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Chebotarev sets

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1. Introduction. In this paper, we consider the following problem, and its generalizations:

Let p_n denote the nth prime, π the set of all primes, and P_{odd} the set consisting of every other prime:

$$P_{\mathrm{odd}} = \{p_n \in \pi \mid n \text{ odd}\} = \{2, 5, 11, 17, 23, \ldots\}. \tag{1.1}$$

Can the set P_{odd} be realized as a finite union of primes in residue classes, even if we are willing to allow a finite number of exceptions?

More generally, can we realize a given set of primes ideals in a number field K with rational density relative to the full set of primes ideals of K as a finite union of prime ideals that arise in the Chebotarev density theorem, i.e. of Frobenius conjugacy classes in the Galois groups of finite Galois extensions of K?

In the following we will identify the (positive) prime number p with the ideal that it generates in the ring of integers.

The natural instinct is that the set P_{odd} above cannot be realized in this manner. In fact, the cardinality of subsets of primes with density 1/2 is uncountable, for example we can pick subsets by performing a random coin flip at each prime, yet the number of residue classes (or, more generally, Frobenius conjugacy classes) is countable. Therefore, most subsets of primes will fail to arise from residue or conjugacy classes.

In this article we prove that P_{odd} cannot be realized as a finite union of primes in residue/conjugacy classes, even if we are willing to allow a finite number of exceptions.

We will show that the set P_{odd} is too 'quiet' relative to the set of all primes to arise from such arithmetic sets. Primes in progressions, or in

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Frobenius conjugacy classes, are quantifiably irregular, as a result of the non-trivial zeros of the L-functions that govern them.

1.1. Chebotarev sets. Let K be a number field, and let $\pi(K)$ denote the set of all non-zero prime ideals of K. Let L/K be a finite Galois extension of number fields with Galois group $\mathrm{Gal}(L/K) = G$. For a prime ideal $\mathfrak{p} \in \pi(K)$, unramified in L/K, let $(L/K, \mathfrak{p})$ denote the conjugacy class of Frobenius automorphisms of primes $\mathfrak{P} \in \pi(L)$ dividing $\mathfrak{p}$. If $C \subset G$ is a conjugacy class in G, let $\pi(L/K, C)$ denote the set of primes $\mathfrak{p} \in \pi(K)$ such that $\mathfrak{p}$ is unramified in L/K and $(L/K, \mathfrak{p})$ is equal to C.

For two sets S_1 and S_2 the symmetric difference $S_1 \triangle S_2$ is the set $(S_1 \setminus S_2) \cup (S_2 \setminus S_1)$. We will say two sets S_1 and S_2 are *equal up to finite sets* if $S_1 \triangle S_2$ is finite.

Definition 1. Call a set of primes $P \subseteq \pi(K)$ a *Chebotarev set* for K if there are finitely many finite Galois extensions L_i/K and conjugacy classes $C_i \subset \mathrm{Gal}(L_i/K)$ such that $P = \bigcup_i \pi(L_i/K, C_i)$ up to finite sets. That is, $P \triangle \bigcup_i \pi(L_i/K, C_i)$ is finite.

We allow the possibility that P is the empty set (i.e. the number of fields L_i is zero).

Suppose that $K \subseteq L' \subseteq L$ is a tower of finite extensions with L'/K and L/K both Galois. Then a conjugacy class in $\mathrm{Gal}(L'/K)$ can be lifted to a finite union of conjugacy classes in $\mathrm{Gal}(L/K)$. Hence P is a Chebotarev set for K if and only if there is a finite Galois extension L/K and a finite number of distinct conjugacy classes $C_i \subset \mathrm{Gal}(L/K)$ such that $P = \bigcup_i \pi(L/K, C_i)$ up to finite sets.

Given Chebotarev sets $P, Q \subseteq \pi(K)$ the following properties are consequences of the Chebotarev density theorem:

- the natural and Dirichlet densities $\delta(P)$ exist and are equal to
$$\frac{\sum_i |C_i|}{|G|};$$
- if $\delta(P) = 0$ or 1 then P is finite or co-finite;
- if P, Q are Chebotarev sets then so are $P \cup Q$, $P \cap Q$, $\pi(K) \setminus P$, $P \triangle Q$.

Thus, any subset of $\pi(K)$ with no density or irrational density cannot be a Chebotarev set. Also, any set of density 0 (resp. 1) which is infinite (resp. co-infinite) cannot be a Chebotarev set. So, for example, the set of (positive) primes congruent to 3 mod 4 constitute a Chebotarev set of $\mathbb{Q}$, but they are generators of an infinite set of prime ideals of density 0 when viewed in $K = \mathbb{Q}(i)$ and therefore do *not* form a Chebotarev set of K.

Our problem is to produce a set $P \subset \pi(\mathbb{Q})$ of positive rational density which is provably not a Chebotarev set.

For a subset $P \subset \pi(K)$ of prime ideals of K, let $P(x)$ denote the function which counts the number of elements in P and with absolute norm $\leq x$:

$$P(x) = \#\{\mathfrak{p} \in P \mid N_{K/\mathbb{Q}}(\mathfrak{p}) \leq x\}. \tag{1.2}$$

We also let

$$\pi(x, K) = \#\{\mathfrak{p} \in \pi(K) \mid N_{K/\mathbb{Q}}(\mathfrak{p}) \leq x\}. \tag{1.3}$$

We have the following theorem.

THEOREM 1. *Let $\beta \in \mathbb{Q}$ with $0 < \beta < 1$. Assume that $P \subset \pi(K)$ is a Chebotarev set of density β. Then*

$$P(x) - \beta\pi(x, K) = \Omega(x^{1/2}/\log x), \tag{1.4}$$

with the implied constant in the Ω depending on P.

Here, for real functions f, g with $g > 0$, we are using the Ω notation to mean the following: We say that $f(x) = \Omega(g(x))$ if $\limsup |f(x)|/g(x) > 0$, i.e. if there is a constant $c > 0$ and a sequence x_n with $x_n \to \infty$ such that $|f(x_n)| > cg(x_n)$.

COROLLARY 1. *The set P_{odd} is not a Chebotarev set.*

Proof. The set P_{odd} specified in the introduction has natural density (and hence Dirichlet density) $1/2$ in $\pi(\mathbb{Q})$. Because P_{odd} consists of every other prime, we have

$$P_{\mathrm{odd}}(x) - \pi(x)/2 = \begin{cases} 1/2 & \text{if } p_{2j-1} \leq x < p_{2j}, \\ 0 & \text{if } p_{2j} \leq x < p_{2j+1}. \end{cases} \tag{1.5}$$

Therefore Theorem 1 implies that P_{odd} cannot be realized as a Chebotarev set. ■

As we will explain in the proof of this theorem, the same conclusion holds if we replace the counting function $\beta\pi(x, K)$ by $\beta\,\mathrm{Li}(x)$, or with the counting function $sQ(x)$ of another Chebotarev set $Q \subseteq \pi(K)$ of non-zero density $\delta(Q)$, which is essentially distinct from P, so long as $s\delta(Q) = \beta$. By essentially distinct, we mean that the symmetric difference $P \triangle Q$ is infinite.

THEOREM 2. *Let $P \subset \pi(K)$ be a Chebotarev set of density β, $0 < \beta < 1$, and let $f(x)$ stand for $\beta\,\mathrm{Li}(x)$ or $sQ(x)$, as above. Then*

$$P(x) - f(x) = \Omega(x^{1/2}/\log x), \tag{1.6}$$

with the implied constant depending on P and f.

In a personal communication with the authors, Serre raised the question of whether our techniques allow us to address the size of a summatory function on prime ideals of K with complex valued weight function that is constant on Frobenius conjugacy classes of G, i.e. a sum of the form

$$\sum_{\substack{\mathfrak{p}:\, N\mathfrak{p}\le X \\ \mathfrak{p}\ \text{unramified}}} h(\mathfrak{p}) \tag{1.7}$$

with h complex valued and taking on finitely many values according to conjugacy classes of G. Such a summatory function can be expressed as a linear combination of counting functions $\pi(x, L/K, C)$.

More specifically, let $C_1, \ldots, C_r$ be the distinct conjugacy classes of G. Let $\eta_1, \ldots, \eta_r$ be complex numbers not all 0, let η be defined by

$$\eta := \sum_{j=1}^{r} \eta_j \delta(C_j) = \frac{1}{|G|} \sum_{j=1}^{r} \eta_j |C_j|, \tag{1.8}$$

and let $\pi(x, L/K, C)$ be defined by (3.2).

Theorem 3. *Let*

$$F(x) = \sum_{j=1}^{r} \eta_j \pi(x, L/K, C_j). \tag{1.9}$$

Then

$$F(x) - \eta\,\mathrm{Li}(x) = \Omega(x^{1/2}/\log x), \tag{1.10}$$

with the implied constant depending on the choice of the set $\{\eta_j\}$. *Furthermore, the same result holds if we replace* $\mathrm{Li}(x)$ *by* $\pi(x, K)$, *though one then needs the additional restriction that not all the* η_j *are equal.*

The key idea used to prove these theorems is that the functions on the left hand side of (1.4), (1.6), and (1.10) are discontinuous at an infinite number of values of x. Consequently, when expressed as a linear combination of explicit formulas, infinitely many of the non-trivial zeros of the relevant L-functions must survive. These zeros (in fact we only need one non-trivial zero to enter) are responsible for making these differences large on average, which we show by considering their mean square on a logarithmic scale.

Note that the statements of Theorems 1–3 do not assume the Generalized Riemann Hypothesis. In fact, if the GRH does not hold, stronger Ω results than these hold, hence we have stated these theorems unconditional on the GRH.

The precise statement of the Ω bound in the case that the GRH fails requires some discussion concerning the location of the zeros of the relevant

L-functions, and how these zeros interact upon taking certain linear combinations of the logarithmic derivatives of these L-functions. This discussion and the corresponding result can be found in Section 2.5 and in Theorem 6 at the end of Section 3.1.

Also observe that in our theorems we do not prove $\Omega_{\pm}$ results, i.e. we do not address the question of sign changes. Without further assumptions, such as linear independence of the non-trivial zeros over $\mathbb{Q}$ (other than those possibly occurring at $s = 1/2$), we cannot, in general, prove the existence of sign changes. For a discussion on issues related to sign changes see [7].

DEFINITION 2. Call a set of primes $P \subseteq \pi(K)$ an *almost Chebotarev set* for K if there is a Chebotarev set $Q \subseteq \pi(K)$ such that $P = Q$ up to sets of density zero. That is, $P \triangle Q$ has density zero.

It seems much more difficult to prove the existence of a set which is not almost Chebotarev—although we suspect that our example, P_{odd}, is one such set.

We conclude the introduction by noting that Serre studied 'Frobenian' (i.e. named differently than here) sets and functions in his paper [8] and book [9, Chapter 3], the latter in relation to the problem of counting the number of solutions modulo p to a system of polynomial equations. Lagarias defined a similar notion of Chebotarev sets in [5], also for studying solutions to polynomial congruences modulo p. See Lemma 3.1 in his paper for the equivalence of his definition to ours, though without allowing for finitely many exceptions.

2. The classical case.

In this section we consider the more classical situation of sets P of rational primes that are realized using residue classes. We will essentially establish Theorem 1 for the special case of residue classes, rather than Frobenius conjugacy classes.

The techniques that we develop will serve as a model, in Section 3, where we will modify our approach to the general setting of Chebotarev sets.

Assume that

$$P = P_0 \cup \bigcup_{j=1}^{r} \pi(q_j, a_j) \setminus P_1 \tag{2.1}$$

where P_0, P_1 consist of finitely many elements, i.e. the possible exceptions in excess and deficiency, and

$$\pi(q, a) = \{p \text{ prime} \mid p = a \bmod q\} \tag{2.2}$$

consists of rational primes in the residue class $a \bmod q$.

As noted in the introduction, there is a single positive integer q and distinct residue classes a_j mod q such that, after relabelling r as needed,

$$P = P_0 \cup \bigcup_{j=1}^{r} \pi(q, a_j) \setminus P_1. \tag{2.3}$$

We can also assume that $\gcd(a_j, q) = 1$, that P_0 is disjoint from $\bigcup_{j=1}^{r} \pi(q_j, a_j)$, and P_1 is contained within this union.

Next, we define, as usual,

$$\pi(x, q, a) = \#\{p \le x \mid p = a \bmod q\}. \tag{2.4}$$

From (2.3), we have, for x larger than all the elements of P_0 and P_1,

$$P(x) = \lambda + \sum_{j=1}^{r} \pi(x, q, a_j), \tag{2.5}$$

where

$$\lambda = |P_0| - |P_1|. \tag{2.6}$$

The prime number theorem states that

$$\pi(x) \sim \mathrm{Li}(x), \tag{2.7}$$

where

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}. \tag{2.8}$$

The prime number theorem for arithmetic progressions, proven by Hadamard and de la Vallée Poussin, asserts, for $\gcd(a, q) = 1$, that primes are equidistributed amongst the residue classes modulo q that are relatively prime to q:

$$\pi(x, q, a) \sim \pi(x)/\phi(q), \tag{2.9}$$

where $\phi(q) = \#\{a \bmod q \mid (a, q) = 1\}$. Therefore,

$$P(x) \sim \frac{r}{\phi(q)} \pi(x). \tag{2.10}$$

We also make the assumption that

$$r/\phi(q) < 1, \tag{2.11}$$

so that P omits a positive proportion of the primes.

2.1. Outline of proof. We now provide a summary of our proof of Theorem 1 in the case of residue classes. We begin with the classical formula (2.15) which uses Dirichlet characters to count prime powers in arithmetic progressions, weighted by the von Mangoldt function. A summation by parts, and isolating the squares and higher powers of primes, allows us to pass to counting primes in arithmetic progressions in formula (2.23). This leads to formula (2.12) for $P(x) - r\pi(x)/\phi(q)$ in Lemma 2.1.

We then apply the explicit formula, in Section 2.3, to derive equation (2.29) in Lemma 2.2. That lemma expresses $P(x)-r\pi(x)/\phi(q)$ in terms of non-trivial zeros of Dirichlet L-functions, along with two complicated, but innocuous, terms which are denoted by $A(x)$ and $B(x)$. We then exploit the fact that $P(x) - r\pi(x)/\phi(q)$ is discontinuous at the primes to show, in Lemma 2.3, that the sum over zeros is non-empty.

Having at least one non-trivial zero appear in (2.29), i.e. with $\alpha_\rho \neq 0$, guarantees the Ω bound of Theorem 1, whether the Generalized Riemann Hypothesis is false or not. The former case is considered in Theorem 4 of Section 2.5. The latter case is handled in Sections 2.6–2.8 by considering mean squares of $P(x) - r\pi(x)/\phi(q)$.

Finally, we generalize our results from the classical case of residue classes to Chebotarev sets in Section 3.

2.2. Counting primes in arithmetic progressions. Our first step is to derive the following lemma, which expresses $P(x) - r\pi(x)/\phi(q)$ in terms of Dirichlet characters.

LEMMA 2.1. *For x larger than all the elements of P_0 and P_1,*

$$
\begin{aligned}
(2.12)\qquad & P(x) - \frac{r}{\phi(q)}\pi(x) \\
&= \lambda + \sum_{j=1}^{r} \pi(x,q,a_j) - \frac{r}{\phi(q)}\pi(x) \\
&= \lambda + \sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} c_\chi \biggl(\frac{\psi(x,\chi)}{\log x} + \int_2^x \frac{\psi(t,\chi)}{t(\log t)^2}\,dt \biggr) + \frac{rR(x,1)}{\phi(q)} - \sum_{j=1}^{r} R(x,q,a_j) \\
&\quad - \frac{r}{\phi(q)} \biggl(\frac{\sum_{p^k\le x,\, p\mid q} \log p}{\log x} + \int_2^x \frac{\sum_{p^k\le t,\, p\mid q}\log p}{t(\log t)^2}\,dt \biggr),
\end{aligned}
$$

where λ is defined in (2.6)*, $R(x,1)$ and $R(x,q,a)$ are defined, below, in* (2.25) *and* (2.19)*, and*

$$c_\chi = \frac{1}{\phi(q)} \sum_{j=1}^{r} \bar\chi(a_j). \tag{2.13}$$

Proof. For a given q, let χ be a Dirichlet character modulo q. We will denote the principal character by χ_0.

As in the proof of Dirichlet's theorem, it is easier to count prime powers weighted by the von Mangoldt function than it is to simply count primes.

Thus, let

$$\psi(x,\chi) := \sum_{n\le x} \chi(n)\Lambda(n), \tag{2.14}$$

where $\Lambda(n) = \log p$ if $n = p^m$ for some $m \in \mathbb{Z}$, and $\Lambda(n) = 0$ otherwise.

We have

$$\begin{aligned}\psi(x,q,a) &:= \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n)\\ &= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{n\le x} \Lambda(n)\chi(n)\\ &= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a)\psi(x,\chi).\end{aligned} \tag{2.15}$$

The main contribution to $\psi(x,q,a)$ comes from the principal character:

$$\psi(x,\chi_0) = \sum_{\substack{p^k\le x\\ p\nmid q}} \log p = \psi(x) - \sum_{\substack{p^k\le x\\ p\mid q}} \log p. \tag{2.16}$$

Therefore

$$\psi(x,q,a) = \frac{1}{\phi(q)}\Big(\psi(x) + \sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} \bar{\chi}(a)\psi(x,\chi) - \sum_{\substack{p^k\le x\\ p\mid q}} \log p\Big). \tag{2.17}$$

We define

$$\begin{aligned}\Pi(x,q,a) &:= \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \frac{\Lambda(n)}{\log n} = \sum_{\substack{p^k\le x\\ p^k\equiv a \bmod q}} \frac{1}{k}\\ &= \pi(x,q,a) + R(x,q,a)\end{aligned} \tag{2.18}$$

with

$$R(x,q,a) = \sum_{\substack{p^k\le x\\ k\ge 2\\ p^k\equiv a \bmod q}} \frac{1}{k}. \tag{2.19}$$

Therefore,

$$\pi(x,q,a) = \Pi(x,q,a) - R(x,q,a). \tag{2.20}$$

Now, summing by parts yields

$$\Pi(x,q,a) = \frac{\psi(x,q,a)}{\log x} + \int_2^x \frac{\psi(t,q,a)}{t(\log t)^2}\,dt, \tag{2.21}$$

so that

$$\pi(x,q,a) = \frac{\psi(x,q,a)}{\log x} + \int_2^x \frac{\psi(t,q,a)}{t(\log t)^2}\,dt - R(x,q,a). \tag{2.22}$$

Substituting (2.17) into the above, we get

$$\begin{aligned}(2.23)\qquad & \pi(x,q,a)\\ &= \frac{1}{\phi(q)}\bigg(\frac{\psi(x)}{\log x}+\int_2^x \frac{\psi(t)}{t(\log t)^2}\,dt-\frac{\sum_{p^k\le x,\,p\mid q}\log p}{\log x}-\int_2^x\frac{\sum_{p^k\le t,\,p\mid q}\log p}{t(\log t)^2}\,dt\bigg)\\ &\quad+\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}}\bar{\chi}(a)\bigg(\frac{\psi(x,\chi)}{\log x}+\int_2^x\frac{\psi(t,\chi)}{t(\log t)^2}\,dt\bigg)-R(x,q,a).\end{aligned}$$

The special case of $q=1$ (and any value for a) of (2.22) is

$$\pi(x)=\frac{\psi(x)}{\log x}+\int_2^x\frac{\psi(t)}{t(\log t)^2}\,dt-R(x,1), \tag{2.24}$$

with

$$R(x,1)=\sum_{k\ge2}\frac{\pi(x^{1/k})}{k}. \tag{2.25}$$

With these formulas in hand, consider again the difference $P(x)-\frac{r}{\phi(q)}\pi(x)$. Subtracting $\frac{r}{\phi(q)}\pi(x)$ from (2.5), and then substituting (2.23) and (2.24), gives the lemma.

Note, that, on summing over r values of a_j and subtracting $r/\phi(q)$ times (2.24), we cancelled the main term

$$\frac{r}{\phi(q)}\bigg(\frac{\psi(x,\chi_0)}{\log x}+\int_2^x\frac{\psi(t,\chi_0)}{t(\log t)^2}\,dt\bigg).\ \blacksquare \tag{2.26}$$

2.3. Applying the explicit formula. We will show that the right hand side of (2.12) can get as large, in absolute value, as $\gg x^{1/2}/\log x$. This will be independent of the GRH. In fact, if GRH fails, the lower bound that we can prove is at least as large.

To accomplish this, we will write an explicit formula for (2.12) in terms of the zeros of the Dirichlet L-functions, $L(s,\chi)$, where χ runs over all Dirichlet characters for the modulus q.

The explicit formula for $\psi(x,\chi)$, where $\chi\ne\chi_0$ is a primitive character, takes the form, for $x>1$ not a prime power,

$$\psi(x,\chi)=-\sum_{\rho_\chi}\frac{x^{\rho_\chi}}{\rho_\chi}-(1-\mathfrak{a}_\chi)\log x-b(\chi)+\sum_{m=1}^{\infty}\frac{x^{\mathfrak{a}_\chi-2m}}{2m-\mathfrak{a}_\chi}, \tag{2.27}$$

where ρ_χ runs over the non-trivial zeros of $L(s,\chi)$, with the sum over zeros taken as $\lim_{X\to\infty}|\Im\rho_\chi|<X$. Also, $\mathfrak{a}_\chi=1$ if $\chi(-1)=-1$ and 0 otherwise, and $b(\chi)$ is a constant depending on χ, namely the constant term in the Laurent expansion about $s=0$ of $(L'/L)(s,\chi)$ (Taylor expansion if

$\chi(-1) = -1$). If x is a prime power, the right hand side above converges to $\psi(x,\chi) - \Lambda(x)\chi(x)/2$, i.e. one needs to subtract half of the last term in the sum defining $\psi(x,\chi)$. For a derivation of this explicit formula see, for instance, Davenport [2, pp. 115–120].

When χ is an imprimitive character, say induced by $\chi_1 \bmod q_1$, then

$$\psi(x,\chi) = \psi(x,\chi_1) - \sum_{\substack{p^k \le x \\ p \mid q}} (\log p)\chi_1(p^k). \tag{2.28}$$

For notational convenience, in the case of imprimitive χ, we set $\mathfrak{a}_\chi = \mathfrak{a}_{\chi_1}$ and also $b(\chi) = b(\chi_1)$.

Therefore, we have shown that we can rewrite (2.12) in the following form.

Lemma 2.2. *For $x > \max(\lambda, 1)$, and not a prime power,*

$$P(x) - \frac{r}{\phi(q)}\pi(x) = \frac{1}{\log x}\sum_\rho \alpha_\rho \frac{x^\rho}{\rho} + A(x) + B(x), \tag{2.29}$$

where $\alpha_\rho \in \mathbb{C}$ (described below in (2.32)), and the sum over ρ is taken over the union over the non-trivial zeros of all $L(s,\chi)$, $\chi \bmod q$, $\chi \neq \chi_0$.

Here, the function $A(x)$ gathers together all the remaining terms that are discontinuous:

$$A(x) = \frac{rR(x,1)}{\phi(q)} - \sum_{j=1}^{r} R(x,q,a_j) - \frac{r}{\phi(q)}\,\frac{\sum_{p^k \le x,\, p\mid q} \log p}{\log x} - \frac{1}{\log x} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0 \\ \chi \text{ imprimitive}}} c_\chi \sum_{\substack{p^k \le x \\ p \mid q}} (\log p)\chi_1(p^k), \tag{2.30}$$

and $B(x)$ incorporates the rest:

$$B(x) = \lambda - \frac{r}{\phi(q)}\int_2^x \frac{\sum_{p^k \le t,\, p\mid q} \log p}{t(\log t)^2}\,dt + \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} c_\chi \biggl(\int_2^x \frac{\psi(t,\chi)}{t(\log t)^2}\,dt - (1-\mathfrak{a}_\chi) + \frac{b(\chi)}{\log x} - \frac{1}{\log x}\sum_{m=1}^{\infty} \frac{x^{\mathfrak{a}_\chi - 2m}}{2m - \mathfrak{a}_\chi}\biggr). \tag{2.31}$$

Recall that c_χ is given in (2.13). The role of the sum over imprimitive characters in $A(x)$ is to account for (2.28).

We assume that the ρ in (2.29) are distinct, by grouping equal ρ under the same α_ρ. In the case of imprimitive characters, the non-trivial zeros of $L(s,\chi)$ coincide with those of the Dirichlet L-function $L(s,\chi_1)$, corresponding to the inducing character χ_1. More precisely, for given ρ,

$$\alpha_\rho = -\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} c_\chi m_\chi(\rho), \tag{2.32}$$

where $m_\chi(\rho)$ is the multiplicity of the zero ρ for $L(s,\chi)$.

We believe that

$$\alpha_\rho = O_q(1) \tag{2.33}$$

as $c_\chi = O(1)$, and we expect, for given χ and ρ, that $m_\chi(\rho) = 0$ or 1 (i.e. we expect ρ to be at most a simple zero of $L(s,\chi)$). More should be true: distinct $L(s,\chi)$, for primitive χ, presumably have distinct zeros.

For our purposes, we only need an estimate for the number of zeros in an interval, and we will use the following estimate:

$$\sum_{|\Im\rho|<T} |\alpha_\rho| = O_q(T\log T). \tag{2.34}$$

The above bound follows from the asymptotic formula for the number of zeros of $L(s,\chi)$ given in (2.56), which also implies the following estimate that we will use:

$$\sum_{n\le|\Im\rho|<n+1} |\alpha_\rho| = O_q(\log n) \quad \text{as } n\to\infty. \tag{2.35}$$

Lemma 2.3. *Infinitely many α_ρ in* (2.29) *are non-zero.*

Proof. Notice that $A(x)$ has jump discontinuities at a relatively thin set of prime powers: $R(x,1)$ and $R(x,q,a_j)$ jump when x is a prime power p^k with $k\ge 2$. The remaining terms in $A(x)$ jump at prime powers p^k with $p\mid q$, $k\ge 1$. Hence, overall, $A(x)$ has only finitely many jump discontinuities at the primes, namely the primes p that divide q. Furthermore, every term that appears in $B(x)$ is continuous with respect to x. But $P(x) - \frac{r}{\phi(q)}\pi(x)$ is discontinuous at all primes (our assumption that $r/\phi(q) < 1$ enters here).

Therefore, the sum over ρ in (2.29) must have infinitely many terms with $\alpha_\rho \neq 0$, otherwise the sum over ρ would be a continuous function for all x. ∎

2.4. Ω results. The fact that at least one α_ρ is non-zero is a crucial point, and we are now in a position to obtain our Ω results.

Let Θ be the lim sup of the real parts of the zeros ρ such that $\alpha_\rho\neq 0$, i.e. the zeros that appear in (2.29):

$$\Theta = \limsup\{\Re\rho \mid \alpha_\rho \neq 0\}. \tag{2.36}$$

Equivalently, from (2.12), Θ is the lim sup of the real parts of the poles of the function

$$-\sum_{\chi\neq\chi_0} c_\chi \frac{L'(s,\chi)}{L(s,\chi)}, \tag{2.37}$$

and also of the real parts of the singularities of

$$\sum_{\chi\neq\chi_0} c_\chi \log L(s,\chi). \tag{2.38}$$

Notice that $\Theta \geq 1/2$, since the zeros of $L(s,\chi)$ that occur off the half-line (assuming GRH fails) come in pairs, ρ and $1-\overline{\rho}$, symmetric about the line $\Re s = 1/2$.

2.5. Ω bound, assuming $\Theta > 1/2$. We first assume that $\Theta > 1/2$, i.e. the GRH fails, and at least one zero to the right of $\Re(s) = 1/2$ survives in the explicit formula on taking the linear combination in (2.12).

We will prove the following theorem.

THEOREM 4. *Assume that $\Theta > 1/2$. Then, for every $\delta > 0$,*

$$P(x) - \frac{r}{\phi(q)}\pi(x) = \Omega(x^{\Theta-\delta}), \tag{2.39}$$

with the implied constant depending on δ and q.

Proof. We do so by establishing, assuming $\Theta > 1/2$, the estimate

$$\sum_{j=1}^{r} \Pi(x,q,a_j) - \frac{r}{\phi(q)}\Pi(x) = \Omega(x^{\Theta-\delta}). \tag{2.40}$$

The left hand side above is easier to work with than (2.39), since L-functions naturally count prime powers rather than just primes. Notice that $\Pi(x,q,a) - \pi(x,q,a) = O_q(x^{1/2}/\log x)$ and $\Pi(x) - \pi(x) = O(x^{1/2}/\log x)$ (see (2.47) and (2.48) below). Hence, for $0 < \delta < \Theta - 1/2$, we see that (2.40) implies (2.39). The bound (2.39) then holds for every $\delta > 0$, since taking δ larger gives a weaker bound.

For a contradiction, assume that (2.40) does not hold, i.e. there exists a $\delta > 0$ such that $|\sum_{j=1}^{r} \Pi(x,q,a_j) - \frac{r}{\phi(q)}\Pi(x)| \ll x^{\Theta-\delta}$. Consider the Dirichlet integral (akin to Dirichlet series, see [4, Chapter 5])

$$\begin{aligned} &\int_1^\infty \frac{\sum_{j=1}^{r} \Pi(x,q,a_j) - \frac{r}{\phi(q)}\Pi(x)}{x^{s+1}}\,dx \\ &\qquad = \frac{1}{s}\sum_{\chi} c_\chi \log L(s,\chi) - \frac{r}{s\phi(q)}\log\zeta(s) \\ &\qquad = \frac{1}{s}\sum_{\chi\neq\chi_0} c_\chi \log L(s,\chi) + \frac{r}{s\phi(q)}\sum_{p\mid q}\log(1-p^{-s}). \end{aligned} \tag{2.41}$$

One can prove this identity, when $\Re s > 1$, by observing that the numerator of the integrand is a step function with steps at prime powers, and then integrating termwise the contribution from each prime power. The assumption that $\Re s > 1$ is used to rearrange integration and summation, and also to identify the resulting Dirichlet series with the right hand side above.

Notice that the right hand side above has singularities (branch cuts) coming from the zeros of $L(s,\chi)$, specifically from the ρ with $\alpha_\rho \neq 0$, i.e. those that survive the linear combination $\sum_{\chi\neq\chi_0} c_\chi \log L(s,\chi)$. There are also some additional singularities originating on the line $\Re s = 0$.

Now, if the numerator of the above integrand is $\ll x^{\Theta-\delta}$, then the left hand side of (2.41) defines an analytic function for $\Re s > \Theta - \delta$. But this contradicts, from the definition of Θ, the fact that the right hand side has singularities in this half-plane. Therefore,

$$\sum_{j=1}^{r} \Pi(x,q,a_j) - \frac{r}{\phi(q)}\Pi(x) = \Omega(x^{\Theta-\delta}) \tag{2.42}$$

for all $\delta > 0$.

This establishes the theorem. By taking δ sufficiently small, it also proves Theorem 1, for residue classes, in the case that $\Theta > 1/2$. ∎

2.6. Ω estimate in the classical case, assuming $\Theta = 1/2$. In this subsection, we assume that $\Theta = 1/2$. This can occur in two ways: either if the GRH holds, or if the only zeros surviving the linear combination of explicit formulas arising from (2.12) are on the half-line.

Theorem 5. *If $\Theta = 1/2$, then*

$$P(x) - \frac{r}{\phi(q)}\pi(x) = \Omega(x^{1/2}/\log x). \tag{2.43}$$

While we could modify the approach given in the previous subsection, it is complicated by the presence of the squares of primes. We could adapt the approach described in Ingham [4] for the problem of $\psi(x) - x$, and prove that

$$\sum_{j=1}^{r} \Pi(x,q,a_j) - \frac{r}{\phi(q)}\Pi(x) = \Omega_{\pm}(x^{1/2}/\log x), \tag{2.44}$$

i.e. the difference of these prime *power* counting functions gets, in size, as large as a constant times $x^{1/2}/\log x$, and points in *both* positive and negative directions for infinite sequences of $x \to \infty$. Now the squares of primes contribute an amount to $P(x) - \frac{r}{\phi(q)}\pi(x)$ that is asymptotically a constant times $x^{1/2}/\log x$, i.e. of the same size as (2.44), but always pointing in one direction. Hence, estimate (2.44) would establish (2.43).

Instead, however, we will take an alternate approach that yields more information. We will consider two mean square averages of the remainder term, each giving a separate proof of (2.43). Both averages are of interest in their own right. To do so we first prove the following lemma which provides for a more manageable formula in comparison to Lemma 2.2.

Lemma 2.4. *Let*

$$\kappa = \frac{1}{\phi(q)} \sum_{j=1}^{r} \sum_{b^2 = a_j \bmod q} 1. \tag{2.45}$$

Then, writing $\rho = 1/2 + i\gamma$, *we have*

$$P(x) - \frac{r}{\phi(q)}\pi(x) = \frac{x^{1/2}}{\log x}\Bigg(\sum_{0<|\gamma|<X} \alpha_\rho \frac{x^{i\gamma}}{\rho} + \nu + O\bigg(\frac{x^{1/2}(\log X)^2}{X} + \frac{1}{\log x}\bigg)\Bigg), \tag{2.46}$$

where ν *is equal to* $r/\phi(q) - \kappa$ *plus, if the term* $\rho = 1/2$ *appears in* (2.29), $2\alpha_{1/2}$.

Proof. We first bound each term that appears in $A(x)$, $B(x)$. The prime number theorem and (2.25) give

$$R(x,1) = \pi(x^{1/2})/2 + O(x^{1/3}/\log x) = x^{1/2}/\log x + O(x^{1/2}/(\log x)^2). \tag{2.47}$$

Similarly, from the prime number theorem for arithmetic progressions,

$$\sum_{j=1}^{r} R(x,q,a_j) = \kappa x^{1/2}/\log x + O_q(x^{1/2}/(\log x)^2), \tag{2.48}$$

with the implied constant in the O depending on q, and with κ defined above.

Finally, there are only finitely many $p \mid q$. Furthermore, $p^k \le x$ implies that $k \le (\log x)/\log p$. Hence $\sum_{p^k \le x,\, p|q} \log p = O_q(\log x)$, and so

$$-\frac{1}{2}\frac{\sum_{p^k\le x,\, p|q} \log p}{\log x} - \frac{1}{\log x} \sum_{\substack{\chi \bmod q \\ \chi \ne \chi_0 \\ \chi \text{ imprimitive}}} c_\chi \sum_{\substack{p^k \le x \\ p|q}} (\log p)\chi_1(p^k) = O_q(1). \tag{2.49}$$

Putting these together gives

$$A(x) = (r/\phi(q) - \kappa)x^{1/2}/\log x + O_q(x^{1/2}/(\log x)^2). \tag{2.50}$$

To estimate $B(x)$, notice that

$$\int_2^x \frac{\sum_{p^k \le t,\, p|q} \log p}{t(\log t)^2}\, dt \ll_q \int_2^x \frac{dt}{t \log t} \ll \log\log x. \tag{2.51}$$

Thus, because λ and the second line of (2.31) are bounded, we have

$$B(x) \ll_q \left| \sum_{\chi \neq \chi_0} c_\chi \int_2^x \frac{\psi(t,\chi)}{t(\log t)^2}\, dt \right| + \log\log x. \tag{2.52}$$

Let

$$G(x,\chi) = \int_2^x \frac{\psi(t,\chi)}{t}\, dt. \tag{2.53}$$

Equation (5), p. 117 of Davenport [2] gives the explicit formula with a rate of convergence:

$$\psi(x,\chi) = -\sum_{|\Im \rho_\chi| < X} \frac{x^{\rho_\chi}}{\rho_\chi} + O_q(x(\log xX)^2/X + \log x), \tag{2.54}$$

valid for $x \geq 2$ (or else for $x > 1$ by adding a 1 to the O term). Note that this formula, with the $O(\log x)$ included, is true for both primitive and imprimitive characters, and whether x is equal to a prime power or not. We have also absorbed the last three terms of (2.27) into the O term above. Thus, integrating and letting $X \to \infty$, we obtain

$$G(x,\chi) = -\sum_{\rho_\chi} \frac{x^{\rho_\chi}}{\rho_\chi^2} + O_q((\log x)^2). \tag{2.55}$$

The above series over ρ_χ converges absolutely, as can be seen from the asymptotic formula for the number of zeros [2, p. 101],

$$N(T,\chi) := \#\{\rho_\chi \mid |\Im \rho_\chi| \leq T\} = \frac{T}{\pi} \log \frac{qT}{2\pi} - \frac{T}{\pi} + O(\log T + \log q). \tag{2.56}$$

Summing over χ we get

$$\sum_{\chi \neq \chi_0} c_\chi \int_2^x \frac{\psi(t,\chi)}{t}\, dt = -\sum_\rho \alpha_\rho \frac{x^\rho}{\rho^2} + O_q((\log x)^2), \tag{2.57}$$

with the sum over all non-trivial zeros ρ of all $L(s,\chi)$ for the modulus q. The same coefficients α_ρ appear here as in (2.29) because the same linear combination of the terms involving the zeros ρ appears as in the sum $\sum_{\chi \neq \chi_0} c_\chi \psi(x,\chi)$.

It follows, on integrating by parts, that

$$\sum_{\chi \neq \chi_0} c_\chi \int_2^x \frac{\psi(t,\chi)}{t(\log t)^2}\, dt \ll_q \frac{x^{1/2}}{(\log x)^2} \sum_\rho \frac{|\alpha_\rho|}{|\rho|^2} \ll_q \frac{x^{1/2}}{(\log x)^2}. \tag{2.58}$$

The last bound follows, on summing by parts, from (2.34).

Thus, returning to (2.52), we get

$$B(x) \ll_q \frac{x^{1/2}}{(\log x)^2}. \tag{2.59}$$

Thus, our estimates (2.50) and (2.59) for $A(x)$ and $B(x)$ give

$$(2.60)\qquad P(x) - \frac{r}{\phi(q)}\pi(x) = \frac{1}{\log x}\left(\sum_{\rho} \alpha_\rho \frac{x^\rho}{\rho} + \left(\frac{r}{\phi(q)} - \kappa\right) x^{1/2} + O(x^{1/2}/\log x)\right),$$

By (2.54), for $2 \le x < X$, we can write this as a finite sum over ρ, as expressed in (2.46). The assumption $x < X$ is used here to simplify, in (2.54), $\log(xX)$ by $\log X$. We also use it below when estimating the contribution from the above O term.

Finally, we need to deal with the possibility of non-trivial zeros at $s = 1/2$. Such terms contribute $2\alpha_{1/2} x^{1/2}/\log x$ to the sum in (2.46). ∎

2.7. A mean square estimate of the average difference. We continue with our proof of Theorem 5. Let

$$(2.61)\qquad \Delta(x) := \frac{\log x}{x^{1/2}}\left(P(x) - \frac{r}{\phi(q)}\pi(x)\right).$$

Rather than work with $\Delta(x)$ directly, it is technically easier to work with its average:

$$(2.62)\qquad M(x) := \frac{1}{x}\int_2^x \Delta(t)\,dt = \sum_{\rho \ne 1/2} \alpha_\rho \frac{x^{i\gamma}}{\rho(i\gamma + 1)} + \nu + O(1/\log x).$$

The latter equality can be derived by integrating the bracketed expression in (2.46) termwise, and letting $X \to \infty$. Recall that ν is defined in (2.46).

The lemma below will be used to prove our Ω bound (2.43).

Lemma 2.5. *The following holds:*

$$(2.63)\qquad \lim_{Y\to\infty} \frac{1}{Y}\int_{\log 2}^{Y} |M(e^y)|^2\,dy = \sum_{|\gamma|>0} \frac{|\alpha_\rho|^2}{|\rho(i\gamma+1)|^2} + \nu^2 > 0.$$

Proof. For any $\epsilon > 0$, there exists $T = T(\epsilon)$ such that

$$(2.64)\qquad M(x) = \sum_{0<|\gamma|<T} \alpha_\rho \frac{x^{i\gamma}}{\rho(i\gamma+1)} + \nu + V(x),$$

where

$$(2.65)\qquad V(x) < \epsilon$$

for all x sufficiently large. This can be obtained using estimate (2.35) to show that the sum in (2.62) converges absolutely, and hence uniformly in x.

The natural scale at which to analyze the explicit formula is logarithmic. Set $y = \log x$, and consider

$$\frac{1}{Y}\int_{\log 2}^{Y} |M(e^y)|^2\,dy. \tag{2.66}$$

Substitute the right hand side of (2.64) for $M(e^y)$. Now,

$$\lim_{Y\to\infty}\frac{1}{Y}\int_{\log 2}^{Y}\Big|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho(i\gamma+1)}+\nu\Big|^2 dy = \sum_{0<|\gamma|<T}\frac{|\alpha_\rho|^2}{|\rho(i\gamma+1)|^2}+\nu^2, \tag{2.67}$$

which follows by multiplying

$$\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho(i\gamma+1)}+\nu \tag{2.68}$$

by its conjugate, expanding, and noting that only the diagonal terms survive the limit $Y \to \infty$. Next, the expression in (2.68) is bounded for $y \in \mathbb{R}$, and combining with (2.65) gives

$$\frac{1}{Y}\int_{\log 2}^{Y}\bigg(2\Big|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho(i\gamma+1)}+\nu\Big||V(e^y)|+|V(e^y)|^2\bigg)\,dy \ll \epsilon+\epsilon^2 \tag{2.69}$$

for all Y sufficiently large.

Since we may make ϵ as small as we wish, we get the equality expressed in (2.63). We also have the inequality stated in (2.63) because, by Lemma 2.3, at least one α_ρ is non-zero. ∎

Hence,

$$M(e^y) = \Omega(1), \tag{2.70}$$

i.e.

$$M(x) = \Omega(1), \tag{2.71}$$

which implies, from (2.62), that

$$\Delta(x) = \Omega(1), \tag{2.72}$$

and hence from (2.61) we get (2.43) of Theorem 5.

2.8. Unsmoothed mean square estimate. In this subsection we give an alternate proof of the Ω bound (2.43) by working out an unsmoothed mean square.

Substituting $y = \log x$ and $Y = \log X$ in (2.46), we consider

$$P(e^y) - \frac{r}{\phi(q)}\pi(e^y) = \frac{e^{y/2}}{y}\bigg(\sum_{0<|\gamma|<e^Y}\alpha_\rho\frac{e^{i\gamma y}}{\rho}+\nu+O\Big(\frac{e^{y/2}Y^2}{e^Y}+\frac{1}{y}\Big)\bigg). \tag{2.73}$$

Unlike $M(x)$ which was uniformly approximated by the finite sum (2.64), the above diverges absolutely and cannot be uniformly approximated. However, we can show that we can approximate the sum with finitely many terms so that the remainder term is uniformly small in mean square.

Thus, we truncate the sum over ρ at some large but fixed T (i.e. independent of y), and consider the mean square of the remainder. This is essentially Lemma 2.2 of [7], but we provide slightly more details here. Thus, for $T \geq 1$ and $\log 2 \leq y$, we have

$$P(e^y) - \frac{r}{\phi(q)}\pi(e^y) = \frac{e^{y/2}}{y}\Bigg(\sum_{0<|\gamma|<T} \alpha_\rho \frac{e^{i\gamma y}}{\rho} + \nu + r(y,T)\Bigg), \tag{2.74}$$

where, for all $Y \geq y$,

$$r(y,T) = \sum_{T\leq|\gamma|<e^Y} \alpha_\rho \frac{e^{i\gamma y}}{\rho} + O\bigg(\frac{e^{y/2}Y^2}{e^Y} + \frac{1}{y}\bigg). \tag{2.75}$$

The following lemma gives a bound on the mean square of the remainder $r(y,T)$.

LEMMA 2.6. *Let $T > 1$ and $Y > T^{1/2}/\log T$. Then*

$$\frac{1}{Y/2}\int_{Y/2}^{Y} |r(y,T)|^2\,dy \ll_q \frac{(\log T)^2}{T}. \tag{2.76}$$

Proof. Substitute (2.75) into the integrand, and use the inequality

$$|a+b|^2 \leq 2(|a|^2 + |b|^2), \tag{2.77}$$

which follows from the arithmetic geometric mean inequality $2|ab| \leq |a|^2 + |b|^2$, to get

$$\int_{Y/2}^{Y} |r(y,T)|^2\,dy \ll \int_{Y/2}^{Y} \bigg|\sum_{T\leq|\gamma|\leq e^Y} \alpha_\rho \frac{e^{iy\gamma}}{1/2+i\gamma}\bigg|^2\,dy + \frac{1}{Y}. \tag{2.78}$$

The term $1/Y$ comes about from integrating the square of the O term in (2.73). Multiplying out the sum above by its conjugate, estimating the resulting integral, and extending the double sum to infinity, we see that the right hand side above becomes

$$\begin{aligned} &\sum_{\substack{T\leq|\gamma_1|\leq e^Y \\ T\leq|\gamma_2|\leq e^Y}} \frac{\alpha_{\rho_1}\bar{\alpha}_{\rho_2}}{\rho_1\bar{\rho}_2}\int_{Y/2}^{Y} e^{iy(\gamma_1-\gamma_2)}\,dy + \frac{1}{Y} \\ &\qquad \ll \sum_{\substack{T\leq|\gamma_1|\leq\infty \\ T\leq|\gamma_2|\leq\infty}} \frac{|\alpha_{\rho_1}|\,|\alpha_{\rho_2}|}{|\rho_1|\,|\rho_2|}\min\bigg(Y, \frac{1}{|\gamma_1-\gamma_2|}\bigg) + \frac{1}{Y}. \end{aligned} \tag{2.79}$$

Breaking up the sum over zeros into unit intervals $|\gamma| \in [n, n+1)$ with $n \in \mathbb{Z}$, $n \geq T-1$, and using (2.35) shows that the above sum is bounded by

$$\ll Y \sum_{n \geq T-1} \frac{(\log n)^2}{n^2} + \sum_{\substack{n \geq T-1 \\ m \geq n+1}} \frac{\log m}{m} \frac{\log n}{n} \frac{1}{m-n}. \tag{2.80}$$

The first sum accounts for the contribution of the diagonal terms, i.e. where $|\gamma_1|$ lies in an interval $[n, n+1)$ and $|\gamma_2|$ lies in $[m, m+1)$, with $|m-n| \leq 1$. For such pairs of zeros, which could potentially be very close, we use Y as an upper bound for $\min(Y, 1/|\gamma_1 - \gamma_2|)$. For all other pairs of zeros the quantity $1/|\gamma_1 - \gamma_2| \ll 1/|m-n|$ is much smaller than Y. This gives the second sum above, i.e. the off-diagonal terms. We have also exploited symmetry in taking half the terms, i.e. we have dropped $n \geq m+1$.

By comparing with the integral $\int_T^\infty (\log t)^2/t^2\, dt$, we get, on integrating by parts,

$$\sum_{n \geq T-1} \frac{(\log n)^2}{n^2} \ll \frac{(\log T)^2}{T}. \tag{2.81}$$

To bound the off-diagonal contribution, break up the sum over m into the terms $n+1 \leq m \leq 2n$ and the tail $m > 2n$. The first portion can be estimated as follows:

$$\begin{aligned} \sum_{\substack{n \geq T-1 \\ n+1 \leq m \leq 2n}} \frac{\log m}{m} \frac{\log n}{n} \frac{1}{m-n} &\ll \sum_{n \geq T-1} \frac{(\log n)^2}{n^2} \sum_{n+1 \leq m \leq 2n} \frac{1}{m-n} \\ &\ll \sum_{n \geq T-1} \frac{(\log n)^3}{n^2} \ll \frac{(\log T)^3}{T}. \end{aligned} \tag{2.82}$$

For the contribution from the tail, use $1/|m-n| < 2/m$ when $m > 2n$:

$$\begin{aligned} \sum_{\substack{n \geq T-1 \\ m > 2n}} \frac{\log m}{m} \frac{\log n}{n} \frac{1}{m-n} &\ll \sum_{n \geq T-1} \frac{\log n}{n} \sum_{m > 2n} \frac{\log m}{m} \frac{1}{m} \\ &\ll \sum_{n \geq T-1} \frac{(\log n)^2}{n^2} \ll \frac{(\log T)^2}{T}, \end{aligned} \tag{2.83}$$

where we used $\sum_{m > 2n} (\log m)/m^2 \ll (\log n)/n$ in passing from the second expression to the third.

Putting these bounds together gives

$$\int_{Y/2}^{Y} |r(y,T)|^2\, dy \ll_q Y \frac{(\log T)^2}{T} + \frac{(\log T)^3}{T} + \frac{1}{Y}. \tag{2.84}$$

For given T and all $Y > T^{1/2}/\log T$, the first term on the right hand side dominates. Dividing by $1/(Y/2)$ gives the lemma. ∎

Returning to (2.74), we consider the mean square:

$$\frac{1}{Y/2}\int_{Y/2}^{Y}\left|\left(P(e^y)-\frac{r}{\phi(q)}\pi(e^y)\right)\frac{y}{e^{y/2}}\right|^2 dy = \frac{1}{Y/2}\int_{Y/2}^{Y}\left|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho}+\nu+r(y,T)\right|^2 dy. \tag{2.85}$$

The above equals

$$\frac{1}{Y/2}\int_{Y/2}^{Y}\left|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho}+\nu\right|^2 dy + E \tag{2.86}$$

where

$$|E| \ll \frac{1}{Y/2}\int_{Y/2}^{Y}\left|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho}+\nu\right||r(y,T)|\,dy+\frac{1}{Y/2}\int_{Y/2}^{Y}|r(y,T)|^2\,dy. \tag{2.87}$$

By multiplying the expression inside the absolute value of (2.86) by its conjugate, and integrating termwise, we get

$$\frac{1}{Y/2}\int_{Y/2}^{Y}\left|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho}+\nu\right|^2 dy = \nu^2+\sum_{0<|\gamma|<T}\frac{|\alpha_\rho|^2}{|\rho|^2}+O_T(1/Y). \tag{2.88}$$

Next we estimate E. The bound (2.87) gives

$$|E| \ll \max_{Y/2\le y\le Y}\left|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho}+\nu\right|\frac{1}{Y/2}\int_{Y/2}^{Y}|r(y,T)|\,dy + \frac{1}{Y/2}\int_{Y/2}^{Y}|r(y,T)|^2\,dy. \tag{2.89}$$

Lemma 2.6 gives an estimate for the second integral:

$$\frac{1}{Y/2}\int_{Y/2}^{Y}|r(y,T)|^2\,dy \ll \frac{(\log T)^2}{T}. \tag{2.90}$$

Now, from (2.35),

$$\left|\sum_{0<|\gamma|<T}\alpha_\rho\frac{e^{i\gamma y}}{\rho}+\nu\right| \ll (\log T)^2. \tag{2.91}$$

Furthermore, the Cauchy–Schwarz inequality gives

$$\frac{1}{Y/2}\int_{Y/2}^{Y}|r(y,T)|\,dy \ll +\frac{1}{Y/2}\left(\int_{Y/2}^{Y}dy\cdot\int_{Y/2}^{Y}|r(y,T)|^2\,dy\right)^{1/2}, \tag{2.92}$$

which, by Lemma 2.6, is

$$\ll (\log T)/T^{1/2}. \tag{2.93}$$

Since we may choose T as large as we please, on combining the above estimates we see that, as $Y \to \infty$,

$$\frac{1}{Y/2} \int_{Y/2}^{Y} \left| \left(P(e^y) - \frac{r}{\phi(q)} \pi(e^y) \right) \frac{y}{e^{y/2}} \right|^2 dy \to \nu^2 + \sum_{\rho \neq 1/2} \frac{|\alpha_\rho|^2}{|\rho|^2}. \tag{2.94}$$

As in the previous subsection, the right hand side above is positive because at least one of the α_ρ is non-zero. Therefore,

$$\left(P(e^y) - \frac{r}{\phi(q)} \pi(e^y) \right) \frac{y}{e^{y/2}} = \Omega(1), \tag{2.95}$$

hence giving another proof of Theorem 5.

Note that it is important that $r/\phi(q) < 1$, since if we take P to be the set of all primes then it *can* be realized, in many ways, as a union of primes in residue classes by taking all residue classes modulo q, for any positive integer q. The reason the proof fails in this case is that $P(x) - \pi(x)$ is then identically zero (and hence continuous), giving a mean square for $P(x) - \pi(x)$, and more precisely of (2.94), which is always zero. The positivity of the right hand side of that equation requires there to be at least one non-zero term appearing on the right hand side. However, from (2.13), all the c_χ and hence α_ρ are 0, and similarly for the term $r/\phi(q) - \kappa$ which then equals 0.

3. Generalization to Chebotarev sets. Here we generalize the problem to prime ideals and Chebotarev sets.

Therefore, let L be a Galois extension of K with Galois group $G = \mathrm{Gal}(L/K)$. For a prime ideal $\mathfrak{p} \in K$, we let the Artin symbol $(L/K, \mathfrak{p})$ denote the conjugacy class of Frobenius automorphisms corresponding to the prime ideals $\mathfrak{P} \in L$ that divide $\mathfrak{p}$.

Given a conjugacy class C of G, we let

$$\pi(L/K, C) = \{\mathfrak{p} \in \pi(K) \mid \mathfrak{p} \text{ unramified in } L,\ (L/K, \mathfrak{p}) = C\} \tag{3.1}$$

consist of the unramified prime ideals $\mathfrak{p} \in K$, and Frobenius conjugacy class in G equal to C. We define the counting function

$$\pi(x, L/K, C) := \sum_{\substack{N_{K/\mathbb{Q}}(\mathfrak{p}) \le x \\ \mathfrak{p} \in \pi(L/K, C)}} 1 \tag{3.2}$$

to be the number of prime ideals in $\pi(L/K, C)$ with norm less than or equal to x. Throughout what follows, we simply write $N\mathfrak{a}$ rather than $N_{K/\mathbb{Q}}(\mathfrak{a})$.

The Chebotarev density theorem states that

$$\pi(x, L/K, C) \sim \frac{|C|}{|G|}\,\mathrm{Li}(x), \tag{3.3}$$

and the prime number theorem for prime ideals in K states that

$$\pi(x, K) := \{\mathfrak{p} \in \pi(K) \mid N\mathfrak{p} \le x\} \sim \mathrm{Li}(x). \tag{3.4}$$

Therefore, say we have a subset P of prime ideals in $\pi(K)$ that is realized, up to finitely many exceptions, as a finite union of Frobenius conjugacy classes in the Galois group G of some Galois extension L of K. We can restrict ourselves to the case of a single Galois extension L for similar reasons that we were able to restrict ourselves to a single modulus q in the previous section. See the comments in the introduction in Section 1.1.

3.1. Proof of Theorems 1 and 2. All the formulas used in the classical situation of residue classes in Section 2 have analogues in the case of number fields. In particular, the explicit formula for our situation has been worked out, with remainder terms, by Lagarias and Odlyzko [6]. We develop and collect below the needed formulas.

Define

$$\psi(x, L/K, C) := \sum_{\substack{N\mathfrak{p}^m \le x \\ \mathfrak{p}\ \text{unramified} \\ (L/K,\mathfrak{p})^m = C}} \log N\mathfrak{p}, \tag{3.5}$$

$$\begin{aligned} \mathit{\Pi}(x, L/K, C) &:= \sum_{\substack{N\mathfrak{p}^m \le x \\ \mathfrak{p}\ \text{unramified} \\ (L/K,\mathfrak{p})^m = C}} \frac{1}{m} \\ &= \pi(x, L/K, C) + R(x, L/K, C), \end{aligned} \tag{3.6}$$

where

$$R(x, L/K, C) := \sum_{\substack{N\mathfrak{p}^m \le x \\ \mathfrak{p}\ \text{unramified},\, m \ge 2 \\ (L/K,\mathfrak{p})^m = C}} \frac{1}{m}, \tag{3.7}$$

so that

$$\begin{aligned} \pi(x, L/K, C) &= \mathit{\Pi}(x, L/K, C) - R(x, L/K, C) \\ &= \frac{\psi(x, L/K, C)}{\log x} + \int_2^x \frac{\psi(t, L/K, C)}{t(\log t)^2}\,dt - R(x, L/K, C). \end{aligned} \tag{3.8}$$

Likewise, define

$$\mathit{\Pi}(x, K) := \sum_{N\mathfrak{p}^m \le x} \frac{1}{m} = \pi(x, K) + R(x, K), \tag{3.9}$$

where

$$R(x,K) := \sum_{\substack{N\mathfrak{p}^m \le x \\ m\ge 2}} \frac{1}{m}. \tag{3.10}$$

Thus,

$$\begin{aligned}\pi(x,K) &= \Pi(x,K) - R(x,K) \\ &= \frac{\psi(x,K)}{\log x} + \int_2^x \frac{\psi(t,K)}{t(\log t)^2}\,dt - R(x,K).\end{aligned} \tag{3.11}$$

We will also use

$$\begin{aligned}R(x,K) &= \sum_{N\mathfrak{p}^2\le x} \frac{1}{2} + \sum_{\substack{N\mathfrak{p}^m\le x \\ m\ge 3}} \frac{1}{m} \\ &= x^{1/2}/\log x + O(x^{1/3}/\log x),\end{aligned} \tag{3.12}$$

which follows from the prime number theorem for ideals, with the implied constant in the O depending on K. Similarly, from the Chebotarev density theorem, we have

$$\sum_{j=1}^{r} R(x, L/K, C_j) = \kappa x^{1/2}/\log x + O(x^{1/2}/(\log x)^2), \tag{3.13}$$

with the implied constant depending on L/K and the C_j, and, overriding the notation for κ used earlier,

$$\kappa = \frac{1}{|G|}\sum_{j=1}^{r} |C_j| \sum_{b^2\in C_j} 1, \tag{3.14}$$

the inner sum counting the number of conjugacy class representatives $b \in G$ that, when squared, lie in C_j.

To obtain an explicit formula for $\psi(x, L/K, C)$, Lagarias and Odlyzko mimic the approach taken in Davenport for primes in arithmetic progression, using the following linear combination of logarithmic derivatives of Artin L-functions in order to extract primes ideals (and their powers) lying in the conjugacy class C:

$$\begin{aligned}F_C(s) &:= -\frac{|C|}{|G|}\sum_{\phi} \bar{\phi}(g)\frac{L'}{L}(s,\phi,L/K) \\ &= \sum_{\mathfrak{p}^m} \theta(\mathfrak{p}^m)\log(N\mathfrak{p})(N\mathfrak{p})^{-ms},\end{aligned} \tag{3.15}$$

where g is any element of the conjugacy class C, ϕ runs over the irreducible characters of G, and, for unramified $\mathfrak{p}$,

$$\theta(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } (L/K,\mathfrak{p})^m = C, \\ 0 & \text{otherwise,}\end{cases} \tag{3.16}$$

while, for ramified $\mathfrak{p}$, $|\theta(\mathfrak{p}^m)| \le 1$. Notice that, while the right hand side of (3.15) resembles the Dirichlet series that gives the counting function in (2.15), there is a minor difference. Above, and also in (3.17) below, the characters are primitive. The way to interpret (2.15) so that it matches with the formula here, is that each χ in (2.15) should be replaced by its inducing character at a cost of $O(\log x)$ to $\psi(x,q,a)$ coming from the primes that ramify.

Brauer [1] proved that each Artin L-function can be written as a ratio of Hecke L-functions, hence the linear combination of logarithmic derivatives of Artin L-functions above can be written in terms of Hecke L-functions. In our situation, the particular linear combination turns out, nicely, to have a similar form to (3.15). Lagarias and Odlyzko use a construction (Lemma 4.1 in their paper) of Deuring [3] to write

$$F_C(s) = -\frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\frac{L'}{L}(s,\chi,L/E), \tag{3.17}$$

where χ runs over the irreducible Hecke characters of $H = \langle g\rangle$, the cyclic subgroup generated by g, and E is the fixed field of H.

The advantage of writing $F_C(s)$ in terms of Hecke characters is that the analytic properties of Hecke L-functions are well established. Lagarias and Odlyzko carry out a Perron integral in order to extract the Dirichlet coefficients, with $N\mathfrak{p}^m \le x$, of $F_C(s)$.

Restricting to $2 < x < X$, equation (7.4) of [6] gives

$$\begin{aligned}\psi(x, L/K, C) \\ = \frac{|C|}{|G|}\Big(x - \sum_{\chi}\bar{\chi}(g)\sum_{|\Im\rho_\chi|<X}\frac{x_\chi^{\rho}}{\rho_\chi}\Big) + \text{remainder}(x,X,L/K,C),\end{aligned} \tag{3.18}$$

where

$$\text{remainder}(x,X,L/K,C) = O(x(\log X)^2/X + \log x), \tag{3.19}$$

with the implied constant depending on L/K and C. Here, ρ runs over all the non-trivial zeros of $L(s,\chi,L/E)$. The main term $(|C|/|G|)x$ arises from the principal character χ_0 since $L(s,\chi_0,L/E)$ has, up to finitely many Euler factors, $\zeta(s)$ as one of its factors, and hence a simple pole at $s=1$.

Our remainder term is simpler than in (7.4) of Lagarias and Odlyzko because we are taking L/K to be fixed. Furthermore, remainder$(x,X,L/K,C)$ is a piecewise continuous function, with $O(\log x)$ discontinuities at the points $x = N\mathfrak{p}^m$, where $\mathfrak{p}$ runs over the ramified primes in K.

Substitute (3.18) into (3.8), apply estimate (3.13), and then substitute all into

$$P(x) = \lambda + \sum_{j=1}^{r}\pi(x,L/K,C_j), \tag{3.20}$$

where $\lambda \in \mathbb{Z}$ accounts for finitely many exceptions, and x is sufficiently large (so that x exceeds the norm of any of these exceptions). Letting

$$\beta = \sum_{j=1}^{r} |C_j|/|G|, \tag{3.21}$$

we deduce, on subtracting the analogous formula for $\beta\pi(x,K)$ and cancelling the main term coming from the pole at $s=1$ of the factor of $\zeta(s)$ in the Dedekind zeta function ζ_K, that, for $2 \le x < X$ and assuming GRH,

$$\begin{aligned} &P(x) - \beta\pi(x,K) \\ &\quad = \frac{x^{1/2}}{\log x}\biggl(\sum_{|\gamma|<X} \alpha_\rho \frac{x^{i\gamma}}{\rho} + (\beta-\kappa) + O\biggl(\frac{x^{1/2}(\log X)^2}{X} + \frac{1}{\log x}\biggr)\biggr), \end{aligned} \tag{3.22}$$

with some $\alpha_\rho \in \mathbb{C}$, and the sum over ρ is over the non-trivial zeros of all relevant L-functions, namely the Hecke L-functions, for each C_j in (3.17). More precisely,

$$\alpha_\rho = -\frac{1}{|G|}\sum_{j=1}^{r} |C_j| \sum_{\chi\ne\chi_0} \bar{\chi}(g_j) m_\chi(\rho), \tag{3.23}$$

with $m_\chi(\rho)$ the multiplicity of the zero ρ for $L(s,\chi,L/E_j)$.

Bounds (2.34) and (2.35) continue to hold, though with the implied constants depending on L, K, and C_j rather than on q.

As in the classical case, if the term $\rho = 1/2$ appears in the sum, we absorb it into the constant term. Thus, let $\mu = \beta - \kappa$ plus, in the event that $\rho = 1/2$ appears in the sum, $2\alpha_{1/2}$. The above then becomes

$$\begin{aligned} &P(x) - \beta\pi(x,K) \\ &\quad = \frac{x^{1/2}}{\log x}\biggl(\sum_{0<|\gamma|<X} \alpha_\rho \frac{x^{i\gamma}}{\rho} + \mu + O\biggl(\frac{x^{1/2}(\log X)^2}{X} + \frac{1}{\log x}\biggr)\biggr). \end{aligned} \tag{3.24}$$

The cancellation of the main term deserves some elaboration. The L-function corresponding to the principal character in (3.17) factors as the product of $\zeta_E(s)$ and Hecke L-functions, and, because $K \subseteq E$, $\zeta_E(s)$ itself has $\zeta_K(s)$ as a factor. The latter Dedekind zeta function is responsible for cancellation in (3.22) of the main term. We therefore see that we could, in the statement of Theorem 1, replace $\beta\pi(x,K)$ with just $\beta\,\mathrm{Li}(x)$, since this would have the same effect of cancelling the main term, with no further impact on the form of the remaining terms. Similarly, we could replace the counting function by $\frac{\beta}{\delta(Q)}Q(x)$ where $Q \subseteq \pi(K)$ is another Chebotarev set with density $\delta(Q)$, or more generally, take any difference of the form (1.10), since the choice of η there ensures cancellation of the main term.

Next, the jump discontinuities of the left hand side of (3.24), up to given x, outnumber those of the O term of the right hand side, for the same reason as in the case of residue classes mod q: the discontinuities of the remainder term occur at $x = N\mathfrak{p}^m$, $m \geq 2$, for $\mathfrak{p} \in \pi(K)$, coming from the terms $R(x, L/K, C_j)$ and $R(x, K)$, of which, because $m \geq 2$, there are $O(x^{1/2}/\log x)$ many. The other discontinuities come from the ramified primes of which there are finitely many (and $O(\log x)$ of their powers, but these powers are already counted as discontinuities of $R(x, K)$). On the other hand, the left hand side has jump discontinuities at all prime ideals $\pi \in \pi(K)$ of which there are asymptotically $x/\log x$ many. We therefore conclude, as previously, that infinitely many of the α_ρ must be non-zero or else the sum over zeros would be a continuous function and the right hand side would not have sufficiently many discontinuities.

Again this holds when we replace $\beta\pi(x, K)$ with $\beta\,\mathrm{Li}(x)$ or $Q(x)$ as above, though in the latter case we must also ensure that Q does not essentially coincide with P, namely that the symmetric difference $P \triangle Q$ is infinite. In each case the difference between $P(x)$ and any of these counting functions has discontinuities at a positive proportion of $N\mathfrak{p}$ for primes ideals $\mathfrak{p} \in \pi(K)$, i.e. at $\gg x/\log x$ points. The same applies to the difference in (1.10).

As in the previous section, define

$$\Theta = \limsup\{\Re\rho \mid \alpha_\rho \neq 0\}. \tag{3.25}$$

Then $\Theta \geq 1/2$.

If $\Theta = 1/2$, we have two mean square estimates analogous to those in Sections 2.7 and 2.8. In order to carry out these estimates, we also need a bound, as before, for the number of non-trivial zeros of a Hecke L-function, $N(L, T) = |\{\rho \mid L(\rho) = 0,\ |\Im\rho| \leq T,\ 0 < \Re\rho < 1\}|$, in intervals of length one. Lagarias and Odlyzko [6, Lemma 5.4] prove that $N(L, T+1) - N(L, T) = O(\log T)$, with the implied constant depending on the L-function, hence the method used to obtain the mean square estimate in the case of Dirichlet L-functions and residue classes follows through as before, and we summarize the formulas.

Adapting the notation used in Section 2.7, let

$$\Delta(x) := \frac{\log x}{x^{1/2}}(P(x) - \beta\pi(x, K)) \tag{3.26}$$

and

$$M(x) := \frac{1}{x}\int_2^x \Delta(t)\, dt. \tag{3.27}$$

Then

$$\lim_{Y\to\infty} \frac{1}{Y}\int_{\log 2}^{Y} |M(e^y)|^2\, dy = \sum_{|\gamma|>0} \frac{|\alpha_\rho|^2}{|\rho(i\gamma+1)|^2} + \mu^2 \tag{3.28}$$

and

$$\lim_{Y\to\infty}\frac{1}{Y/2}\int\limits_{Y/2}^{Y}\left|(P(e^y)-r\pi(e^y,K))\frac{y}{e^{y/2}}\right|^2 dy=\mu^2+\sum_{\rho\neq 1/2}\frac{|\alpha_\rho|^2}{|\rho|^2}.$$

And, because at least one α_ρ is non-zero, both mean squares are positive. From either, we can thus conclude as in Section 2.7 or 2.8 that

$$P(x)-\beta\pi(x,K)=\Omega(x^{1/2}/\log x). \tag{3.29}$$

This concludes our proof of Theorems 1–3.

We also get the following theorem, depending on the value of Θ:

THEOREM 6. *For every* $\delta>0$,

$$P(x)-\beta\pi(x,K)=\Omega(x^{\Theta-\delta}), \tag{3.30}$$

with the implied constant depending on δ *and* P.

Proof. If $\Theta=1/2$ then Theorem 1 provides a stronger result and the above therefore holds. If $\Theta>1/2$, the theorem follows, as in Section 2.5, from the fact that at least one α_ρ is non-zero and from the identity, initially derived with the assumption that $\Re s>1$,

$$\int\limits_{1}^{\infty}\frac{\sum_{j=1}^{r}\Pi(x,L/K,C_j)-\beta\Pi(x,K)}{x^{s+1}}\,dx = \frac{1}{s}\sum_{j=1}^{r}\sum_{\chi_j}\bar{\chi}(g_j)\log L(s,\chi_j,L/E_j)-\beta\log\zeta_K(s), \tag{3.31}$$

where, for $1\le j\le r$, χ_j runs over all the irreducible Hecke characters of $H_j=\langle g_j\rangle$ and E_j is the fixed field of H_j. ■

Finally, a similar theorem holds for a variety of counting functions. As before, let $Q\subseteq\pi(K)$ be a Chebotarev set with the same density β as P, such that the symmetric difference $P\triangle Q$ is infinite, and F any finite extension of $\mathbb{Q}$. Let $f(x)$ stand for $\beta\,\mathrm{Li}(x)$ or $Q(x)$. For each such choice of $f(x)$, the difference $P(x)-f(x)$ can be expressed as a linear combination of explicit formulas having the same form as (3.22), though with the constant term $\beta-\kappa$ replaced by $-\kappa$ in the case of $f(x)=\beta\,\mathrm{Li}(x)$, and by $\kappa_2-\kappa$ when $f(x)=Q(x)$, where κ_2 is the analogue of (3.14) for the Chebotarev set Q. Thus, defining Θ_f to be the analogue, for a given f, of (3.25), and likewise Θ_F for the difference in (1.10), we have the following theorem:

THEOREM 7. *Let* $f(x)$ *be as in the above paragraph, and* $F(x)$ *and* η *be given by* (1.9) *and* (1.8). *Then, for every* $\delta>0$,

$$P(x)-f(x)=\Omega(x^{\Theta_f-\delta}), \tag{3.32}$$

with the implied constant depending on δ, P, *and* f, *and*

$$F(x) - \eta \operatorname{Li}(x) = \Omega(x^{\Theta_F - \delta}), \tag{3.33}$$

with the implied constant depending on Θ *and* η. *The latter result also holds* (*with* Θ_F *adjusted accordingly*) *if we replace* $\operatorname{Li}(x)$ *by* $\pi(x, K)$, *with the restriction that not all* η_j *are equal.*

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