On the first sign change in Mertens’ theorem

by

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1. Introduction. Mertens’ Theorem states that

$$\Delta_M(x) := \sum_{p \leq x} \frac{1}{p} - \log \log(x) - M = O(\log(x)^{-1})$$

for $x \to \infty$, where $M = 0.26149\ldots$ denotes the Mertens constant [8]. Rosser and Schoenfeld observed that $\Delta_M(x)$ is positive for $1 \leq x \leq 10^8$ and posed the question whether this would always be the case [12, p. 72f]. This has been answered by Robin who showed that $\Delta_M(x)$ changes sign infinitely often [10].

In this paper we show that the first sign change occurs before $\exp(495.702833165) = 1.909875\ldots \times 10^{215}$. More specifically, we prove

**Theorem 1.1.** There exists an $x_0 \in [\exp(495.702833109), \exp(495.702833165)]$ such that $\Delta_M(x) < 0$ for all $x \in [x_0 - \exp(239.046541), x_0]$.

This problem is similar to bounding the Skewes number, the number in $[2, \infty)$ where the first sign change of $\Delta(x) = \pi(x) - \text{li}(x)$ occurs [14]; this number is by now known to lie between $10^{19}$ (see [2]) and $\exp(727.951335792)$ (see [13]). The functions $\Delta(x)$ and $\Delta_M(x)$ are closely related and the Prime Number Theorem, $\Delta(x) = o(\text{li}(x))$ for $x \to \infty$, is in fact equivalent to $\Delta_M(x) = o(\log(x)^{-1})$ for $x \to \infty$. But since $\Delta(x)$ and $\Delta_M(x)$ are biased in opposite directions, there is no correlation between the sign changes of the two functions. On the Riemann Hypothesis, sign changes of $\Delta_M(x)$ rather occur at points where $\Delta(x) \approx -2\sqrt{x}/\log(x)$.

Theorem 1.1 is proven by an adaption of the Lehman method for bounding the Skewes number [8], using explicit formulas and numerical approximations to part of the zeros of the Riemann zeta function from [4]. In doing so, the kernel function in Lehman’s method is replaced by the Logan function [7],

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which appears to be more suitable for this problem. This is done in such
generality that it can easily be reapplied to the original Lehman method.

2. Notation. As usual, \( \zeta(s) \) denotes the Riemann zeta function and
zeros of \( \zeta(s) \) are denoted by \( \rho = \beta + i\gamma \) with \( \beta, \gamma \in \mathbb{R} \). The Euler constant
is denoted by \( C_0 = 0.57721 \ldots \) and the Mertens constant by
\[
M = C_0 - \sum_{p} \sum_{m=2}^{\infty} \frac{1}{mp^m} = 0.26149 \ldots.
\]

We use the symbol \( \sum' \) to define normalized summatory functions, i.e. we
define
\[
\sum'_{x<n<y} a_n := \frac{1}{2} \sum_{x<n<y} a_n + \frac{1}{2} \sum_{x\leq n \leq y} a_n.
\]

Moreover, we define the Mertens prime-counting functions
\[
\pi_M(x) = \sum'_{p<x} \frac{1}{p} \quad \text{and} \quad \pi^*_M(x) = \sum_{m=1}^{\infty} \frac{\pi_M(x^{1/m})}{m}.
\]

The Fourier transform of a function \( f \) is denoted by \( \hat{f} \) and defined by
\[
\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-itx} \, dt.
\]

Finally, we will use Turing’s big theta notation for explicit estimates and
write \( f(x) = \Theta(g(x)) \) for \( |f(x)| \leq g(x) \).

3. Description of the method. The method we use is similar to the
Lehman method for finding regions where \( \pi(x) - \text{li}(x) \) is positive \([6]\). We
aim to calculate upper bounds for a weighted mean value
\[
\frac{\omega+\varepsilon}{\omega-\varepsilon} \int_{\omega-\varepsilon}^{\omega+\varepsilon} K(y-\omega)ye^{y^2/2}[\pi_M(e^y) - \log(y) - M] \, dy,
\]
where \( K(y) \) is a non-negative kernel function. By using explicit formulas
this mean value can be expressed as a sum over the non-trivial zeros of \( \zeta(s) \),
which can be approximated numerically. Then, if an \( \omega \) can be found for which
the value in \((3.1)\) is negative, there must exist an \( x \in [\exp(\omega-\varepsilon), \exp(\omega+\varepsilon)] \)
such that \( \pi_M(x) - \log \log(x) - M \) is negative.

Lehman’s method uses the Gaussian function as a kernel function but
we prefer to use dilatations of the function
\[
K_c(y) := \begin{cases} \frac{c}{2\sinh(c)}I_0(c\sqrt{1-y^2}), & |y| < 1, \\ 0, & \text{otherwise}, \end{cases}
\]
where \( I_0(t) := \sum_{n=0}^{\infty} (t/2)^{2n}/(n!)^2 \) denotes the 0th modified Bessel function. The Fourier transform of \( K_c \) is given by the Logan function (see [4, Proposition 4.1])

\[
\hat{K}_c(x) = \ell_c(x) := \frac{c}{\sinh(c)} \frac{\sin(\sqrt{t^2 - c^2})}{\sqrt{t^2 - c^2}},
\]

which satisfies an optimality property well-suited for this problem [7], and which outperforms the Gaussian function in the similar context of calculating the prime-counting function analytically [4].

We define

\[
K_{c,\varepsilon}(y) := \frac{1}{\varepsilon} K_c(y/\varepsilon) \quad \text{and} \quad \ell_{c,\varepsilon}(x) := \hat{K}_{c,\varepsilon}(x) = \ell_c(\varepsilon x).
\]

Then our main result is

**Theorem 3.1.** Let \( 0 < \varepsilon < 10^{-3}, c \geq 3, \omega - \varepsilon > 200, \) and let \( H \geq c/\varepsilon \) be a number such that \( \beta = 1/2 \) holds for all zeros \( \rho = \beta + i\gamma \) of the Riemann zeta function with \( 0 < \gamma \leq H \). Furthermore, let \( h = 0 \) if the Riemann Hypothesis holds and \( h = 1 \) otherwise. Then

\[
\int_{\omega - \varepsilon}^{\omega + \varepsilon} K_{c,\varepsilon}(y - \omega) y e^{y/2} \left[ \pi_M(e^y) - \log(y) - M \right] dy 
\leq \sum_{|\gamma| \leq c/\varepsilon} e^{-i\gamma \omega} \ell_{c,\varepsilon}(\gamma) \left( \frac{1}{\rho} - \frac{1}{\omega \rho^2} \right) + 1 + 5.4 \times 10^{-10} + E_1 + E_2 + E_3,
\]

where

\[
E_1 \leq 0.33 e^{0.71 \sqrt{\varepsilon}} \frac{\varepsilon^{0.071 \sqrt{\varepsilon}}}{\sinh(c)} \log(3c) \log \left( \frac{c}{\varepsilon} \right),
\]

\[
E_2 \leq \frac{3.36 + 126 \varepsilon}{1000 \omega^2} + 2.8 \left( \frac{e}{2H} \right)^{\omega/2 - 1} \log(H),
\]

\[
E_3 \leq \frac{e^{\omega/2}}{1.99 H} \log(H) \left( \frac{c e^{3.12 \sqrt{\varepsilon}}}{\omega \sinh(c)} + \left( \frac{e \varepsilon}{\omega} \right)^{\omega/2} \right).
\]

Moreover, if \( a \in (0, 1) \) satisfies \( ac/\varepsilon \geq 10^3 \) in addition to the previous conditions, then

\[
\sum_{ac/\varepsilon < |\gamma| \leq c/\varepsilon} \left| e^{-i\gamma \omega} \ell_{c,\varepsilon}(\gamma) \left( \frac{1}{\rho} - \frac{1}{\omega \rho^2} \right) \right| 
\leq \frac{0.32 + 3.51 \varepsilon}{ca^2} \log \left( \frac{c}{\varepsilon} \right) \frac{c \log(c \sqrt{1 - a^2})}{\sinh(c)}.
\]

The proof needs some preparation.
4. The explicit formula for $\pi_M^*(x)$. The first ingredient is the explicit formula for $\pi_M^*(x)$. We define the auxiliary function

$$\tilde{E}(z) = \int_0^\infty \frac{e^{-t}}{z-t} dt,$$

which coincides with the exponential integral $Ei(z)$ in $\mathbb{R} \setminus \{0\}$, and which occurs naturally in explicit formulas for prime-counting functions.

**Lemma 4.1.** Let $x > 1$. Then

$$\pi_M^*(x) = \log \log(x) + C_0 - \sum^*_{\rho} \tilde{E}(\rho \log(x)) + \int_x^\infty \frac{dt}{t^2 \log(t)(t^2 - 1)},$$

where $\sum^*$ means that the sum over zeros is calculated as

$$\lim_{T \to \infty} \sum_{|\gamma| < T} \tilde{E}(\rho \log(x)).$$

**Proof.** The argument is similar to the original proof of the Riemann explicit formula [15]. Let

$$\psi(x, r) = \sum'_{p^m < x} \frac{\log(p)}{p^{mr}}.$$

Then we have

$$\pi_M^*(x) = \int_1^\infty \psi(x, r) dr.$$

From [5, (39)] we get the explicit formula

$$\psi(x, r) = \frac{x^{1-r}}{1-r} - \sum^*_{\rho} \frac{x^{\rho-r}}{\rho - r} - \sum_{n=1}^\infty \frac{x^{-2n-r}}{-2n - r} - \frac{\zeta'\zeta}{\zeta^2}(r).$$

Since $Ei(-x) = \log(x) + C_0 + o(x)$ for $x \searrow 0$ [9, p. 40], and since $\log(\zeta(1+\varepsilon)) = -\log(\varepsilon) + o(1)$ for $\varepsilon \searrow 0$, we have

$$\int_1^\infty \left( \frac{x^{1-r}}{1-r} - \frac{\zeta'}{\zeta}(r) \right) dr = \lim_{\varepsilon \searrow 0} [Ei(-\varepsilon \log(x)) + \log(\zeta(1+\varepsilon))] = \log \log(x) + C_0.$$

The sum over zeros takes the form

$$\int_1^\infty \sum^*_{\rho} \frac{x^{\rho-r}}{\rho - r} dr = \sum^*_{\rho} \tilde{E}(\rho(1) \log(x)) = \sum^*_{\rho} \tilde{E}(\rho \log(x)).$$
and for the sum over the trivial zeros we find
\[
\int_{1}^{\infty} \sum_{n=1}^{\infty} \frac{x^{-2n-r}}{2n + r} \, dr = \int_{1}^{\infty} \sum_{n=1}^{\infty} x^{-(2n+1)r} \, \frac{dr}{r} = \int_{1}^{\infty} \frac{x^{-3r}}{1 - x^{-2r}} \, dr
\]
\[
= \int_{x}^{\infty} \frac{dt}{t^2 \log(t)(t^2 - 1)}. \quad \blacksquare
\]

5. The difference \( \pi^*_M(x) - \pi_M(x) \). By definition of the Mertens constant \(2.1\) we have
\[
\pi_M(x) = \pi^*_M(x) + M - C_0 + r_M(x), \quad \text{where} \quad r_M(x) = \sum'_{\substack{p^m > x \geq 2}} \frac{1}{mp^m}.
\]
The term \( r_M(x) \) is responsible for the positive bias in Mertens’ Theorem and needs to be bounded from above.

**Lemma 5.1.** Let \( \log(x) > 200 \). Then
\[
r_M(x) \leq \frac{1 + 5.3 \times 10^{-10}}{\sqrt{x} \log(x)}.
\]

**Proof.** First we consider the contribution of the squares of prime numbers which yield the main term. Let \( r(t) = \psi(t) - t \), where \( \psi(t) := \psi(t,0) \) in the sense of \(4.2\) denotes the normalized Chebyshev function, and assume \( |r(t)| < \epsilon t \) for \( t \geq \sqrt{x} \) and some \( \epsilon > 0 \). Then partial summation gives
\[
\sum'_{p \geq \sqrt{x}} \frac{1}{p^2} < \left[ \frac{-r(t)}{t^2 \log(t)} \right]^{\infty}_{\sqrt{x}} + \int_{\sqrt{x}}^{\infty} \frac{dt}{t^2 \log(t)} - \int_{\sqrt{x}}^{\infty} r(t) \frac{d}{dt} \left( \frac{1}{t^2 \log(t)} \right) \, dt \leq 2 \frac{1 + 3\epsilon}{\sqrt{x} \log(x)}.
\]

For \( 3 \leq m \leq \log(x) \) we use
\[
\sum_{p \geq x^{1/m}} \frac{1}{p^m} \leq \frac{1}{x} + \int_{x^{1/m}}^{\infty} \frac{dt}{t^m} = \frac{1}{x} + \frac{1}{m - 1} x^{1/m - 1},
\]
which gives
\[
\sum_{\substack{p^m > x \geq 2 \atop 3 \leq m \leq \log(x)}} \frac{1}{mp^m} \leq \frac{\log(x)}{x} + (\zeta(2) - 1)x^{-2/3} < \frac{10^{-12}}{\sqrt{x} \log(x)}.
\]

For \( m > \log(x) \) we estimate trivially:
\[
\sum_{p} \frac{1}{p^m} \leq \sum_{n=3}^{\infty} n^{-m} + 2^{-m} \leq 2^{-m} + \int_{2}^{\infty} \frac{dt}{t^m} = 2^{-m} \left( 1 + \frac{2}{m - 1} \right).
\]
Therefore, we get

\[
\sum_{\substack{p^m \geq x \\\text{m > } \log x}} \frac{1}{mp^m} \leq \frac{1.01}{\log(x)} \sum_{m \geq \log(x)} 2^{-m} \leq \frac{2.02 \times 2^{-\log(x)}}{\log(x)} < \frac{10^{-16}}{\sqrt{x \log(x)}}.
\]

By [3, Table 1], (5.1) holds with \(\varepsilon = 1.752 \times 10^{-10}\) and so the assertion follows.

6. Evaluating the sum over zeros. The next problem is to approximate the following integral of the sum over zeros:

\[
\varepsilon \int_{-\varepsilon}^{\varepsilon} K_{c,\varepsilon}(y - \omega)ye^{y/2} \sum_{\rho}^{*} \tilde{Ei}(-\rho y) \, dy.
\]

Here, integral and sum may be interchanged, since the sum converges locally in \(L^1\). Therefore, we may treat each summand individually.

6.1. Asymptotic expansion of the summands. Since the Logan kernel should also be of interest for the question of finding regions where \(\pi(x) - \text{li}(x)\) is positive, the following lemma is presented in a more general version, which also covers the classical case.

**Lemma 6.1.** Let \(0 < \varepsilon < \omega\), and let \(K \in L^1([-\varepsilon, \varepsilon])\) satisfy \(\|K\|_{L^1} = 1\). Let \(a \in [0, 1]\), let \(\rho = \beta + i\gamma\), where \(0 \leq \beta \leq 1\) and \(\gamma \in \mathbb{R} \setminus \{0\}\), and let

\[
\Phi_{\omega,\rho,a} = \int_{\omega-\varepsilon}^{\omega+\varepsilon} K(y - \omega)ye^{(1/2-a)y} \tilde{Ei}((a - \rho)y) \, dy.
\]

Then

\[
(6.1) \quad \Phi_{\omega,\rho,a} = \sum_{j=1}^{k} (j - 1)! \frac{F_{\omega,\rho}^{(-j)}(0)}{(\rho - a)^j} + \Theta\left(\frac{k!e^{\varepsilon/2}e^{(1/2-\beta)\omega}}{(\omega - \varepsilon)^k|\gamma|^{k+1}}\right),
\]

where \(F_{\omega,\rho}^{(-1)}(0) = -e^{(1/2-\rho)\omega} \hat{K}(\frac{\rho}{i} - \frac{1}{2i})\) and for \(j \geq 2\) and any \(m \geq 0\),

\[
(6.2) \quad F_{\omega,\rho}^{(-j)}(0) = (-1)^j e^{(1/2-\rho)\omega} \sum_{n=0}^{m} \binom{n + j - 2}{n} (-i)^n \hat{K}^{(n)}(\frac{\rho}{i} - \frac{1}{2i}) \omega^{n+j-1} + \Theta\left(\frac{e^{j-2+\varepsilon/2}e^{(1/2-\beta)\omega}}{\omega^{j-1}} \frac{(\varepsilon/\omega)^{m+1}}{1 - \varepsilon/\omega}\right).
\]
Proof. By definition of \( E \) we have

\[
(6.3) \quad \Phi_{\omega,\rho,a} = \int_{\omega-\epsilon}^{\omega+\epsilon} K(y - \omega) ye^{(1/2-\alpha)y} \int_{0}^{\infty} e^{(a-\rho-r)y} dr \, dy
\]

\[
= \int_{0}^{\infty} \frac{1}{a-\rho-r} \int_{\omega-\epsilon}^{\omega+\epsilon} K(y - \omega) ye^{(1/2-\rho-r)y} dy \, dr.
\]

Now let

\[
F_{\omega,\rho}^{(-j)}(r) := (-1)^j \int_{\omega-\epsilon}^{\omega+\epsilon} y^{1-j} K(y - \omega) e^{(1/2-\rho-r)y} dy,
\]

which is well defined since \( \omega > \epsilon \), and satisfies \( \frac{d}{dr} F_{\omega,\rho}^{(-j)} = F_{\omega,\rho}^{(-1-j)} \). Then partial summation gives

\[
\Phi_{\omega,\rho,a} = -\int_{0}^{\infty} \frac{F_{\omega,\rho}^{(0)}(r)}{r + \rho - a} dr = \sum_{j=1}^{k} (j - 1)! \frac{F_{\omega,\rho}^{(-j)}(0)}{(\rho - a)^j} - k! \int_{0}^{\infty} \frac{F_{\omega,\rho}^{(-k)}(r)}{(r + \rho - a)^{k+1}} dr.
\]

Here, the trivial bound

\[
|F_{\omega,\rho}^{(-k)}(r)| \leq \frac{\epsilon}{\omega + y} |K(y)| e^{(1/2-\beta-r)(y+\omega)} dy \leq \frac{e^{\epsilon/2}}{(\omega - \epsilon)^{k+1}} e^{(1/2-\beta)\omega} \epsilon^{\epsilon/\gamma},
\]

yields

\[
\left| \int_{0}^{\infty} \frac{F_{\omega,\rho}^{(-k)}(r)}{(r + \rho - a)^{k+1}} dr \right| \leq \frac{e^{\epsilon/2} e^{(1/2-\beta)\omega}}{(\omega - \epsilon)^{k+1}},
\]

which confirms (6.1). It remains to evaluate \( F_{\omega,\rho}^{(-j)}(0) \). For \( j = 1 \) we find

\[
F_{\omega,\rho}^{(-1)}(0) = -e^{(1/2-\rho)\omega} \int_{-\epsilon}^{\epsilon} K(y) e^{-i(y - \omega/2)} dy = -e^{(1/2-\rho)\omega} \hat{K}\left(\frac{\rho}{i} - \frac{1}{2i}\right).
\]

For larger values of \( j \) we use the Taylor series expansion

\[
\frac{1}{(\omega + y)^u} = \sum_{n=0}^{\infty} \binom{u + n - 1}{n} \frac{(-y)^n}{\omega^{u+n}}
\]

and

\[
\int_{-\epsilon}^{\epsilon} K(y) y^n e^{-i(y - \omega/2)} dy = i^n \hat{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right),
\]

which gives

\[
F_{\omega,\rho}^{(-j)}(0) = (-1)^j e^{(1/2-\rho)\omega} \sum_{n=0}^{\infty} \binom{j + n - 2}{n} \frac{(-i)^n \hat{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right)}{\omega^{n+j-1}}.
\]
From (6.4) we also get
\[ \left| \hat{K}^{(n)} \left( \frac{\rho}{i} - \frac{1}{2i} \right) \right| \leq e^{\varepsilon/2} \varepsilon^n; \]
moreover the inequality \( \binom{a}{b} \leq \left( \frac{ea}{b} \right)^b \), which follows from Stirling’s lower bound for \( b! \), implies
\[ \binom{j + n - 2}{n} \leq e^n \left( 1 + \frac{j - 2}{n} \right)^n \leq e^{n+j-2}. \]
Thus, we have
\[ \sum_{n=m+1}^{\infty} \binom{j + n - 2}{n} \left| \hat{K}^{(n)} \left( \frac{\rho}{i} - \frac{1}{2i} \right) \right| \leq \frac{e^{j-2+\varepsilon/2}}{\omega^{j-1}} \sum_{n=m+1}^{\infty} \left( \frac{e\varepsilon}{\omega} \right)^n \]
\[ = \frac{e^{j-2+\varepsilon/2}}{\omega^{j-1}} \frac{(e\varepsilon/\omega)^{m+1}}{1 - e\varepsilon/\omega}, \]
which confirms the bound in (6.2).

6.2. Bounds for the kernel function. We need some bounds to estimate the tails of the sum over zeros. These are provided by the following two lemmas from [1] and [3]:

**Lemma 6.2** ([3, Lemma 2]). Let \( 0 < \varepsilon < 10^{-3} \) and \( c \geq 3 \). Then
\[ \sum_{|\gamma| > c/\varepsilon} \left| \ell_{c,\varepsilon} \left( \frac{\rho}{i} - \frac{1}{2i} \right) \right| \leq 0.32 e^{0.71 \sqrt{c\varepsilon}} \log(3c) \log \left( \frac{c}{\varepsilon} \right). \]

**Lemma 6.3** ([1, Lemma 5.5]). Let \( 0 < \varepsilon < 10^{-3} \) and \( c \geq 3 \), and let \( a \in (0, 1) \) satisfy \( ac/\varepsilon > 10^3 \). Then
\[ \sum_{ac/\varepsilon < |\gamma| \leq c/\varepsilon} \left| \ell_{c,\varepsilon}(\gamma) \right| \leq \frac{1 + 11c\varepsilon}{\pi ca^2} \log \left( \frac{c}{\varepsilon} \right) \cosh(c\sqrt{1-a^2}) \sinh(c). \]

We also need bounds for the derivatives \( \ell_{c,\varepsilon}^{(n)} \left( \frac{\rho}{i} - \frac{1}{2i} \right) \) occurring in (6.2), for calculations not assuming the Riemann Hypothesis.

**Lemma 6.4.** Let \( 0 < \varepsilon \leq \delta < c/100 \), and let \( z \in \mathbb{C} \) satisfy \( |\Re(z)| \geq c/\varepsilon \) and \( |\Im(z)| \leq 1/2 \). Then
\[ |\ell_{c,\varepsilon}^{(n)}(z)| \leq n! c e^{1.56 \sqrt{3c}} \left( \frac{2\varepsilon}{\delta} \right)^n. \]

**Proof.** The bound follows from the Cauchy formula
\[ \ell_{c,\varepsilon}^{(n)}(z) = \frac{n!}{2\pi i} \int_{|z-\xi|=\delta/(2\varepsilon)} \frac{\ell_{c,\varepsilon}(\xi)}{(z-\xi)^{n+1}} d\xi \]
if we show that

\[(6.7) \quad |\ell_{c,\varepsilon}(\xi)| \leq \frac{cc^{1.56\sqrt{\delta}}}{\sinh(c)}\]

in the range of integration. By basic properties of \(\ell_{c,\varepsilon}\) it suffices to prove this bound for \(\varepsilon = 1\) under the conditions \(\Re(\xi) \geq c - \delta, \ 0 \leq \Im(\xi) \leq \delta\), and we may also assume \(\delta < c/100\). Since we have

\[
|\Im(\sqrt{(\xi^2 - c^2)})| \leq |\Im(\sqrt{(\xi - c)^2 - c^2})|
\]

\[
\leq \sqrt{2}|1 + i\delta c \sinh\left(\frac{\pi}{4} + \frac{1}{2} \arctan\left(\frac{\delta c - \delta^2}{\delta c}\right)\right)|
\]

\[
\leq 2^{3/4} \sinh(1.181) \sqrt{\delta c} \leq 1.56 \sqrt{\delta c}
\]

under these conditions, the desired bound follows from

\[
\left|\left|\frac{\sin(z)}{z}\right|\right| \leq e^{\Im(z)}.
\]

7. Proof of Theorem 3.1. By Lemmas 4.1 and 5.1 we have

\[
\pi_M(e^y) - \log(y) - M = \pi_M^*(e^y) - \log(y) - C_0 + r_M(e^y)
\]

\[
\leq - \sum_{\rho}^\ast \tilde{Ei}(-\rho y) + \frac{1 + 5.4 \times 10^{-10}}{y} e^{-y/2}
\]

for \(y > 200\), where we have estimated the integral in (4.1) trivially by \(e^{-3y}\). Therefore

\[
\int_{\omega - \varepsilon}^{\omega + \varepsilon} K_{c,\varepsilon}(y - \omega) y e^{y/2} [\pi_M(e^y) - \log(y) - M] \, dy
\]

\[
\leq - \sum_{\rho} \Phi_{\omega,\rho,0} + 1 + 5.4 \times 10^{-10},
\]

with \(\Phi_{\omega,\rho,0}\) as defined in Lemma 6.1 with \(K = K_{c,\varepsilon}\) and \(\hat{K} = \ell_{c,\varepsilon}\). We subdivide the sum over zeros into two parts. For \(0 < \gamma \leq H\) we choose \(k = 2\) and \(m = 0\) in Lemma 6.1 which gives

\[(7.1) \quad - \sum_{|\gamma| \leq H} \Phi_{\omega,\rho,0} \leq \sum_{|\gamma| \leq c/\varepsilon} e^{-i\gamma\omega} \ell_{c,\varepsilon}(\gamma)\left(\frac{1}{\rho} - \frac{1}{\omega \rho^2}\right)
\]

\[+ \sum_{c/\varepsilon < |\gamma| \leq H} \left|\ell_{c,\varepsilon}(\gamma)\right| \left(1 + \frac{\varepsilon}{c\omega}\right) + \frac{1}{\omega^2} \sum_{|\gamma| < H} \left(\frac{2.72\varepsilon}{\gamma^2} + \frac{2.01}{|\gamma|^3}\right),
\]
where we have used $\varepsilon \leq 10^{-3}$. For $\gamma > H$ we have

\begin{align}
(7.2) \quad & \sum_{|\gamma| > H} |\Phi_{\omega, \rho, 0}| \leq e^{h\omega/2} \sum_{|\gamma| > H} \left| \ell_{c, \varepsilon} \left( \frac{\rho}{\gamma} \right) \right| \sum_{j=1}^{k} \frac{(j-1)!}{\omega^{j-1}} H^{1-j} \\
& + e^{h\omega/2} \sum_{|\gamma| > H} \sum_{j=2}^{k} \frac{(j-1)!}{|\gamma|^j} \left( \sum_{n=1}^{m} \binom{n + j - 2}{n} \frac{|\ell_{c, \varepsilon} \left( \frac{\rho}{\gamma} \right)|}{\omega^{n+j-1}} \right) \left( \frac{e^{(n)} \left( \frac{\rho}{\gamma} \right)}{\omega^{n+j-1}} \right) + e^{j-2+e/2(e\varepsilon)^{m+1}} \sum_{|\gamma| > H} \frac{k! e^{j/2}}{(\omega - \varepsilon)^k |\gamma|^{k+1}}.
\end{align}

for arbitrary $k \geq 2$ and $m \geq 1$, where $h = 0$ if the Riemann Hypothesis holds and $h = 1$ otherwise. So the inequality in (3.2) holds with

\begin{align}
(7.3) \quad & E_1 = \sum_{c/\varepsilon < |\gamma| \leq H} \left| \ell_{c, \varepsilon} (\gamma) \right| \left( 1 + \frac{\varepsilon}{c\omega} \right) \\
& + e^{h\omega/2} \sum_{|\gamma| > H} \left| \ell_{c, \varepsilon} \left( \frac{\rho}{\gamma} \right) \right| \sum_{j=1}^{k} \frac{(j-1)!}{\omega^{j-1}} H^{1-j},
\end{align}

\begin{align}
(7.4) \quad & E_2 = \frac{1}{\omega^2} \sum_{\rho} \left( \frac{2.72\varepsilon}{\gamma^2} + \frac{2.01}{|\gamma|^3} \right) + e^{h\omega/2} \sum_{|\gamma| > H} \frac{k! e^{j/2}}{(\omega - \varepsilon)^k |\gamma|^{k+1}},
\end{align}

\begin{align}
(7.5) \quad & E_3 = e^{\omega/2} \sum_{|\gamma| > H} \sum_{j=2}^{k} \frac{(j-1)!}{|\gamma|^j} \\
& \times \left( \sum_{n=1}^{m} \binom{n + j - 2}{n} \frac{|\ell_{c, \varepsilon} \left( \frac{\rho}{\gamma} \right)|}{\omega^{n+j-1}} \right) \left( \frac{e^{(n)} \left( \frac{\rho}{\gamma} \right)}{\omega^{n+j-1}} \right) + e^{j-2+e/2(e\varepsilon)^{m+1}} \sum_{|\gamma| > H} \frac{k! e^{j/2}}{(\omega - \varepsilon)^k |\gamma|^{k+1}}.
\end{align}

We proceed by bounding $E_k$. To this end we choose $k = m = |\omega/2|$. In (7.3) we take $H = c/\varepsilon$, which gives

\begin{align}
(7.6) \quad & E_1 \leq e^{h\omega/2} \sum_{c/\varepsilon < |\gamma|} \left| \ell_{c, \varepsilon} (\gamma) \right| \sum_{j=0}^{k-1} \frac{j!}{\omega^j} \left( \frac{\varepsilon}{c} \right)^j,
\end{align}

where the inner sum is bounded by

\begin{align}
\sum_{j=0}^{\infty} \left( \frac{\varepsilon}{2c} \right)^j \leq \left( 1 - \frac{1}{6000} \right)^{-1} \leq 1.0002,
\end{align}

since $c \geq 3$. Using this and (6.5) in (7.6) gives (3.3).

In (7.4) we use the bounds $\sum_{\gamma} \gamma^{-2} < 0.0463$ and $\sum_{\gamma} |\gamma|^{-3} < 0.00167$ from [11, Lemma 17], the bound
(7.7) \[ \sum_{|\gamma| > T} |\gamma|^{-k} \leq T^{1-k} \log(T) \]

for \( T \geq 2\pi e \) and \( k \geq 2 \) from [6, Lemma 2], and the inequality \((\omega - \varepsilon)^k \geq e^{-\varepsilon}\omega^k\), which follows from \( k \leq \omega/2 \), and get

\[
\mathcal{E}_2 \leq \frac{0.00336 + 0.126\varepsilon}{\omega^2} + \frac{e^{\omega/2} e^{2\varepsilon k!}}{(\omega H)^k} \log(H)
\leq \frac{3.36 + 126\varepsilon}{1000\omega^2} + 2.8 \left( \frac{e}{2H} \right)^{\omega/2-1} \log(H).
\]

In (7.5) we use (7.7) again and the bound from Lemma 6.4, where we choose \( \delta = 4\varepsilon \), which gives

(7.8) \[ \mathcal{E}_3 \leq e^{\omega/2} \sum_{j=2}^{k} H^{1-j} \log(H) \left( \frac{c e^{3.12\sqrt{c\varepsilon}}}{\sinh(c)} \sum_{n=1}^{m} \frac{j-1}{\omega} \frac{(n+j-2)!}{\omega^{n+j-2}2^n} \right.
+ \frac{1.002e^{j-1}}{e} \left( \frac{j-1}{\omega^{j-1}} \right)^{m+1},
\]

Since \( n+j-2 \leq \omega \) we have \((n+j-2)!/\omega^{n+j-2} \leq 1/\omega \), so the inner sum is bounded by \(1/(2\omega)\). In the second summand, we use \( (j-1)!/\omega^{j-1} \leq 2^{1-j} \).

Since \( \sum_{j=1}^{\infty} H^{-j} \leq 1.001/H \), \( \sum_{j=1}^{\infty} (2H/e)^{-j} \leq 1.001e/(2H) \), and \( m + 1 \geq \omega/2 \), we obtain the bound in (3.5).

Finally, the estimate in (3.6) follows from (6.6) since

\[
\sum_{ac/\varepsilon < |\gamma| \leq c/\varepsilon} \left| \frac{\ell_{c,\varepsilon}(\gamma)}{\rho} \left( 1 - \frac{1}{\omega\rho} \right) \right| \leq \left( 1 + \frac{1}{200 \times 1000} \right) \sum_{ac/\varepsilon < |\gamma| \leq c/\varepsilon} \left| \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \right|.
\]

8. Numerical results. To locate potential regions where the left hand side of (3.2) should be small, the function

\[ \sigma_T(y) = \sum_{|\gamma| \leq T} \frac{e^{i\gamma y}}{1/2 - i\gamma} \]

has been evaluated for \( T = 10^6 \) at all points in \( 10^{-7} \mathbb{Z} \cap [1, 2500] \). Since \( \ell_{c,\varepsilon}(\gamma) = 1 + O((\varepsilon\gamma)^2/c) \) for \( \gamma \to 0 \), this gives a reasonably good approximation to the first part of the sum in (3.2), and the objective is thus to find regions where \( \sigma_T(y) \) is smaller than \(-1\).

The evaluation has been done using the method for fast multiple evaluation of trigonometric sums from [4]. A more detailed search with \( T = 10^8 \) around 495.7028078, the first point where \( \sigma_{10^6}(y) \) turned out to be promisingly small, revealed a short region of length \( \approx 2.8 \times 10^{-8} \) about 495.702833137 where \( \sigma_{10^8}(y) < -1 \).
Proof of Theorem 1.1. The assertion now follows by an application of Theorem 3.1 with \( \omega = 495.702833137 \), \( c = 280 \), \( \varepsilon = 2.8 \times 10^{-8} \), \( H = 10^{11} \) (which has been reported in [4]) and \( a = 0.4 \).

Table 1. Values of \( y \in [1, 2500] \) for which \( \sigma_{10^6}(y) < -0.95 \)

<table>
<thead>
<tr>
<th>( y )</th>
<th>( \sigma_{10^6}(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>495.7028078</td>
<td>-0.9972...</td>
</tr>
<tr>
<td>1423.957207</td>
<td>-0.9740...</td>
</tr>
<tr>
<td>1623.9204309</td>
<td>-0.9807...</td>
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<tr>
<td>1859.1291846</td>
<td>-1.0511...</td>
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<tr>
<td>2107.5263606</td>
<td>-1.0214...</td>
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<tr>
<td>2285.3917834</td>
<td>-1.0454...</td>
</tr>
<tr>
<td>2430.3039554</td>
<td>-1.0172...</td>
</tr>
<tr>
<td>2447.6661764</td>
<td>-1.0028...</td>
</tr>
</tbody>
</table>

The sum over zeros was calculated using approximations to the zeros with imaginary part up to \( 4 \times 10^9 \) which were given within an absolute accuracy of \( 2^{-64} \). The sum was evaluated using multiple precision arithmetic, which gave the bound

\[
\sum_{|\gamma| \leq 4 \times 10^9} e^{-i \omega \ell(c, \varepsilon)} \left( \frac{1}{\rho} - \frac{1}{\omega \rho^2} \right) \leq -1.00015419.
\]

The sum in (3.6) is then bounded by \( 1.2 \times 10^{-11} \) and we have

\[
\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \leq 1.2 \times 10^{-12} + 1.37 \times 10^{-8} + 1.6 \times 10^{-24} \leq 1.38 \times 10^{-8}.
\]

Thus, the left hand side of (3.2) is bounded by

\[-1.00015419 + 1.2 \times 10^{-11} + 1 + 5.4 \times 10^{-10} + 1.38 \times 10^{-8} < -0.000154.\]

Consequently, there exists an \( x \in [\exp(w - \varepsilon), \exp(w + \varepsilon)] \) such that

\[
\pi_M(x) - \log \log(x) - M < -0.000154/(\sqrt{x \log(x)}).
\]

Obviously, we have

\[
\pi_M(x - y) - \log \log(x - y) - M \leq \pi_M(x) - \log \log(x) - M + \int_{x-y}^{x} \frac{dt}{t \log t} \leq -\frac{0.000154 \sqrt{x \log(x)}}{y} + \frac{y}{(x-y) \log(x-y)},
\]

which is negative for \( y \leq 0.00015 \sqrt{x} \). Since \( 0.00015 \sqrt{x} > \exp(239.046541) \), the theorem follows. ■

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First sign change in Mertens’ theorem

References


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