) *Analytical aspect.* In order to get formulas for $\Delta_{\Omega_k}(x; q, a, b)$ and $\Delta_{\omega_k}(x; q, a, b)$, our strategy is to apply Perron's formula to the associated Dirichlet series $F_{\Omega_k}(s, \chi)$ and $F_{\omega_k}(s, \chi)$, then we choose special contours to avoid the singularities of these Dirichlet series. See Section 3 for the details.

First, we have a look at the case of counting primes in arithmetic progressions. If we only count primes, by (2-4), we have

$$F_{\Omega_1}(s, \chi) = F(s, \chi) = \sum_p \frac{\chi(p)}{p^s} = \log L(s, \chi) - \tfrac{1}{2}\log L(2s, \chi^2) + G(s).$$

The main contributions for $\Delta_{\Omega_1}(x; q, a, b)$ are from the first two terms,

$$\log L(s, \chi) - \tfrac{1}{2}\log L(2s, \chi^2).$$

The first term $\log L(s, \chi)$ counts all the primes with weight 1 and prime squares with weight $\frac{1}{2}$. The higher order powers of primes are negligible since they only contribute $O(x^{1/3})$. The singularities of $\log L(s, \chi)$, i.e., the zeros of $L(s, \chi)$, on the critical line contribute the oscillating terms in our result. In our proof, we use special Hankel contours to avoid the singularities of $\log L(s,\chi)$ and extract these oscillating terms (Lemma 12). See Sections 3 and 4 for the details of how to handle these singularities. The second term $-\frac{1}{2}\log L(2s,\chi^2)$ counts the prime squares with weight $-\frac{1}{2}$ and contributes the bias term. When χ is a real character, the point $s=\frac{1}{2}$ is a pole of $L(2s,\chi^2)$, and hence the integration of $-\frac{1}{2}\log L(2s,\chi^2)$ over the Hankel contour around $s=\frac{1}{2}$ contributes a bias term with order of magnitude $\sqrt{x}/\log x$. Using the orthogonality of Dirichlet characters, and the formula $\sum_{\chi \text{ real}}(\bar{\chi}(a)-\bar{\chi}(b)) = N(q,a)-N(q,b)$, we get the expected size of the bias.

Another natural and convenient function to consider is $-L'(s,\chi)/L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\Lambda(n)/n^s$, which is much easier to analyze than $\log L(s,\chi)$. This weighted form counts each prime p and its powers with weight $\log p$. Similar to $\log L(s,\chi)$, all the singularities of the function $-L'(s,\chi)/L(s,\chi)$ on the critical line are the nontrivial zeros of $L(s,\chi)$ and thus there is no bias for this weighted counting function

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)-\sum_{\substack{n\le x\\ n\equiv b \bmod q}}\Lambda(n).$$

Thus, partial summation is used to extract the sum

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\frac{\Lambda(n)}{\log n}-\sum_{\substack{n\le x\\ n\equiv b \bmod q}}\frac{\Lambda(n)}{\log n}$$

from the above weighted form, which is possible because $\log n$ is a smooth function. However, there is no way to do this with the analogue (1-3) to recover the unweighted counting function $\Delta_{\Omega_k}(x;q,a,b)$ or $\Delta_{\omega_k}(x;q,a,b)$.

If we count all the prime powers with the same weight 1, by (2-7), we have

$$F_{\omega_1}(s,\chi)=\tilde{F}(s,\chi)=\log L(s,\chi)+\tfrac{1}{2}\log L(2s,\chi^2)+\tilde{G}_1(s).$$

In this case, the bias is from the second term $\frac{1}{2}\log L(2s,\chi^2)$ for real character χ which counts the prime squares with positive weight $\frac{1}{2}$. This is why the bias is opposite to the case of counting only primes.

For the general case, when we derive the formula for $\Delta_{\Omega_k}(x;q,a,b)$ using analytic methods, by (2-6) in Lemma 6, the main contributions for $F_{\Omega_k}(s,\chi)$ will be from $\frac{1}{k!}F^k(s,\chi)$, which is essentially

$$\frac{1}{k!}\big(\log L(s,\chi)-\tfrac{1}{2}\log L(2s,\chi^2)\big)^k.$$

In the expansion of the above formula, the term $\frac{1}{k!}\log^k L(s,\chi)$ contributes the oscillating terms (see (4-9) and (4-13))

$$\frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\sum_{L\left(\frac{1}{2}+i\gamma_\chi,\chi\right)=0}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}.$$

When χ is real, the term

$$\frac{1}{k!}\big(-\tfrac{1}{2}\log L(2s,\chi^2)\big)^k=\frac{(-1)^k}{k!2^k}(\log L(2s,\chi^2))^k$$

contributes a bias term (see (4-10) and (4-14))

$$\frac{1}{(k-1)!}\frac{(-1)^k}{2^{k-1}}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}.$$

Then summing over all the real characters, we get the expected bias term in our formula for $\Delta_{\Omega_k}(x;q,a,b)$. The factor $(-1)^k/2^{k-1}$ explains why the bias has different directions depending on the parity of k and why the bias decreases as k increases. Other terms with factors of the form $\log^{k-j} L(s,\chi)\log^j(2s,\chi^2)$ for $1\le j\le k-1$ only contribute oscillating terms with lower orders of $\log\log x$ which can be put into the error term in our formula (see Lemma 14).

Similarly, for the case of $\Delta_{\omega_k}(x;q,a,b)$, by (2-10) in Lemma 7, the main contributions for $F_{\omega_k}(s,\chi)$ are from

$$\frac{1}{k!}\tilde{F}^k(s,\chi)=\frac{1}{k!}\big(\log L(s,\chi)+\tfrac{1}{2}\log L(2s,\chi^2)+\tilde{G}_1(s)\big)^k.$$

The main terms are from the contributions of the terms $\frac{1}{k!}\log^k L(s,\chi)$ and $\frac{1}{k!}\big(\frac{1}{2}\log L(2s,\chi^2)\big)^k$. Thus, the main oscillating terms are the same as that of $\Delta_{\Omega_k}(x;q,a,b)$, and the bias term has the same size without direction change.

Through the above analysis, we see that the biases are mainly affected by the powers of $\pm\frac{1}{2}\log L(2s,\chi^2)$ for real characters which count the products of prime squares.

(2) *Combinatorial aspect.* Instead of giving precise prediction of the size of the bias as above, here we use a simpler combinatorial intuition to roughly explain the behavior of the bias. We borrowed this combinatorial explanation from Hudson [1980].

Pick a large number X. Let S_1 be the set of primes $p\equiv 1 \bmod 4$ up to X, and S_2 be the set of primes $p\equiv 3 \bmod 4$ up to X. Using these primes, we generate the set $V^{(2)}:=\{pq : p,q\in S_1\cup S_2,\ p \text{ and } q \text{ can be the same}\}$.

Let $V_1^{(2)}:=\{n\in V^{(2)}:n\equiv 1 \bmod 4\}$, and $V_2^{(2)}:=\{n\in V^{(2)}:n\equiv 3 \bmod 4\}$. Then, the integers in $V_1^{(2)}$ come from either products of two primes from S_1 or products of two primes from S_2. The integers in $V_2^{(2)}$ are the product of two primes pq with $p\in S_1$ and $q\in S_2$. Thus,

$$|V_1^{(2)}|=\binom{|S_1|}{2}+|S_1|+\binom{|S_2|}{2}+|S_2|=\frac{|S_1|^2+|S_2|^2}{2}+\frac{|S_1|+|S_2|}{2},$$

and

$$|V_2^{(2)}|=|S_1|\cdot|S_2|.$$

It is clear that $|V_1^{(2)}|>|V_2^{(2)}|$. Note that $\frac{1}{2}(|S_1|+|S_2|)$ counts the squares of primes with weight $\frac{1}{2}$ which makes a crucial difference between $V_1^{(2)}$ and $V_2^{(2)}$.

Let $V_1^{(0)}=\{1\}$ and $V_2^{(0)}=\varnothing$. For any $k\ge 1$, denote

$$V_1^{(k)}:=\{n=p_1\cdots p_k : p_j\in S_1\cup S_2 \text{ for all } 1\le j\le k,\ n\equiv 1 \bmod 4\},$$
$$V_2^{(k)}:=\{n=p_1\cdots p_k : p_j\in S_1\cup S_2 \text{ for all } 1\le j\le k,\ n\equiv 3 \bmod 4\},$$

where the p_j can be the same. Note that $V_1^{(1)}=S_1$ and $V_2^{(1)}=S_2$.

We give inductive formulas for $|V_1^{(k)}|$ and $|V_2^{(k)}|$. The elements of $V_1^{(k)}$ and $V_2^{(k)}$ are generated by integers of the form $p^j n_{k-j}$ for $p \in S_1$ or S_2 and $n_{k-j} \in V_1^{(k-j)}$ or $V_2^{(k-j)}$ $(1 \le j \le k)$. By (2-1) in Lemma 5, we have

$$k|V_1^{(k)}| = \underbrace{(|V_1^{(k-1)}|\cdot|S_1|+|V_2^{(k-1)}|\cdot|S_2|)}_{pn_{k-1}} + \underbrace{|V_1^{(k-2)}|(|S_1|+|S_2|)}_{p^2n_{k-2}} + \underbrace{(|V_1^{(k-3)}|\cdot|S_1|+|V_2^{(k-3)}|\cdot|S_2|)}_{p^3n_{k-3}} + \cdots$$

and

$$k|V_2^{(k)}| = \underbrace{(|V_2^{(k-1)}|\cdot|S_1|+|V_1^{(k-1)}|\cdot|S_2|)}_{pn_{k-1}} + \underbrace{|V_2^{(k-2)}|(|S_1|+|S_2|)}_{p^2n_{k-2}} + \underbrace{(|V_2^{(k-3)}|\cdot|S_1|+|V_1^{(k-3)}|\cdot|S_2|)}_{p^3n_{k-3}} + \cdots .$$

Thus,

$$\begin{aligned}
k(|V_1^{(k)}| - |V_2^{(k)}|)
&= (|V_1^{(k-1)}|\cdot|S_1| + |V_2^{(k-1)}|\cdot|S_2|) - (|V_2^{(k-1)}|\cdot|S_1| + |V_1^{(k-1)}|\cdot|S_2|) + (|V_1^{(k-2)}| - |V_2^{(k-2)}|) \\
&\qquad (|S_1| + |S_2|) + (|V_1^{(k-3)}|\cdot|S_1| + |V_2^{(k-3)}|\cdot|S_2|) - (|V_2^{(k-3)}|\cdot|S_1| + |V_1^{(k-3)}|\cdot|S_2|) + \cdots \\
&= (|V_1^{(k-1)}| - |V_2^{(k-1)}|)(|S_1| - |S_2|) + (|V_1^{(k-2)}| - |V_2^{(k-2)}|)(|S_1| + |S_2|) \\
&\qquad + (|V_1^{(k-3)}| - |V_2^{(k-3)}|)(|S_1| - |S_2|) + \cdots .
\end{aligned} \tag{2-11}$$

For $1 \le j \le k-1$, suppose $|V_1^{(j)}| < |V_2^{(j)}|$ for odd j and $|V_1^{(j)}| > |V_2^{(j)}|$ for even j. Therefore, by (2-11) and induction, we deduce that $|V_1^{(k)}| < |V_2^{(k)}|$ for odd k and $|V_1^{(k)}| > |V_2^{(k)}|$ for even k. This provides us a heuristic explanation for the bias oscillation of $\Delta_{\Omega_k}(x; 4, 3, 1)$.

This heuristic also works for the quadratic residues and nonresidues modulo q whenever q has a primitive root. Because in this case, all the residues form a cyclic group (we thank the referee for pointing this out). When q has no primitive root, one should consider the group structure of the residue classes if we want to give a similar heuristic as above.

For example, for $q = 8$, we know that $1 \bmod 8$ is the only quadratic residue, and

$$3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \bmod 8, \quad 3\cdot 5 \equiv 7 \bmod 8, \quad 5\cdot 7 \equiv 3 \bmod 8, \quad \text{and} \quad 3\cdot 7 \equiv 5 \bmod 8. \tag{2-12}$$

If we define $V_j^{(1)} := \{p \equiv j \bmod 8,\ p \le X\}$ for $j = 1, 3, 5,$ and 7. For any $k \ge 2$, let

$$V_j^{(k)} := \{n = p_1 \cdots p_k \equiv j \bmod 8,\ p_i \in V_1^{(1)} \cup V_3^{(1)} \cup V_5^{(1)} \cup V_7^{(1)}, 1 \le i \le k\}, \quad \text{for } j = 1, 3, 5, 7.$$

Then similar to $q = 4$, using (2-12) and Lemma 5, we have

$$\begin{aligned}
k|V_1^{(k)}| = \underbrace{\big(|V_1^{(k-1)}|\cdot|V_1^{(1)}| + |V_3^{(k-1)}|\cdot|V_3^{(1)}| + |V_5^{(k-1)}|\cdot|V_5^{(1)}| + |V_7^{(k-1)}|\cdot|V_7^{(1)}|\big)}_{pn_{k-1}} \\
+ \underbrace{|V_1^{(k-2)}|\big(|V_1^{(1)}| + |V_3^{(1)}| + |V_5^{(1)}| + |V_7^{(1)}|\big)}_{p^2n_{k-2}} + \cdots
\end{aligned}$$

$$k|V_3^{(k)}| = \underbrace{\big(|V_3^{(k-1)}|\cdot|V_1^{(1)}| + |V_1^{(k-1)}|\cdot|V_3^{(1)}| + |V_5^{(k-1)}|\cdot|V_7^{(1)}| + |V_7^{(k-1)}|\cdot|V_5^{(1)}|\big)}_{pn_{k-1}} + \underbrace{|V_3^{(k-2)}|\big(|V_1^{(1)}| + |V_3^{(1)}| + |V_5^{(1)}| + |V_7^{(1)}|\big)}_{p^2 n_{k-2}} + \cdots$$

and similarly for $k|V_5^{(k)}|$ and $k|V_7^{(k)}|$. For $1 \le l \le k-1$, suppose $|V_1^{(l)}| < |V_3^{(l)}| \sim |V_5^{(l)}| \sim |V_7^{(l)}|$ for odd l, by the formulas for $k|V_j^{(k)}|$ $(j = 1, 3, 5, 7)$ and induction on k, we will derive the expected bias phenomenon.

3. Contour integral representation

In this section, we express the inner sums in (1-1) and (1-2) as integrals over truncated Hankel contours (see Lemma 10 below).

Let

$$\psi_{f_k}(x, \chi) := \sum_{\substack{n \le x \\ f(n)=k}} \chi(n),$$

where $f = \Omega$ or ω. By Perron's formula [Karatsuba 1993, Chapter V, Theorem 1] we have the following lemma.

Lemma 8. *For any $T \ge 2$,*

$$\psi_{f_k}(x, \chi) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F_{f_k}(s, \chi) \frac{x^s}{s}\, ds + O\Big(\frac{x \log x}{T} + 1\Big),$$

where $c = 1 + 1/\log x$, and $f = \Omega$ or ω.

Starting from Lemma 8, we will shift the contour to the left, in a way which avoids the singularities of the integrand. We will then require estimates of the integrand along the various parts of the new contour.

Lemma 9. *Assume* $\mathrm{ERH_q}$. *Then, for any $0 < \delta < \frac{1}{6}$ and for all $\chi \ne \chi_0 \bmod q$, there exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^{\infty}$ satisfying $n \le T_n \le n+1$ such that, for $T \in \mathcal{T}$,*

$$F_{f_k}(\sigma + iT) = O(\log^k T), \quad \big(\tfrac{1}{2} - \delta < \sigma < 2\big)$$

where $f = \Omega$ or ω.

Proof. Using the similar method as in [Titchmarsh 1986, Theorem 14.16], one can show that, for any $\epsilon > 0$ and for all $\chi \ne \chi_0 \bmod q$, there exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^{\infty}$ satisfying $n \le T_n \le n+1$ such that, $T_n^{-\epsilon} \ll |L(\sigma + iT_n, \chi)| \ll T_n^{\delta+\epsilon}$), $\big(\frac{1}{2} - \delta < \sigma < 2\big)$. Hence, by formulas (2-4), (2-6), (2-7), (2-8), and (2-10), we get the conclusion of this lemma. □

Let ρ be a zero of $L(s, \chi)$, Δ_ρ be the distance of ρ to the nearest other zero, and $D_\gamma := \min_{T \in \mathcal{T}}(|\gamma - T|)$. For each zero ρ, and $X > 0$, let $\mathcal{H}(\rho, X)$ denote the truncated Hankel contour surrounding the point $s = \rho$ with radius $0 < r_\rho \le \min\big(\frac{1}{x}, \frac{1}{3}\Delta_\rho, \frac{1}{2}D_\gamma, \frac{1}{3}\big|\rho - \frac{1}{2}\big|\big)$, which includes the circle $|s - \rho| = r_\rho$ excluding the

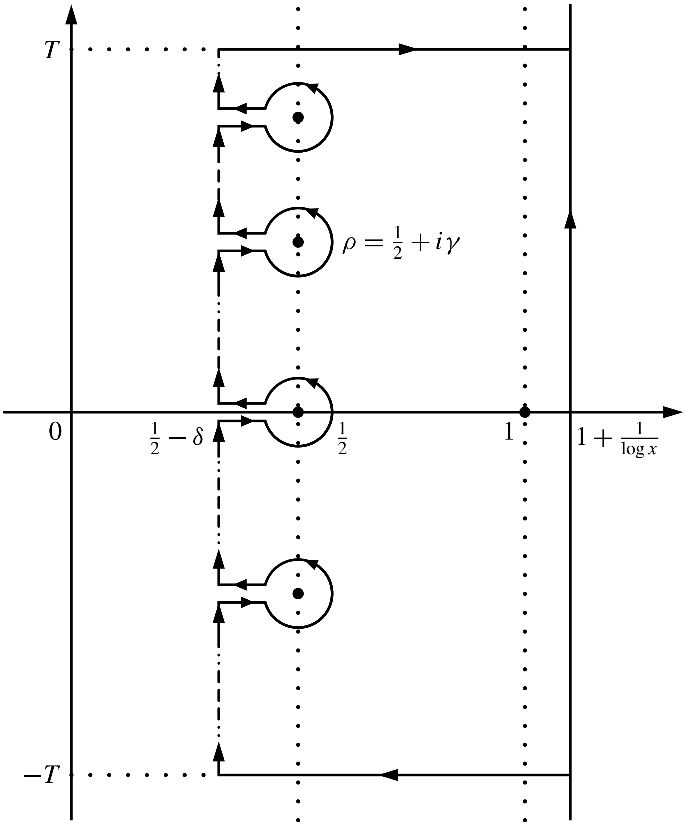

Figure 3.1. Integration contour

point $s=\rho-r_\rho$, and the half-line $(\rho-X,\rho-r]$ traced twice with arguments $+\pi$ and $-\pi$ respectively. Let Δ_0 be the distance of $\frac{1}{2}$ to the nearest zero. Let $\mathcal{H}(\frac{1}{2},X)$ denote the corresponding truncated Hankel contour surrounding $s=\frac{1}{2}$ with radius $r_0=\min\left(\frac{1}{x},\frac{\Delta_0}{3}\right)$.

Take $\delta=\frac{1}{10}$. By Lemma 8, we pull the contour to the left to the line $\Re(s)=\frac{1}{2}-\delta$ using the truncated Hankel contour $\mathcal{H}(\rho,\delta)$ to avoid the zeros of $L(s,\chi)$ and using $\mathcal{H}\left(\frac{1}{2},\delta\right)$ to avoid the point $s=\frac{1}{2}$. See Figure 3.1.

Then we have the following lemma.

Lemma 10. *Assume* $\mathrm{ERH_q}$, *and* $L\left(\frac{1}{2},\chi\right)\neq 0$ $(\chi\neq\chi_0)$. *Then, for any fixed* $k\geq 1$, *and* $T\in\mathcal{T}$,

$$\psi_{f_k}(x,\chi)=\sum_{|\gamma|\leq T}\frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)}F_{f_k}(s,\chi)\frac{x^s}{s}\,ds+a(\chi)\frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac{1}{2},\delta\right)}F_{f_k}(s,\chi)\frac{x^s}{s}\,ds + O\left(\frac{x\log x}{T}+\frac{x(\log T)^k}{T}+x^{1/2-\delta}(\log T)^{k+1}\right),$$

where $a(\chi)=1$ *if* χ *is real*, 0 *otherwise*, *and* $f=\omega$ *or* Ω.

Proof. By formulas (2-6) and (2-10), if χ is not real, $s=\frac{1}{2}$ is not a singularity of $F_{f_k}(s,\chi)$. Hence the second term is zero if χ is not real. By Lemma 9, the integral on the horizontal line is

$$\ll (\log T)^k \int_{\frac{1}{2}-\delta}^{c} \frac{x^\sigma}{|\sigma+iT|}\,d\sigma \ll \frac{x^c(\log T)^k}{T} \ll \frac{x(\log T)^k}{T}. \tag{3-1}$$

Under the assumption $\mathrm{ERH_q}$, the integral on the vertical line $\Re(s)=\frac{1}{2}-\delta$ is

$$\ll \int_{-T}^{T} \frac{x^{1/2-\delta}\log^k(|t|+2)}{\left|\frac{1}{2}-\delta+it\right|}\,dt \ll x^{1/2-\delta}(\log T)^{k+1}. \tag{3-2}$$

By (3-1), (3-2), and Lemma 8, we get the desired error term in this lemma. □

4. Proof of the main theorems

In this section, we give the full proof of Theorems 1 and 2, by quoting Lemmas 12, 13, 14, and 15 of which the proofs will appear in later sections.

Proof of Theorems 1 and 2. Let γ be the imaginary part of a zero of $L(s,\chi)$ in the critical strip. We have the following lemma.

Lemma 11 [Ford and Sneed 2010, Lemma 2.2]. *Let χ be a Dirichlet character modulo q. Let $N(T,\chi)$ denote the number of zeros of $L(s,\chi)$ with $0<\Re(s)<1$ and $|\Im(s)|<T$. Then*

(1) $N(T,\chi)=O(T\log(qT))$ *for* $T\ge 1$,

(2) $N(T,\chi)-N(T-1,\chi)=O(\log(qT))$ *for* $T\ge 1$,

(3) *uniformly for* $s=\sigma+it$ *and* $\sigma\ge -1$,

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{|\gamma-t|<1} \frac{1}{s-\rho} + O(\log q(|t|+2)). \tag{4-1}$$

For simplicity, we denote

$$\frac{1}{\Gamma_j(u)} := \left[\frac{d^j}{dz^j}\left(\frac{1}{\Gamma(z)}\right)\right]_{z=u}.$$

The following lemma is the starting lemma to give us the bias terms and oscillating terms in our main theorems. This lemma may have independent use, we will give the proof in Section 8.

Lemma 12. *Let $\mathcal{H}(a,\delta)$ be the truncated Hankel contour surrounding a complex number a ($\Re(a)>2\delta$) with radius $0<r\ll\frac{1}{x}$. Then, for any integer $k\ge 1$,*

$$\frac{1}{2\pi i}\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\frac{x^s}{s}\,ds = \frac{(-1)^k x^a}{a\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\} + O_k\left(\frac{|x^{a-\delta/3}|}{|a|}\right)$$
$$+ O_k\left(\frac{|x^a|}{|a|^2\log^2 x}(\log\log x)^{k-1}\right) + O_k\left(\frac{|x^a|}{|a|^2|\Re(a)-\delta|}\,\frac{(\log\log x)^{k-1}}{(\log x)^3}\right).$$

Remark. By (5-3) in the proof of Lemma 16, one can easily show that

$$\left|\frac{1}{\Gamma_j(0)}\right| \ll \Gamma(j+1). \tag{4-2}$$

By Lemma 10, we need to examine the integration over the truncated Hankel contours $\mathcal{H}(\rho,\delta)$ and $\mathcal{H}\big(\frac{1}{2},\delta\big)$. By (2-4) and (2-7), and the assumptions of our theorems, on each truncated Hankel contour $\mathcal{H}(\rho,\delta)$, we integrate the formula (4-1) in Lemma 11 to obtain

$$F(s,\chi) = \log(s-\rho) + H_\rho(s), \tag{4-3}$$

$$\tilde{F}(s,\chi) = \log(s-\rho) + \tilde{H}_\rho(s), \tag{4-4}$$

where

$$H_\rho(s) = \sum_{0<|\gamma'-\gamma|\le 1} \log(s-\rho') + O(\log|\gamma|),$$

$$\tilde{H}_\rho(s) = \sum_{0<|\gamma'-\gamma|\le 1} \log(s-\rho') + O(\log|\gamma|).$$

If χ is real, $s=\frac{1}{2}$ is a pole of $L(2s,\chi^2)$. So, by (2-4) and (2-7), on the truncated Hankel contour $\mathcal{H}\big(\frac{1}{2},\delta\big)$, for a real character χ, we write

$$F(s,\chi) = \tfrac{1}{2}\log\big(s-\tfrac{1}{2}\big) + H_B(s), \tag{4-5}$$

$$\tilde{F}(s,\chi) = -\tfrac{1}{2}\log\big(s-\tfrac{1}{2}\big) + \tilde{H}_B(s), \tag{4-6}$$

where $H_B(s)=O(1)$ and $\tilde{H}_B(s)=O(1)$.

Denote

$$I_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} k!F_{\Omega_k}(s,\chi)\frac{x^s}{s}\,ds, \qquad I_B(x) := \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} k!F_{\Omega_k}(s,\chi)\frac{x^s}{s}\,ds,$$

and

$$\tilde{I}_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} k!F_{\omega_k}(s,\chi)\frac{x^s}{s}\,ds, \qquad \tilde{I}_B(x) := \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} k!F_{\omega_k}(s,\chi)\frac{x^s}{s}\,ds.$$

We define a function $T(x)$ as follows: for $T_{n'}\in\mathcal{T}$ satisfying $e^{2^{n+1}} \le T_{n'} \le e^{2^{n+1}}+1$, let $T(x)=T_{n'}$ for $e^{2^n}\le x\le e^{2^{n+1}}$. In particular, we have

$$x \le T(x) \le 2x^2, \quad (x \ge e^2).$$

Thus, by Lemma 10, for $T=T(x)$,

$$\psi_{\Omega_k}(x,\chi) = \frac{1}{k!}\sum_{|\gamma|\le T} I_\rho(x) + \frac{a(\chi)}{k!}I_B(x) + O(x^{1/2-\delta/2}), \tag{4-7}$$

$$\psi_{\omega_k}(x,\chi) = \frac{1}{k!}\sum_{|\gamma|\le T} \tilde{I}_\rho(x) + \frac{a(\chi)}{k!}\tilde{I}_B(x) + O(x^{1/2-\delta/2}). \tag{4-8}$$

We will see later that $\sum_{|\gamma|\le T} I_\rho(x)$ and $\sum_{|\gamma|\le T} \tilde{I}_\rho(x)$ will contribute the oscillating terms, i.e., the summation over zeros, in our theorems, and $I_B(x)$ and $\tilde{I}_B(x)$ will contribute the bias terms.

Next, we want to find the main contributions for $I_\rho(x)$, $I_B(x)$, $\tilde{I}_\rho(x)$, and $\tilde{I}_B(x)$. By (2-6) and (4-3), we have

$$\begin{aligned} I_\rho(x) &= \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{j=1}^{k}\binom{k}{j}(\log(s-\rho))^{k-j}(H_\rho(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{m=0}^{k-2} F^m(s,\chi)F_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: I_{M_\rho}(x) + E_{M_\rho}(x) + E_{R_\rho}(x), \end{aligned} \tag{4-9}$$

and by (2-6) and (4-5),

$$\begin{aligned} I_B(x) &= \frac{1}{2^k}\frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \left(\log\left(s-\tfrac12\right)\right)^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{j=1}^{k}\binom{k}{j}\left(\tfrac12\log\left(s-\tfrac12\right)\right)^{k-j}(H_B(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{m=0}^{k-2} F^m(s,\chi)F_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: B_M(x) + E_B(x) + E_R(x). \end{aligned} \tag{4-10}$$

Here, $I_{M_\rho}(x)$ and $B_M(x)$ will make main contributions to $I_\rho(x)$ and $I_B(x)$, respectively. Similarly, by (2-10) and (4-4), we have

$$\begin{aligned} \tilde{I}_\rho(x) &= \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{j=1}^{k}\binom{k}{j}(\log(s-\rho))^{k-j}(\tilde{H}_\rho(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{m=0}^{k-2} \tilde{F}^m(s,\chi)\tilde{F}_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: \tilde{I}_{M_\rho}(x) + \tilde{E}_{M_\rho}(x) + \tilde{E}_{R_\rho}(x), \end{aligned} \tag{4-11}$$

and by (2-10) and (4-6),

$$\begin{aligned} \tilde{I}_B(x) &= \frac{(-1)^k}{2^k}\frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \left(\log\left(s-\tfrac12\right)\right)^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{j=1}^{k}\binom{k}{j}\left(-\tfrac12\log\left(s-\tfrac12\right)\right)^{k-j}(\tilde{H}_B(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{m=0}^{k-2} \tilde{F}^m(s,\chi)\tilde{F}_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: \tilde{B}_M(x) + \tilde{E}_B(x) + \tilde{E}_R(x). \end{aligned} \tag{4-12}$$

Here, $\tilde{I}_{M_\rho}(x)$ and $\tilde{B}_M(x)$ will make main contributions to $\tilde{I}_\rho(x)$ and $\tilde{I}_B(x)$, respectively.

Applying Lemma 12, we have

$$I_{M_\rho}(x) = \tilde{I}_{M_\rho}(x) = \frac{(-1)^k\sqrt{x}}{\log x}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\} + O_k\left(\frac{1}{|\gamma|^2}\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right) + O_k\left(\frac{x^{1/2-\delta/3}}{|\gamma|}\right), \tag{4-13}$$

$$B_M(x) = \frac{(-1)^k\sqrt{x}}{2^{k-1}\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\} + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right) + O_k(x^{1/2-\delta/3}), \tag{4-14}$$

and

$$\tilde{B}_M(x) = (-1)^k B_M(x). \tag{4-15}$$

For the bias terms, by (4-10), (4-12), (4-14), and (4-15), we have

$$I_B(x) = \frac{(-1)^k\sqrt{x}}{2^{k-1}\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\} + E_B(x) + E_R(x) + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right) + O_k(x^{1/2-\delta/3}), \tag{4-16}$$

and

$$\tilde{I}_B(x) = \frac{\sqrt{x}}{2^{k-1}\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\} + \tilde{E}_B(x) + \tilde{E}_R(x) + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right) + O_k(x^{1/2-\delta/3}). \tag{4-17}$$

We will prove the following result in Section 5.

Lemma 13. *For the bias terms,*

$$I_B(x) = \frac{(-1)^k k}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1} + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right),$$

$$\tilde{I}_B(x) = \frac{k}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1} + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right).$$

Then for the oscillating terms, by (4-9), (4-11), and (4-13), and Lemma 11, for $T = T(x)$,

$$\sum_{|\gamma|\le T} I_\rho(x) = \frac{(-1)^k k\sqrt{x}(\log\log x)^{k-1}}{\log x}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \frac{(-1)^k\sqrt{x}}{\log x}\sum_{j=2}^{k}\binom{k}{j}\frac{(\log\log x)^{k-j}}{\Gamma_j(0)}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \sum_{|\gamma|\le T} E_{M_\rho}(x) + \sum_{|\gamma|\le T} E_{R_\rho}(x) + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{\log^2 x}\right), \tag{4-18}$$

and

$$\sum_{|\gamma|\le T} \tilde{I}_\rho(x) = \frac{(-1)^k k\sqrt{x}(\log\log x)^{k-1}}{\log x}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \frac{(-1)^k\sqrt{x}}{\log x}\sum_{j=2}^{k}\binom{k}{j}\frac{(\log\log x)^{k-j}}{\Gamma_j(0)}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \sum_{|\gamma|\le T} \tilde{E}_{M_\rho}(x) + \sum_{|\gamma|\le T} \tilde{E}_{R_\rho}(x) + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{\log^2 x}\right). \tag{4-19}$$

The first terms in the above formulas are the main oscillating terms in our theorems. We will show in Section 6 that the other terms are small in average. For $T = T(x)$, denote

$$\Sigma_1(x;\chi) := \frac{\log x}{\sqrt{x}}\sum_{|\gamma|\le T} E_{M_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{M_\rho}(x), \tag{4-20}$$

$$\Sigma_2(x;\chi) := \frac{\log x}{\sqrt{x}}\sum_{|\gamma|\le T} E_{R_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{R_\rho}(x), \tag{4-21}$$

where $E'_{M_\rho}(x) = E_{M_\rho}(x)/x^\rho$, and $E'_{R_\rho}(x) = E_{R_\rho}(x)/x^\rho$. Similarly, denote

$$\widetilde{\Sigma}_1(x;\chi) := \frac{\log x}{\sqrt{x}}\sum_{|\gamma|\le T} \tilde{E}_{M_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} \tilde{E}'_{M_\rho}(x), \tag{4-22}$$

$$\widetilde{\Sigma}_2(x;\chi) := \frac{\log x}{\sqrt{x}}\sum_{|\gamma|\le T} \tilde{E}_{R_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} \tilde{E}'_{R_\rho}(x), \tag{4-23}$$

where $\tilde{E}'_{M_\rho}(x) = \tilde{E}_{M_\rho}(x)/x^\rho$ and $\tilde{E}'_{R_\rho}(x) = \tilde{E}_{R_\rho}(x)/x^\rho$.

Then we have the following lemma (see Sections 6A and 6B for the proof).

Lemma 14. *For the error terms from the Hankel contours around zeros, we have*

$$\int_2^Y (|\Sigma_1(e^y;\chi)|^2 + |\Sigma_2(e^y;\chi)|^2)\,dy = o(Y(\log Y)^{2k-2}),$$

$$\int_2^Y (|\widetilde{\Sigma}_1(e^y;\chi)|^2 + |\widetilde{\Sigma}_2(e^y;\chi)|^2)\,dy = o(Y(\log Y)^{2k-2}).$$

Moreover, we also need to bound the lower order sum

$$S_1(x;\chi) := (-1)^k \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}, \tag{4-24}$$

and the error from the truncation by a fixed large T_0,

$$S_2(x, T_0; \chi) := \sum_{|\gamma| \le T} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} - \sum_{|\gamma| \le T_0} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}. \tag{4-25}$$

Then we have the following result (See Section 6C for the proof).

Lemma 15. *For the lower order sum and error from the truncation, we have*

$$\int_2^Y |S_1(e^y; \chi)|^2 \, dy = o(Y(\log Y)^{2k-2}),$$

and for fixed large T_0,

$$\int_2^Y |S_2(e^y, T_0; \chi)|^2 \, dy \ll Y \frac{\log^2 T_0}{T_0} + \log Y \frac{\log^3 T_0}{T_0} + \log^5 T_0.$$

Combining Lemmas 13, 14, and 15 with (4-7), (4-8), (4-18), and (4-19), we get, for fixed large T_0,

$$\psi_{\Omega_k}(x, \chi) = \frac{(-1)^k}{(k-1)!} \frac{\sqrt{x}}{\log x} (\log\log x)^{k-1} \Bigg(\sum_{|\gamma| \le T_0} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \Sigma(x, T_0; \chi) \Bigg) + a(\chi) \frac{(-1)^k}{(k-1)!} \frac{\sqrt{x}}{\log x} (\log\log x)^{k-1}, \tag{4-26}$$

where

$$\limsup_{Y \to \infty} \frac{1}{Y} \int_1^Y |\Sigma(e^y, T_0; \chi)|^2 \, dy \ll \frac{\log^2 T_0}{T_0}.$$

Also,

$$\psi_{\omega_k}(x, \chi) = \frac{(-1)^k}{(k-1)!} \frac{\sqrt{x}}{\log x} (\log\log x)^{k-1} \Bigg(\sum_{|\gamma| \le T_0} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \widetilde{\Sigma}(x, T_0; \chi) \Bigg) + a(\chi) \frac{1}{(k-1)!} \frac{\sqrt{x}}{\log x} (\log\log x)^{k-1}, \tag{4-27}$$

where

$$\limsup_{Y \to \infty} \frac{1}{Y} \int_1^Y |\widetilde{\Sigma}(e^y, T_0; \chi)|^2 \, dy \ll \frac{\log^2 T_0}{T_0}.$$

Note that $\sum_{\chi \ne \chi_0} (\bar{\chi}(a) - \bar{\chi}(b)) a(\chi) = N(q, a) - N(q, b)$. Hence, combining (4-26) and (4-27) with (1-1) and (1-2), we get the conclusions of Theorem 1 and Theorem 2. Now we finish the proof of Theorems 1 and 2 modulo the proofs of Lemmas 12, 13, 14, and 15. □

5. The bias terms

In this section, we examine the bias terms and give the proof of Lemma 13.

5A. ***Estimates on the horizontal line.*** In order to examine the corresponding integration on the horizontal line in the Hankel contour, we prove the following estimate which we will use many times later to analyze the error terms in our theorems.

Lemma 16. *For any integers* $k \geq 1$ *and* $m \geq 0$, *we have*

$$\int_0^\delta |(\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k|\sigma^m x^{-\sigma}\,d\sigma \ll_{m,k} \frac{(\log\log x)^{k-1}}{(\log x)^{m+1}}.$$

Proof. Let I represent the integral in the lemma. Then, we have

$$I \leq 2\sum_{j=1}^{k}\binom{k}{j}\pi^j \int_0^\delta |\log\sigma|^{k-j}\sigma^m x^{-\sigma}\,d\sigma \ll_k \sum_{j=1}^{k}\int_0^\delta |\log\sigma|^{k-j}\sigma^m x^{-\sigma}\,d\sigma \tag{5-1}$$

Using a change of variable, $\sigma\log x = t$, we have

$$\begin{aligned}\int_0^\delta |\log\sigma|^{k-j}\sigma^m x^{-\sigma}\,d\sigma &\leq \frac{1}{(\log x)^{m+1}}\int_0^{\delta\log x} |\log t - \log\log x|^{k-j} t^m e^{-t}\,dt\\ &\leq \frac{1}{(\log x)^{m+1}}\sum_{l=0}^{k-j}\binom{k-j}{l}(\log\log x)^{k-j-l}\int_0^{\delta\log x}|\log t|^l t^m e^{-t}\,dt\\ &\ll_k \frac{1}{(\log x)^{m+1}}\sum_{l=0}^{k-j}(\log\log x)^{k-j-l}\int_0^{\delta\log x}|\log t|^l t^m e^{-t}\,dt.\end{aligned}\tag{5-2}$$

Next, we estimate

$$\int_0^{\delta\log x}|\log t|^l t^m e^{-t}\,dt \leq \left(\int_0^1 + \int_1^\infty\right)|\log t|^l t^m e^{-t}\,dt =: I_{l_1} + I_{l_2}. \tag{5-3}$$

For the first integral in (5-3),

$$I_{l_1} = \int_0^1 |\log t|^l t^m e^{-t}\,dt \leq \int_0^1 |\log t|^l\,dt \overset{t\to 1/e^t}{=} \int_0^\infty \frac{t^l}{e^t}\,dt = \Gamma(l+1).$$

For the second integral in (5-3),

$$I_{l_2} = \int_1^\infty \frac{t^m(\log t)^l}{e^t}\,dt \overset{t\to e^t}{=} \int_0^\infty \frac{t^l}{e^{e^t-(m+1)t}}\,dt \ll_m \Gamma(l+1). \tag{5-4}$$

Then, by (5-2)-(5-4), we have

$$\int_0^\delta |\log\sigma|^{k-j}\sigma^m x^{-\sigma}\,d\sigma \ll_k \frac{1}{(\log x)^{m+1}}\sum_{l=0}^{k-j}(\log\log x)^{k-j-l}O_{m,l}(1) \ll_{m,k} \frac{(\log\log x)^{k-j}}{(\log x)^{m+1}}. \tag{5-5}$$

Thus, by (5-1),

$$I \ll_{m,k} \frac{(\log\log x)^{k-1}}{(\log x)^{m+1}}.$$

Hence, we get the conclusion of this lemma. □

5B. ***The bias terms.*** We have the following estimate for the integral over the truncated Hankel contour $\mathcal{H}\big(\tfrac{1}{2},\delta\big)$.

Lemma 17. *Assume the function $f(s)=O(1)$ on $\mathcal{H}\big(\tfrac{1}{2},\delta\big)$. Then, for any integer $m\geq 0$,*

$$\left|\int_{\mathcal{H}(\frac{1}{2},\delta)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^m f(s)\frac{x^s}{s}\,ds\right| \ll_m \frac{\sqrt{x}(\log\log x)^{m-1}}{\log x}.$$

Proof. Since the left-hand side is 0 when $m=0$, we assume $m\geq 1$ in the following proof. By Lemma 16, we have

$$\begin{aligned}\left|\int_{\mathcal{H}(\frac{1}{2},\delta)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^m f(s)\frac{x^s}{s}\,ds\right| &\leq \left|\int_{r_0}^{\delta}\big((\log\sigma-i\pi)^m-(\log\sigma+i\pi)^m\big)f\big(\tfrac{1}{2}-\sigma\big)\frac{x^{1/2-\sigma}}{\frac{1}{2}-\sigma}\,d\sigma\right| \\ &\qquad + O\bigg(\int_{-\pi}^{\pi}\frac{(\log 1/r_0+\pi)^m x^{1/2+r_0}}{\frac{1}{2}-r_0}r_0\,d\alpha\bigg) \\ &\ll \sqrt{x}\bigg(\int_0^{\delta}|(\log\sigma-i\pi)^m-(\log\sigma+i\pi)^m|x^{-\sigma}\,d\sigma+\frac{(\log x+\pi)^m}{x}\bigg) \\ &\ll_m \frac{\sqrt{x}(\log\log x)^{m-1}}{\log x}. \qquad (5\text{-}6)\end{aligned}$$

This completes the proof of this lemma. □

In the following, we prove the asymptotic formulas for the bias terms.

Proof of Lemma 13. Since $H_B(s)=O(1)$, by (4-10), (4-12), and Lemma 17,

$$\begin{aligned}|E_B(x)| &\ll \sum_{j=1}^{k}\left|\int_{\mathcal{H}(\frac{1}{2},\delta)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^{k-j}(H_B(s))^j\frac{x^s}{s}\,ds\right| \\ &\ll \frac{\sqrt{x}}{\log x}\sum_{j=1}^{k}(\log\log x)^{k-j-1} \\ &\ll_k \frac{\sqrt{x}}{\log x}(\log\log x)^{k-2}. \qquad (5\text{-}7)\end{aligned}$$

Similarly,

$$\begin{aligned}|\tilde{E}_B(x)| &\ll \sum_{j=1}^{k}\left|\int_{\mathcal{H}(\frac{1}{2},\delta)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^{k-j}(\tilde{H}_B(s))^j\frac{x^s}{s}\,ds\right| \\ &\ll_k \frac{\sqrt{x}}{\log x}(\log\log x)^{k-2}. \qquad (5\text{-}8)\end{aligned}$$

In the following, we estimate $E_R(x)$ in (4-10) and $\tilde{E}_R(x)$ in (4-12). If χ is not real, $E_R(x)=\tilde{E}_R(x)=0$. If χ is real, by (2-4), on $\mathcal{H}\big(\tfrac{1}{2},\delta\big)$, we write

$$F(2s,\chi^2)=-\log\big(s-\tfrac{1}{2}\big)+H_2(s). \qquad (5\text{-}9)$$

On $\mathcal{H}\big(\frac{1}{2},\delta\big)$, $|H_2(s)| = O(1)$. By (2-6), we have

$$|E_R(x)| \ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n}\in S_m^{(k)}} \left| \int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} F^m(s,\chi) F(\boldsymbol{n}s,\chi) \frac{x^s}{s}\, ds \right|. \tag{5-10}$$

For each $0 \le m \le k-2$, we write

$$F^m(s,\chi)F(\boldsymbol{n}s,\chi) = F^m(s,\chi)F^{m'}(2s,\chi^2)G_{\boldsymbol{n}}(s),$$

where $m+2m' \le k$, and $G_{\boldsymbol{n}}(s) = O(1)$ on $\mathcal{H}(\frac{1}{2},\delta)$. Thus, by (4-5), (5-9), and Lemma 17,

$$\begin{aligned}
&\left| \int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} F^m(s,\chi) F(\boldsymbol{n}s,\chi) \frac{x^s}{s}\, ds \right| \\
&\qquad \ll \left| \int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} \big(\log\big(s-\tfrac{1}{2}\big) + H_B(s)\big)^m \big(\log\big(s-\tfrac{1}{2}\big) - H_2(s)\big)^{m'} G_{\boldsymbol{n}}(s) \frac{x^s}{s}\, ds \right| \\
&\qquad \ll \sum_{j_1=0}^{m} \sum_{j_2=0}^{m'} \left| \int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} \big(\log\big(s-\tfrac{1}{2}\big)\big)^{m+m'-j_1-j_2} (H_B(s))^{j_1} (H_2(s))^{j_2} G_{\boldsymbol{n}}(s) \frac{x^s}{s}\, ds \right| \\
&\qquad \ll \sum_{j_1=0}^{m} \sum_{j_2=0}^{m'} \frac{\sqrt{x}}{\log x} (\log\log x)^{m+m'-j_1-j_2-1} \\
&\qquad \ll_k \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}.
\end{aligned} \tag{5-11}$$

In the last step, we used the conditions $0 \le m \le k-2$ and $m+2m' \le k$.

Combining (5-10) and (5-11), we deduce that

$$|E_R(x)| \ll_k \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}. \tag{5-12}$$

Similarly, if χ is real, by (2-8), we write

$$\tilde{F}(s,\chi;2) = -\log\big(s-\tfrac{1}{2}\big) + \tilde{H}_2(s), \tag{5-13}$$

where $\tilde{H}_2(s) = O(1)$ on $\mathcal{H}\big(\frac{1}{2},\delta\big)$. Using a similar argument as above, by (4-6), (5-13), and Lemma 17, we have

$$|\tilde{E}_R(x)| \ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n}\in S_m^{(k)}} \left| \int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} \tilde{F}^m(s,\chi) \tilde{F}(\boldsymbol{n}s,\chi) \frac{x^s}{s}\, ds \right| \ll_k \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}. \tag{5-14}$$

By (4-10), (4-14), (5-7), and (5-12), we get

$$I_B(x) = \frac{(-1)^k\sqrt{x}}{2^{k-1}\log x} \left\{ k(\log\log x)^{k-1} + \sum_{j=2}^{k} \binom{k}{j} (\log\log x)^{k-j} \frac{1}{\Gamma_j(0)} \right\} + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x} \right).$$

Then, by (4-2),

$$\left|\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right| \ll_k (\log\log x)^{k-2}.$$

Hence,

$$I_B(x) = \frac{(-1)^k k}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1} + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right). \tag{5-15}$$

Similarly, by (4-12), (4-15), (5-8), and (5-14), we have

$$\tilde{I}_B(x) = \frac{k}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1} + O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right). \tag{5-16}$$

This completes the proof of Lemma 13. □

6. Average order of the error terms

In Section 6A and Section 6B, we examine the error terms from the Hankel contours around zeros and give the proof of Lemma 14. In Section 6C, we examine the lower order sum and the error from the truncation, and give the proof of Lemma 15.

6A. *Error terms from the Hankel contours around zeros.* In this section, we give the proof of Lemma 14. The following lemma gives an average estimate for the integral over Hankel contours around zeros, which is the key lemma for our proof.

Lemma 18. *Let ρ be a zero of $L(s,\chi)$. Assume the function $g(s) \ll (\log|\gamma|)^c$ on $\mathcal{H}(\rho,\delta)$ for some constant $c \ge 0$, and*

$$H_\rho(s) = \sum_{0<|\gamma'-\gamma|\le 1}\log(s-\rho') + O(\log|\gamma|) \quad \textit{on } \mathcal{H}(\rho,\delta). \tag{6-1}$$

For any integers $m, n \ge 0$, denote

$$E(x;\rho) := \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^m (H_\rho(s))^n g(s)\frac{x^{s-\rho}}{s}\,ds.$$

Then, for $T = T(x)$, we have

$$\int_2^Y \left| y \sum_{|\gamma|\le T(e^y)} e^{i\gamma y} E(e^y;\rho)\right|^2 dy = o(Y(\log Y)^{2m+2n-2}).$$

We will give the proof of Lemma 18 in next subsection. We use it in this section to prove Lemma 14 first.

Proof of Lemma 14. By (4-20), we have

$$|\Sigma_1(x;\chi)|^2 = \left|\log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{M_\rho}(x)\right|^2 \ll \sum_{j=1}^{k}\left|\log x \sum_{|\gamma|\le T} x^{i\gamma} E_{\rho,j}(x)\right|^2, \tag{6-2}$$

where

$$E_{\rho,j}(x) = \int_{\mathscr{H}(\rho,\delta)} (\log(s-\rho))^{k-j} (H_\rho(s))^j \frac{x^{s-\rho}}{s}\, ds.$$

By Lemma 18, take $m = k - j$, $n = j$, and $g(s) \equiv 1$, (i.e., $c = 0$),

$$\int_2^Y \Big| y \sum_{|\gamma| \le T(e^y)} e^{i\gamma y} E_{\rho,j}(e^y) \Big|^2 dy = o(Y (\log Y)^{2k-2}).$$

Thus,

$$\int_2^Y |\Sigma_1(e^y; \chi)|^2\, dy = o(Y(\log Y)^{2k-2}). \tag{6-3}$$

By definition (4-9) and (4-21), we have

$$\begin{aligned} |\Sigma_2(x;\chi)|^2 &\ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n} \in S_m^{(k)}} \Big| \log x \sum_{|\gamma| \le T} x^{i\gamma} \int_{\mathscr{H}(\rho,\delta)} F^m(s,\chi) F(\boldsymbol{n}s,\chi) \frac{x^{s-\rho}}{s}\, ds \Big|^2 \\ &\ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n} \in S_m^{(k)}} \sum_{j=0}^{m} \Big| \log x \sum_{|\gamma| \le T} x^{i\gamma} E_{m,j}(x,\chi;\boldsymbol{n}) \Big|^2, \end{aligned} \tag{6-4}$$

where

$$E_{m,j}(x,\chi;\boldsymbol{n}) = \int_{\mathscr{H}(\rho,\delta)} (\log(s-\rho))^{m-j} (H_\rho(s))^j F(\boldsymbol{n}s,\chi) \frac{x^{s-\rho}}{s}\, ds.$$

Since on $\mathscr{H}(\rho,\delta)$, we know $F(\boldsymbol{n}s,\chi) = O((\log|\gamma|)^{\frac{1}{2}(k-m)})$, by Lemma 18, we get

$$\int_2^Y \Big| y \sum_{|\gamma| \le T(e^y)} e^{i\gamma y} E_{m,j}(e^y,\chi;\boldsymbol{n}) \Big|^2 dy = o(Y(\log Y)^{2m-2}).$$

Hence, by (6-4), we deduce that

$$\int_2^Y |\Sigma_2(e^y; \chi)|^2\, dy = o(Y(\log Y)^{2k-2}). \tag{6-5}$$

Combining (6-3) and (6-5), we get the first formula in Lemma 14.

For $\widetilde{\Sigma}_1(x;\chi)$ and $\widetilde{\Sigma}_2(x,\chi)$, by (4-22), using a similar argument with Lemma 18,

$$\int_2^Y |\widetilde{\Sigma}_1(e^y;\chi)|^2\, dy \ll \sum_{j=1}^{k} \int_2^Y \Big| y \sum_{|\gamma| \le T(e^y)} e^{i\gamma y} \tilde{E}_{\rho,j}(e^y) \Big|^2 dy = o(Y(\log Y)^{2k-2}), \tag{6-6}$$

where

$$\tilde{E}_{\rho,j}(x) = \int_{\mathscr{H}(\rho,\delta)} (\log(s-\rho))^{k-j} (\tilde{H}_\rho(s))^j \frac{x^{s-\rho}}{s}\, ds.$$

Similarly, by (4-23) and Lemma 18,

$$\int_2^Y |\widetilde{\Sigma}_2(e^y;\chi)|^2\, dy \ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n} \in S_m^{(k)}} \sum_{j=0}^{m} \int_2^Y \Big| y \sum_{|\gamma| \le T(e^y)} e^{i\gamma y} \tilde{E}_{m,j}(e^y,\chi;\boldsymbol{n}) \Big|^2 dy = o(Y(\log Y)^{2k-2}), \tag{6-7}$$

where

$$\tilde{E}_{m,j}(x,\chi;\boldsymbol{n}) = \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^{m-j} (\tilde{H}_\rho(s))^j \tilde{F}(\boldsymbol{n}s,\chi) \frac{x^{s-\rho}}{s}\, ds.$$

Combining (6-6) and (6-7), we get the second formula in Lemma 14. □

6B. *Estimates for integral over Hankel contours around zeros.* We need the following results to finish the proof of Lemma 18.

Lemma 19 [Ford and Sneed 2010, Lemma 2.4]. *Assume $L\big(\frac{1}{2},\chi\big)\neq 0$. For $A\geq 0$ and real $l\geq 0$,*

$$\sum_{\substack{|\gamma_1|,|\gamma_2|\geq A\\ |\gamma_1-\gamma_2|\geq 1}} \frac{\log^l(|\gamma_1|+3)\log^l(|\gamma_2|+3)}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \ll_l \frac{(\log(A+3))^{2l+3}}{A+1}.$$

Lemma 20. *For any integers N, $j\geq 1$, and $0<|\delta_n|\leq 1$, we have*

$$\int_0^\delta \left|\sum_{n=1}^N \log(\sigma+i\delta_n)\right|^j x^{-\sigma}\, d\sigma \ll_j \frac{1}{\log x}\left\{\min\left(N\log\log x, \log\frac{1}{\Delta_N}\right)+N\pi\right\}^j,$$

where $\Delta_N=\prod_{n=1}^N|\delta_n|$.

Proof. Let I denote the integral in the lemma. We consider two cases: $\Delta_N\geq\big(\frac{1}{\log x}\big)^N$ and $\Delta_N<\big(\frac{1}{\log x}\big)^N$.
(1) If $\Delta_N\geq(1/\log x)^N$, we have

$$I \ll \left(\log\frac{1}{\Delta_N}+N\pi\right)^j \int_0^\delta x^{-\sigma}\, d\sigma \ll \frac{1}{\log x}\left(\log\frac{1}{\Delta_N}+N\pi\right)^j \ll \frac{1}{\log x}(N\log\log x+N\pi)^j. \quad (6\text{-}8)$$

(2) If $\Delta_N<(1/\log x)^N$, we write

$$I=\left(\int_0^{(\Delta_N)^{1/N}}+\int_{(\Delta_N)^{1/N}}^{1/\log x}+\int_{1/\log x}^\delta\right)\left|\sum_{n=1}^N\log(\sigma+i\delta_n)\right|^j x^{-\sigma}\, d\sigma =: I_1+I_2+I_3. \quad (6\text{-}9)$$

First, we estimate I_1,

$$I_1 \ll \left(\log\frac{1}{\Delta_N}+N\pi\right)^j \int_0^{(\Delta_N)^{1/N}} x^{-\sigma}\, d\sigma \ll (\Delta_N)^{1/N}\left(\log\frac{1}{\Delta_N}+N\pi\right)^j. \quad (6\text{-}10)$$

For $0<t<1$, consider the function $f(t)=t^{1/N}\big(\log\frac{1}{t}+N\pi\big)^j$. Since the critical point of $f(t)$ is $t=e^{N(\pi-1)}>1$, by (6-10), we have

$$I_1 \ll f\left(\frac{1}{(\log x)^N}\right)=\frac{1}{\log x}(N\log\log x+N\pi)^j \ll \frac{1}{\log x}\left(\log\frac{1}{\Delta_N}+N\pi\right)^j. \quad (6\text{-}11)$$

Next, we estimate I_3. Using the change of variable $\sigma \log x = t$, we get

$$\begin{aligned}
I_3 &\ll \int_{1/\log x}^{\delta} \left(N \log \frac{1}{\sigma} + N\pi\right)^j x^{-\sigma}\, d\sigma \\
&= \frac{1}{\log x} \int_1^{\delta \log x} (N \log\log x - N \log t + N\pi)^j e^{-t}\, dt \\
&= \frac{N^j}{\log x} \sum_{l=0}^{j} \binom{j}{l} (\log\log x + \pi)^{j-l} \int_1^{\delta \log x} (-\log t)^l e^{-t}\, dt \\
&\ll_j \frac{N^j}{\log x} \sum_{l=0}^{j} (\log\log x + \pi)^{j-l} \int_1^{\infty} \frac{t^l}{e^t}\, dt \\
&\ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \\
&\ll \frac{1}{\log x} \left(\log \frac{1}{\Delta_N} + N\pi\right)^j.
\end{aligned} \tag{6-12}$$

For I_2, similar to I_3, using the change of variable $\sigma \log x = t$, we get

$$\begin{aligned}
I_2 &\ll \int_{(\Delta_N)^{1/N}}^{1/\log x} \left(N \log \frac{1}{\sigma} + N\pi\right)^j x^{-\sigma}\, d\sigma \\
&= \frac{1}{\log x} \int_{(\Delta_N)^{1/N} \log x}^{1} (N \log\log x - N \log t + N\pi)^j e^{-t}\, dt \\
&= \frac{N^j}{\log x} \sum_{l=0}^{j} \binom{j}{l} (\log\log x + \pi)^{j-l} \int_{(\Delta_N)^{1/N} \log x}^{1} (-\log t)^l e^{-t}\, dt \quad \left(t \to \frac{1}{e^t}\right) \\
&\ll_j \frac{N^j}{\log x} \sum_{l=0}^{j} (\log\log x + \pi)^{j-l} \int_0^{\infty} \frac{t^l}{e^t}\, dt \\
&\ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \\
&\ll \frac{1}{\log x} \left(\log \frac{1}{\Delta_N} + N\pi\right)^j.
\end{aligned} \tag{6-13}$$

Combining (6-11), (6-12), (6-13), with (6-9), we get

$$I \ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \ll_j \frac{1}{\log x} \left(\log \frac{1}{\Delta_N} + N\pi\right)^j. \tag{6-14}$$

By (6-8) and (6-14), we get the conclusion of this lemma. □

In the following, we use the above lemmas to prove Lemma 18.

Proof of Lemma 18. If $m = 0$, $E(x;\rho) = 0$ and hence the integral is 0. In the following, we assume $m \ge 1$. Let Γ_ρ represent the circle in the Hankel contour $\mathscr{H}(\rho,\delta)$. Then,

$$\begin{aligned}
E(x;\rho) &= \int_{\mathscr{H}(\rho,\delta)} (\log(s-\rho))^m (H_\rho(s))^n g(s) \frac{x^{s-\rho}}{s}\, ds \\
&= \int_{r_\rho}^{\delta} \big((\log\sigma - i\pi)^m - (\log\sigma + i\pi)^m\big)\big(H_\rho\big(\tfrac12 - \sigma + i\gamma\big)\big)^n g\big(\tfrac12 - \sigma + i\gamma\big) \frac{x^{-\sigma}}{\frac12 - \sigma + i\gamma}\, d\sigma \\
&\qquad\qquad + \int_{\Gamma_\rho} (\log(s-\rho))^m (H_\rho(s))^n g(s) \frac{x^{s-\rho}}{s}\, ds \\
&=: E_h(x;\rho) + E_r(x;\rho).
\end{aligned} \tag{6-15}$$

For the second integral in (6-15), since $r_\rho \le \frac{1}{x}$, by Lemma 11,

$$\begin{aligned}
|E_r(x;\rho)| &\ll \frac{(\log|\gamma|)^c r_\rho x^{r_\rho}}{|\gamma|}\left(\log\frac{1}{r_\rho} + \pi\right)^m \left(\sum_{0<|\gamma-\gamma'|\le 1} \log\left(\frac{1}{|\gamma'-\gamma| - r_\rho}\right) + O(\log|\gamma|)\right)^n \\
&\ll \frac{(\log|\gamma|)^c r_\rho x^{r_\rho}}{|\gamma|}\left(\log\frac{1}{r_\rho} + \pi\right)^m (\log|\gamma|)^n \left(\log\left(\frac{1}{r_\rho}\right) + O(1)\right)^n \\
&\ll \frac{(\log|\gamma|)^{n+c}}{|\gamma|} \frac{(\log(1/r_\rho) + \pi)^{m+n}}{1/r_\rho} \ll \frac{(\log|\gamma|)^{n+c}}{|\gamma|} \frac{1}{x^{1-\epsilon}}.
\end{aligned} \tag{6-16}$$

Denote

$$\Sigma(x;\mathrm{g}) := \left|\sum_{|\gamma|\le T} x^{i\gamma} E(x;\rho)\right|^2 \ll \left|\sum_{|\gamma|\le T} x^{i\gamma} E_h(x;\rho)\right|^2 + \left|\sum_{|\gamma|\le T} x^{i\gamma} E_r(x;\rho)\right|^2. \tag{6-17}$$

By (6-16), and $T(x) \ll x^2$, we get

$$\left|\sum_{|\gamma|\le T} x^{i\gamma} E_r(x;\rho)\right|^2 \ll \frac{1}{x^{2-\epsilon}}\left(\sum_{|\gamma|\le T(x)} \frac{(\log|\gamma|)^{n+c}}{|\gamma|}\right)^2 \ll \frac{1}{x^{2-\epsilon}}. \tag{6-18}$$

For the first sum in (6-17),

$$\left|\sum_{|\gamma|\le T} x^{i\gamma} E_h(x;\rho)\right|^2 = \Bigg(\sum_{\substack{|\gamma_1-\gamma_2|\le 1\\ |\gamma_1|,|\gamma_2|\le T}} + \sum_{\substack{|\gamma_1-\gamma_2|>1\\ |\gamma_1|,|\gamma_2|\le T}}\Bigg) x^{i(\gamma_1-\gamma_2)} E_h(x;\rho_1) E_h(x;\bar\rho_2) =: \Sigma_1(x;\mathrm{g}) + \Sigma_2(x;\mathrm{g}).$$

By (6-15),

$$|E_h(x;\rho)| \ll \frac{(\log|\gamma|)^c}{|\gamma|} \sum_{j=1}^{m} \int_0^{\delta} |\log\sigma|^{m-j} \left|H_\rho\big(\tfrac12 - \sigma + i\gamma\big)\right|^n x^{-\sigma}\, d\sigma. \tag{6-19}$$

Let

$$S_j(x) := \int_0^{\delta} |\log\sigma|^{m-j} \left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^n x^{-\sigma}\, d\sigma$$
$$\le \left(\int_0^{\delta} |\log\sigma|^{2(m-j)} x^{-\sigma}\, d\sigma\right)^{\frac{1}{2}} \left(\int_0^{\delta} \left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^{2n} x^{-\sigma}\, d\sigma\right)^{\frac{1}{2}}. \tag{6-20}$$

By (5-5) in the proof of Lemma 16,

$$\int_0^{\delta} |\log\sigma|^{2(m-j)} x^{-\sigma}\, d\sigma \ll \frac{(\log\log x)^{2(m-j)}}{\log x}. \tag{6-21}$$

By condition (6-1) and the Cauchy–Schwarz inequality,

$$\left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^{2n} \ll \left|\sum_{0<|\gamma'-\gamma|\le 1} \log(\sigma+i(\gamma'-\gamma))\right|^{2n} + (\log|\gamma|)^{2n}.$$

Then, by Lemma 20,

$$\int_0^{\delta} \left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^{2n} x^{-\sigma}\, d\sigma \ll \frac{(M_\gamma(x))^{2n} + (\log|\gamma|)^{2n}}{\log x}, \tag{6-22}$$

where $M_\gamma(x) = \min(N(\gamma)\log\log x, \log 1/\Delta_{N(\gamma)})$, $N(\gamma)$ is the number of zeros γ' in the range $0 < |\gamma'-\gamma| \le 1$, and $\Delta_{N(\gamma)} = \prod_{0<|\gamma'-\gamma|\le 1} |\gamma'-\gamma|$.

Thus, by (6-21) and (6-22),

$$S_j(x) \ll \frac{(\log\log x)^{m-j}}{\log x}((M_\gamma(x))^n + (\log|\gamma|)^n).$$

Substituting this into (6-19), we get

$$|E_h(x;\rho)| \ll \frac{(\log|\gamma|)^c}{|\gamma|} \sum_{j=1}^{m} \frac{(\log\log x)^{m-j}}{\log x}((M_\gamma(x))^n + (\log|\gamma|)^n)$$
$$\ll \frac{(\log|\gamma|)^c}{|\gamma|} \frac{(\log\log x)^{m-1}}{\log x}((M_\gamma(x))^n + (\log|\gamma|)^n). \tag{6-23}$$

Then, by Lemma 11, we have

$$|\Sigma_1(x;\mathrm{g})| \ll \sum_{|\gamma|\le T} \log(|\gamma|)\left(\max_{|\gamma'-\gamma|<1} |E_h(x;\rho')|\right)^2$$
$$\ll \frac{(\log\log x)^{2(m-1)}}{\log^2 x} \sum_{\gamma} \frac{(\log|\gamma|)^{2c}}{|\gamma|^2}((M_\gamma(x))^{2n} + (\log|\gamma|)^{2n})$$
$$= \frac{(\log\log x)^{2m+2n-2}}{\log^2 x} o(1). \tag{6-24}$$

Thus, for each positive integer l,

$$\int_{2^l}^{2^{l+1}} \Sigma_1(e^y;\mathrm{g})\, dy = o\left(\frac{l^{2m+2n-2}}{2^l}\right). \tag{6-25}$$

In the following, we examine $\Sigma_2(x;\mathrm{g})$. By (6-15),

$$\Sigma_2(x;\mathrm{g})=\sum_{\substack{|\gamma_1-\gamma_2|>1\\|\gamma_1|,|\gamma_2|\le T}} x^{i(\gamma_1-\gamma_2)}E_h(x;\rho_1)E_h(x;\bar\rho_2). \tag{6-26}$$

For $e^{2^l}\le x\le e^{2^{l+1}}$, $T=T(x)=T_{l'}$ is a constant, and so we define

$$J(x;\mathrm{g}):=\sum_{\substack{|\gamma_1-\gamma_2|>1\\|\gamma_1|,|\gamma_2|\le T_{l'}}} x^{i(\gamma_1-\gamma_2)}\int_{r_{\rho_1}}^{\delta}\int_{r_{\bar\rho_2}}^{\delta} R_{\rho_1}(\sigma_1;x)R_{\bar\rho_2}(\sigma_2;x)\frac{d\sigma_1\,d\sigma_2}{i(\gamma_1-\gamma_2)-(\sigma_1+\sigma_2)}, \tag{6-27}$$

where

$$R_\rho(\sigma;x)=\big((\log\sigma-i\pi)^m-(\log\sigma+i\pi)^m\big)H_\rho^n\big(\tfrac12-\sigma+i\gamma\big)\frac{g\big(\frac12-\sigma+i\gamma\big)x^{-\sigma}}{\frac12-\sigma+i\gamma}.$$

Thus,

$$\int_{e^{2^l}}^{e^{2^{l+1}}}\sum_{\substack{|\gamma_1-\gamma_2|>1\\|\gamma_1|,|\gamma_2|\le T_{l'}}} x^{i(\gamma_1-\gamma_2)}E_h(x;\rho_1)E_h(x;\bar\rho_2)\frac{dx}{x}=J(e^{2^{l+1}};\mathrm{g})-J(e^{2^l};\mathrm{g}). \tag{6-28}$$

By (6-27), (6-19), and (6-23), and Lemma 19, for $e^{2^l}\le x\le e^{2^{l+1}}$

$$\begin{aligned}|J(x;\mathrm{g})|&\ll\sum_{|\gamma_1-\gamma_2|>1}\frac{(\log|\gamma_1|)^c(\log|\gamma_2|)^c}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\bigg(\frac{(\log\log x)^{m-1}}{\log x}\bigg)^2((M_{\gamma_1}(x))^n+(\log|\gamma_1|)^n)((M_{\gamma_2}(x))^n+(\log|\gamma_2|)^n)\\
&\ll\frac{(\log\log x)^{2m+2n-2}}{\log^2 x}\sum_{|\gamma_1-\gamma_2|>1}\frac{(\log|\gamma_1|)^{n+c}(\log|\gamma_2|)^{n+c}}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\\
&\ll\frac{(\log\log x)^{2m+2n-2}}{\log^2 x}.\end{aligned} \tag{6-29}$$

Hence, by (6-26), (6-28), and (6-29), we get, for any positive integer l,

$$\int_{2^l}^{2^{l+1}}\Sigma_2(e^y;\mathrm{g})\,dy=o\bigg(\frac{l^{2m+2n-2}}{2^l}\bigg). \tag{6-30}$$

Therefore, by (6-18), (6-25) and (6-30),

$$\int_2^Y\bigg|y\sum_{|\gamma|\le T(e^y)}e^{i\gamma y}E(e^y;\rho)\bigg|^2dy\ll\sum_{l\le\frac{\log Y}{\log 2}+1}2^{2l}\int_{2^l}^{2^{l+1}}\Sigma(e^y;\mathrm{g})\,dy \tag{6-31}$$

$$\ll 1+\sum_{l\le\frac{\log Y}{\log 2}+1}2^{2l}\int_{2^l}^{2^{l+1}}(\Sigma_1(e^y;\mathrm{g})+\Sigma_2(e^y;\mathrm{g}))\,dy \tag{6-32}$$

$$=o(Y(\log Y)^{2m+2n-2}). \tag{6-33}$$

This completes the proof of Lemma 18. □

6C. ***Lower order sum and error from the truncation.*** In this section, we examine the lower order sum and the error from the truncation by a fixed large T_0, and give the proof of Lemma 15.

Proof of Lemma 15. For the lower order sum, by (4-24), we have

$$\int_2^Y |S_1(e^y;\chi)|^2\,dy \ll \sum_{j=2}^{k}(\log Y)^{2k-2j}\int_2^Y \Bigl|\sum_{|\gamma|\le T(e^y)}\frac{e^{i\gamma y}}{\frac12+i\gamma}\Bigr|^2\,dy.$$

For the inner integral, by Lemma 11 and Lemma 19, and the definition of $T=T(x)$,

$$\begin{aligned}\int_2^Y \Bigl|\sum_{|\gamma|\le T(e^y)}\frac{e^{i\gamma y}}{\frac12+i\gamma}\Bigr|^2\,dy &\le \sum_{l\le\frac{\log Y}{\log 2}+1}\int_{2^l}^{2^{l+1}}\Biggl(\sum_{\substack{|\gamma_1-\gamma_2|\le 1\\ |\gamma_1|,|\gamma_2|\le T_{l'}}}+\sum_{\substack{|\gamma_1-\gamma_2|>1\\ |\gamma_1|,|\gamma_2|\le T_{l'}}}\Biggr)\frac{e^{i(\gamma_1-\gamma_2)y}}{(\frac12+i\gamma_1)(\frac12-i\gamma_2)}\,dy\\ &\ll \sum_{l\le\frac{\log Y}{\log 2}+1}\Biggl(2^l\sum_{\gamma}\frac{\log|\gamma|}{|\gamma|^2}+\sum_{\gamma_1,\gamma_2}\frac{1}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\Biggr)\ll Y.\end{aligned}$$

Thus,

$$\int_2^Y |S_1(e^y;\chi)|^2\,dy \ll \sum_{j=2}^{k} Y(\log Y)^{2k-2j} = o(Y(\log Y)^{2k-2}).$$

Next, we examine $S_2(x,T_0;\chi)$. For fixed T_0, let X_0 be the largest x such that $T=T(x)\le T_0$. Since $x\le T(x)\le 2x^2$, $\log X_0\asymp\log T_0$. By Lemma 11 and Lemma 19,

$$\begin{aligned}\int_2^Y |S_2(e^y,T_0;\chi)|^2\,dy &\le \int_2^{\log X_0}\Bigl|\sum_{|\gamma|\le T_0}\frac{1}{|\gamma|}\Bigr|^2\,dy+\int_{\log X_0}^{Y}\Bigl|\sum_{T_0\le|\gamma|\le T(e^y)}\frac{e^{i\gamma y}}{\frac12+i\gamma}\Bigr|^2\,dy\\ &\ll \log^5 T_0+\sum_{\frac{\log\log X_0}{\log 2}\le l\le\frac{\log Y}{\log 2}+1}\int_{2^l}^{2^{l+1}}\Biggl(\sum_{\substack{|\gamma_1-\gamma_2|\le 1\\ T_0\le|\gamma_1|,|\gamma_2|\le T_{l'}}}+\sum_{\substack{|\gamma_1-\gamma_2|>1\\ T_0\le|\gamma_1|,|\gamma_2|\le T_{l'}}}\Biggr)\frac{e^{i(\gamma_1-\gamma_2)y}}{(\frac12+i\gamma_1)(\frac12-i\gamma_2)}\,dy\\ &\ll \log^5 T_0+\sum_{\frac{\log\log X_0}{\log 2}\le l\le\frac{\log Y}{\log 2}+1}\Biggl(2^l\sum_{|\gamma|\ge T_0}\frac{\log|\gamma|}{|\gamma|^2}+\sum_{|\gamma_1|,|\gamma_2|\ge T_0}\frac{1}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\Biggr)\\ &\ll Y\frac{\log^2 T_0}{T_0}+\log Y\frac{\log^3 T_0}{T_0}+\log^5 T_0.\end{aligned}$$

This completes the proof of this lemma. □

7. Asymptotic formulas for the logarithmic densities

In this section, we give the proof of Theorem 4.

Proof of Theorem 4. For large q, Fiorilli and Martin [2013] gave an asymptotic formula for $\delta_{\Omega_1}(q;a,b)$. Lamzouri [2013] also derived such an asymptotic formula using another method. Here, we want to derive asymptotic formulas for $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$ for fixed q and large k.

Let a be a quadratic nonresidue $\bmod q$ and b be a quadratic residue $\bmod q$, and $(a,q)=(b,q)=1$. Letting $\lambda_k = 1/2^{k-1}$, similar to formula (2.10) of [Fiorilli and Martin 2013], we have, under $\mathrm{ERH_q}$ and $\mathrm{LI_q}$,

$$\delta_{\Omega_k}(q;a,b) = \frac{1}{2} + \frac{(-1)^k}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\lambda_k(N(q;a)-N(q;b))x}{x}\Phi_{q;a,b}(x)\,dx.$$

Noting that $N(q,a)-N(q,b) = -A(q)$,

$$\delta_{\Omega_k}(q;a,b) = \frac{1}{2} + \frac{(-1)^{k-1}}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx. \tag{7-1}$$

For any $\epsilon > 0$,

$$\int_{-\infty}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx = \left(\int_{-\infty}^{1/\lambda_k^{\epsilon}} + \int_{-1/\lambda_k^{\epsilon}}^{1/\lambda_k^{\epsilon}} + \int_{1/\lambda_k^{\epsilon}}^{\infty}\right)\frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx. \tag{7-2}$$

By Proposition 2.17 in [Fiorilli and Martin 2013], $|\Phi_{q;a,b}(t)| \le e^{-0.0454\phi(q)t}$ for $t \ge 200$. So for large enough k,

$$\int_{1/\lambda_k^{\epsilon}}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx \ll \lambda_k \int_{1/\lambda_k^{\epsilon}}^{\infty} e^{-0.0454\phi(q)x}\,dx \ll_{q,J,\epsilon} \lambda_k^J, \quad \text{for any } J > 0. \tag{7-3}$$

The integral over $x \le -1/\lambda_k^{\epsilon}$ is also bounded by λ_k^J.

By Lemma 2.22 in [Fiorilli and Martin 2013], for each nonnegative integer K and real number $C > 1$, we have, uniformly for $|z| \le C$,

$$\frac{\sin z}{z} = \sum_{j=0}^{K}(-1)^j \frac{z^{2j}}{(2j+1)!} + O_{C,K}(|z|^{2K+2}).$$

Thus, the second integral in (7-2) is equal to

$$\begin{aligned}\lambda_k A(q)&\int_{-1/\lambda_k^{\epsilon}}^{1/\lambda_k^{\epsilon}} \frac{\sin(\lambda_k A(q)x)}{\lambda_k A(q)x}\Phi_{q;a,b}(x)\,dx \\ &= \sum_{j=0}^{K}\lambda_k^{2j+1}\frac{(-1)^j A(q)^{2j+1}}{(2j+1)!}\int_{-1/\lambda_k^{\epsilon}}^{1/\lambda_k^{\epsilon}} x^{2j}\Phi_{q;a,b}(x)\,dx + O_{q,K}(\lambda_k^{2K+3-\epsilon}) \\ &= \sum_{j=0}^{K}\lambda_k^{2j+1}\frac{(-1)^j A(q)^{2j+1}}{(2j+1)!}\int_{-\infty}^{\infty} x^{2j}\Phi_{q;a,b}(x)\,dx + O_{q,K,\epsilon}(\lambda_k^{2K+3-\epsilon}). \end{aligned}\tag{7-4}$$

Combining (7-1), (7-3), and (7-4), we get the asymptotic formula (1-4) for $\delta_{\Omega_k}(q;a,b)$. Similarly, or by the results in Theorem 3, we have the asymptotic formula (1-5) for $\delta_{\omega_k}(q;a,b)$. □

8. The source of main terms and proof of Lemma 12

In this section, we give the proof of the main lemma we used for extracting out the bias terms and oscillating terms from the integrals over Hankel contours.

Let $\mathcal{H}(0, X)$ be the truncated Hankel contour surrounding 0 with radius r. Lau and Wu [2002] proved the following lemma.

Lemma 21 [Lau and Wu 2002, Lemma 5]. *For $X > 1$, $z \in \mathbb{C}$ and $j \in \mathbb{Z}^+$, we have*

$$\frac{1}{2\pi i}\int_{\mathcal{H}(0,X)} w^{-z}(\log w)^j e^w\, dw = (-1)^j \frac{d^j}{dz^j}\left(\frac{1}{\Gamma(z)}\right) + E_{j,z}(X),$$

where

$$|E_{j,z}(X)| \le \frac{e^{\pi|\Im(z)|}}{2\pi}\int_X^\infty \frac{(\log t+\pi)^j}{t^{\Re(z)}e^t}\, dt.$$

Proof of Lemma 12. We have the equality

$$\frac{1}{s} = \frac{1}{a} + \frac{a-s}{a^2} + \frac{(a-s)^2}{a^2 s}.$$

With the above equality, we write the integral in the lemma as

$$\frac{1}{2\pi i}\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\left(\frac{1}{a} + \frac{a-s}{a^2} + \frac{(a-s)^2}{a^2 s}\right)x^s\, ds =: I_1 + I_2 + I_3.$$

For I_3, using Lemma 16, we get

$$\begin{aligned}
&\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\frac{(a-s)^2}{a^2 s}x^s\, ds \\
&\le \left|\int_r^\delta ((\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k)\sigma^2 x^{-\sigma}\frac{x^a}{a^2(a-\sigma)}\, d\sigma\right| + \int_{-\pi}^{\pi} x^{\Re(a)+r}\left(\log\tfrac{1}{r}+\pi\right)^k \frac{r^2}{|a|^2|\Re(a)-r|}r, d\alpha \\
&\ll \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\left(\int_0^\delta |(\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k|\sigma^2 x^{-\sigma}\, d\sigma + \frac{(\log\frac{1}{r}+\pi)^k}{(1/r)^3}\right) \\
&\ll_k \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\left(\frac{(\log\log x)^{k-1}}{(\log x)^3} + \frac{1}{x^{3-\epsilon}}\right) \\
&\ll_k \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\frac{(\log\log x)^{k-1}}{(\log x)^3}.
\end{aligned} \tag{8-1}$$

We estimate I_2 similarly. By Lemma 16,

$$\begin{aligned}
&\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\frac{a-s}{a^2}x^s\, ds \\
&\qquad \le \left|\int_r^\delta ((\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k)\sigma x^{-\sigma}\frac{x^a}{a^2}\, d\sigma\right| + \int_{-\pi}^{\pi} x^{\Re(a)+r}\left(\log\tfrac{1}{r}+\pi\right)^k \frac{r}{|a|^2}r\, d\alpha \\
&\qquad \ll \frac{|x^a|}{|a|^2}\left(\int_0^\delta |(\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k|\sigma x^{-\sigma}\, d\sigma + \frac{(\log\frac{1}{r}+\pi)^k}{(1/r)^2}\right) \\
&\qquad \ll_k \frac{|x^a|}{|a|^2}\left(\frac{(\log\log x)^{k-1}}{(\log x)^2} + \frac{1}{x^{2-\epsilon}}\right) \ll_k \frac{|x^a|}{|a|^2}\frac{(\log\log x)^{k-1}}{(\log x)^2}.
\end{aligned} \tag{8-2}$$

For I_1, using change of variable $(s-a)\log x = w$, by Lemma 21, we get

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\frac{1}{\log x}\int_{\mathcal{H}(0,\delta\log x)} (\log w - \log\log x)^k \frac{x^a e^w}{a}\,dw \\
&= \frac{x^a}{a\log x}(-1)^k(\log\log x)^k \frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} e^w\,dw \\
&\quad + (-1)^{k-1}k\frac{x^a}{a\log x}(\log\log x)^{k-1}\frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} e^w\log w\,dw \\
&\quad + \frac{x^a}{a\log x}\sum_{j=2}^{k}\binom{k}{j}\frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} (-\log\log x)^{k-j}(\log w)^j e^w\,dw \\
&= \frac{(-1)^k x^a}{a\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\} \\
&\quad + \frac{x^a}{a\log x}\sum_{j=1}^{k}\binom{k}{j}E_{j,0}(\delta\log x)(-\log\log x)^{k-j}.
\end{aligned} \tag{8-3}$$

By Lemma 21,

$$|E_{j,0}(\delta\log x)| \le \frac{1}{2\pi}\int_{\delta\log x}^{\infty}\frac{(\log t+\pi)^j}{e^t}\,dt \ll_j e^{-\frac{1}{2}\delta\log x}\int_{\frac{1}{2}\delta\log x}^{\infty}\frac{(\log t)^j}{e^{t/2}}\,dt \ll_j x^{-\delta/2}.$$

Hence, we get

$$\left|\frac{x^a}{a\log x}\sum_{j=1}^{k}\binom{k}{j}E_{j,0}(\delta\log x)(-\log\log x)^{k-j}\right| \ll_k \frac{x^{\Re(a)}}{|a|\log x}\sum_{j=1}^{k}x^{-\delta/2}(\log\log x)^{k-j} \ll_k \frac{|x^{a-\delta/3}|}{|a|}. \tag{8-4}$$

Combining (8-1), (8-2), (8-3), and (8-4), we get the conclusion of Lemma 12. □

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