D. Fiorilli (2014) "Elliptic Curves of Unbounded Rank and Chebyshev's Bias,"
International Mathematics Research Notices, Vol. 2014, No. 18, pp. 4997–5024
Advance Access Publication June 10, 2013
doi:10.1093/imrn/rnt103

Elliptic Curves of Unbounded Rank and Chebyshev's Bias

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We establish a conditional equivalence between quantitative unboundedness of the analytic rank of elliptic curves over $\mathbb{Q}$ and the existence of highly biased elliptic curve prime number races. We show that conditionally on a Riemann Hypothesis and on a hypothesis on the multiplicity of the zeros of $L(E,s)$, large analytic ranks translate into an extreme Chebyshev bias. Conversely, we show under a certain linear independence hypothesis on zeros of $L(E,s)$ that if highly biased elliptic curve prime number races do exist, then the Riemann Hypothesis holds for infinitely many elliptic curve L-functions and there exist elliptic curves of arbitrarily large rank.

1 Introduction

Let E be a smooth elliptic curve whose minimal Weierstrass form is

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{1}$$

with $a_i \in \mathbb{Z}$, and let N_E denote its conductor. The set of rational points on this curve $E(\mathbb{Q})$ is a finitely generated abelian group by Mordell's Theorem, and hence is isomorphic to

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}},$$

Received January 14, 2013; Revised April 30, 2013; Accepted May 1, 2013

where $E(\mathbb{Q})_{\mathrm{tors}}$ is the finite set of torsion points. Mazur's Theorem [17] gives the list of 15 different possibilities for $E(\mathbb{Q})_{\mathrm{tors}}$. As for the integer $r = r_{\mathrm{al}}(E)$, the algebraic rank of E, it is a very mysterious invariant of E. A central question in number theory is whether $r_{\mathrm{al}}(E)$ is unbounded as E varies. The highest rank found so far is due to Elkies [8], who explicitly exhibited integer coefficients a_1, a_2, a_3, a_4, and a_6 such that (1) has algebraic rank at least 28. It is conjectured that the set of all ranks of elliptic curves over $\mathbb{Q}$ is unbounded [5, 6, 11, 28]. One approach to this conjecture is to study the L-function of E. The Birch and Swinnerton-Dyer Conjecture states that $r_{\mathrm{al}}(E)$ is equal to the order of vanishing of $L(E, s)$ at $s = 1$. Recall that the trace of the Frobenius endomorphism is given for $p \nmid N_E$ by $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p)$ is the number of projective points on the reduction of E modulo p. Extending the definition of $a_p(E)$ to the whole set of primes by setting

$$a_p(E) := \begin{cases} 1 & \text{if } E \text{ has split multiplicative reduction at } p, \\ -1 & \text{if } E \text{ has nonsplit multiplicative reduction at } p, \\ 0 & \text{if } E \text{ has additive reduction at } p, \end{cases}$$

the L-function of E is defined as

$$L(E, s) := \prod_{p \mid N_E} \left(1 - \frac{a_p(E)}{p^s}\right)^{-1} \prod_{p \nmid N_E} \left(1 - \frac{a_p(E)}{p^s} + \frac{p}{p^{2s}}\right)^{-1}.$$

The Birch and Swinnerton-Dyer Conjecture can be seen as a local-to-global principle, since it asserts that understanding local points on E is sufficient to understand its algebraic rank, which is a global invariant of E. Note that the zeros of $L(E, s)$ come in conjugate pairs and are symmetric about the line $\Re(s) = 1$, because of the functional equation relating $L(E, s)$ to $L(E, 2 - s)$.

Considering this, it is of crucial interest to understand the analytic rank $r_{\mathrm{an}}(E)$, which by definition is the order of vanishing of $L(E, s)$ at $s = 1$. Our main goal is to establish an equivalence between the conjecture that $r_{\mathrm{an}}(E)$ is unbounded as E varies over all elliptic curves over $\mathbb{Q}$ and a statement about the bias of certain prime number races formed with the local points on E. This is related to the initial calculations of Birch and Swinnerton-Dyer [2, 3], who combined counts of local points on E to predict its algebraic rank.

Note that if $r_{\mathrm{an}}(E)$ is unbounded, then it should have a certain growth in terms of conductor, that is there should exist a function $f(N_E)$ tending to infinity as $N_E \to \infty$

such that

$$\limsup_{N_E\to\infty} \frac{r_{\mathrm{an}}(E)}{f(N_E)} > 0. \tag{2}$$

There are two existing conjectures of this kind in the literature. Ulmer [28] has shown the existence of nonisotrivial elliptic curves of arbitrarily large rank over the rational function field $\mathbb{F}_p(t)$ for which the Birch and Swinnerton-Dyer Conjecture holds. His (more general) result shows that Mestre's bound [19] on ranks of elliptic curves is best possible in the function field case. Based on this result, he conjectured that $f(N_E) = \log N_E/\log\log N_E$ is admissible in (2) (Ulmer [28, 29] makes this conjecture on the algebraic rank; however, the Birch and Swinnerton–Dyer Conjecture implies that the corresponding statement on the analytic rank is equivalent.). Note that this is also believed to be best possible, since Mestre's contidional bound on ranks of elliptic curves [19] states that $r_{\mathrm{an}}(E) \ll \log N_E/\log\log N_E$. Recently, Farmer et al. [11] have developed a random matrix theory model for predicting the maximal size of the Riemann zeta function on the critical line, which led them to conjecture that

$$\max_{0\le t\le T}\left|\zeta\left(\frac{1}{2}+it\right)\right| = \mathrm{e}^{(1+o(1))\sqrt{\frac{1}{2}\log T\log\log T}}.$$

Their model also suggests that $f(N_E) = \sqrt{\log N_E \log\log N_E}$ is admissible and best possible in (2).

While the conjectures of Ulmer and Farmer, Gonek and Hughes are incompatible, they both imply the following weaker conjecture.

Conjecture 1.1. We have that

$$\limsup_{N_E\to\infty} \frac{r_{\mathrm{an}}(E)}{\sqrt{\log N_E}} = \infty. \qquad \square$$

We now describe the framework in which we relate Conjecture 1.1 to elliptic curve prime number races. Chebyshev's bias is his observation in a letter to Fuss that there seems to be more primes of the form $4n+3$ than of the form $4n+1$. Chebyshev's prime number race is the study of the oscillatory quantity

$$C(x) := \pi(x; 4, 3) - \pi(x; 4, 1),$$

which is known to have infinitely many sign changes [16]. For an account of the rich history of this subject, the reader is referred to the expository article [13]. Rubinstein and Sarnak [24] established under generalized riemann hypothesis and a linear independence hypothesis on the zeros of Dirichlet L-functions that $C(e^y)$ is positive for $\sim 99.59\%$ of the values of y. More precisely, they established that

$$\lim_{Y\to\infty} \frac{\mathrm{meas}\{y \leq Y : C(e^y) > 0\}}{Y} = 0.9959\ldots$$

One can study a related quantity by considering local points on an elliptic curve E. The celebrated Hasse bound states that counting projective points,

$$|\#E(\mathbb{F}_p) - (p+1)| < 2\sqrt{p}.$$

It is also known that the proportion of primes for which $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$ is positive is equal to the proportion of primes for which it is negative. In the non-CM case, this follows from the Sato–Tate Conjecture for elliptic curves over $\mathbb{Q}$, recently established by Taylor, Clozel, Harris, and Shepherd-Barron [7, 14, 26], which states that when E has no complex multiplication, the numbers $a_p(E)/2\sqrt{p}$ are equidistributed in $[-1, 1]$ with respect to the measure $(2/\pi)\sqrt{1-t^2}\,dt$ (the distribution is simpler when E has complex multiplication). Considering this, Mazur [18] introduced the race between primes for which $a_p(E) > 0$ and primes for which $a_p(E) < 0$. Defining

$$T(t) := \#\{p \leq t : a_p(E) > 0\} - \#\{p \leq t : a_p(E) < 0\},$$

Mazur plotted the graph of $T(t)$ for various elliptic curves. The reader is encouraged to consult [18] in which several other related quantities are studied. Looking at the graphs appearing in Section 2.3 of [18], one readily sees that $T(t)$ exhibits a very erratic behavior. Moreover, it is very apparent in these plots that as $r_{\mathrm{al}}(E)$ increases, $-T(t)$ becomes more and more biased toward positive values. Indeed, in Figure 2.5 of the paper, which is reproduced below in Figure 1, $-T(t)$ does not exhibit any negative value.

Under standard hypotheses, many features of $T(t)$ were explained by Sarnak [25], using the explicit formula for $L(\mathrm{Sym}^n E, s)$. Sarnak also introduced a closely related quantity $S(t)$ defined below, which as he showed can be understood using the explicit formula for $L(E, s)$ alone. Under a Riemann Hypothesis and a Linear Independence Hypothesis, Sarnak deduced an exact formula for the characteristic function of the limiting distribution of $ue^{-\frac{u}{2}}S(e^u)$, and uncovered the direct influence of the analytic rank of E on this quantity.

Building on the work of Sarnak, we shall study the quantity $S(t)$, which compares the primes p for which $a_p(E) < 0$ against those for which $a_p(E) > 0$, weighted by the value

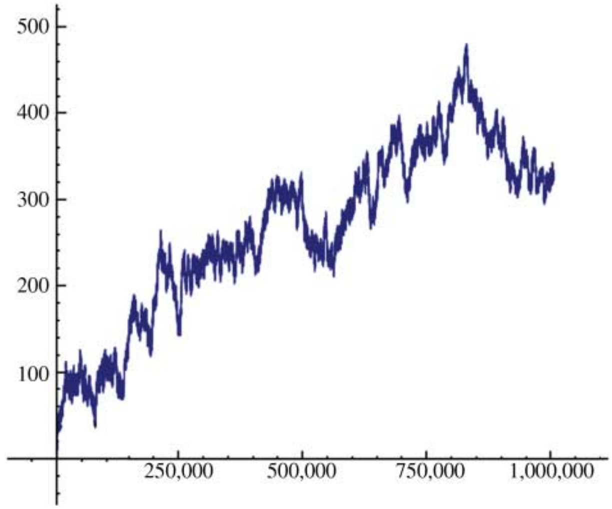

Fig. 1. A plot of $-T(t)$ for the curve $E: y^2 + y = x^3 - 7x + 6$ (data due to William Stein).

of $a_p(E)/\sqrt{p}$. This quantity has a very similar behavior to that of $-T(t)$. Moreover, it can be analyzed using $L(E, s)$ alone [25], in contrast to the analysis of $-T(t)$, which requires hypotheses on symmetric power L-functions of E.

We define the elliptic curve prime number race

$$S(t) := -\sum_{p \le t} \frac{a_p(E)}{\sqrt{p}},$$

and wish to understand the set of t for which $S(t) \ge 0$. To measure the size of this set, that is to measure the bias of $S(t)$ toward either positive or negative values, we define the following lower and upper logarithmic densities:

$$\underline{\delta}(E) := \liminf_{T\to\infty} \frac{1}{\log T} \int_{\substack{2\le t\le T:\\ S(t)\ge 0}} \frac{\mathrm{d}t}{t}, \quad \bar{\delta}(E) := \limsup_{T\to\infty} \frac{1}{\log T} \int_{\substack{2\le t\le T:\\ S(t)\ge 0}} \frac{\mathrm{d}t}{t}.$$

If these two densities are equal, then we denote them both by $\delta(E)$. Under ECRH and LI(E) (see the definitions below), Sarnak [25] has shown that $\delta(E)$ exists, and differs from $\frac{1}{2}$. In other words, $S(t)$ is always biased. Sarnak also discovered the dependence of this bias on the analytic rank of E, and a consequence of his results is that under ECRH and LI(E), elliptic curves of analytic rank zero have $\delta(E) < \frac{1}{2}$ (so $S(t)$ is biased toward negative values), and elliptic curves of analytic rank ≥ 1 have $\delta(E) > \frac{1}{2}$ (so $S(t)$ is biased toward positive values).

Our first main result is that Conjecture 1.1 implies that the quantity $S(t)$ can be arbitrarily biased, under the two following assumptions. Note that the second of these assumption is significantly weaker than the linear independence assumption used in [25].

Hypothesis ECRH (Elliptic Curve Riemann Hypothesis). For any elliptic curve E over $\mathbb{Q}$, the nontrivial zeros of $L(E,s)$ have real part equal to 1. □

Hypothesis BM (Bounded Multiplicity). There exists an absolute constant $C \geq 1$ such that for any elliptic curve E over $\mathbb{Q}$, the nonreal zeros of $L(E,s)$ have multiplicity at most C. □

Theorem 1.2 (Unbounded rank ⇒ arbitrarily biased elliptic curve prime number races). Assume ECRH and BM, and assume Conjecture 1.1 on the analytic rank of elliptic curves over $\mathbb{Q}$. Then, for any $\epsilon > 0$, there exists an elliptic curve E_ϵ over $\mathbb{Q}$ such that

$$1-\epsilon < \underline{\delta}(E_\epsilon) \leq \bar{\delta}(E_\epsilon) < 1.$$

That is to say, there exists arbitrarily biased elliptic curve prime number races. □

Remark 1.3. The proof of Theorem 1.2 does not use the full strength of ECRH and BM. Indeed, it is sufficient to assume that these hypotheses hold for an infinite sequence of elliptic curves $\{E(n)\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{r_{\mathrm{an}}(E(n))}{\sqrt{\log N_{E(n)}}} = \infty.$$ □

Our second main result is a converse result, under a linear independence hypothesis on the zeros of $L(E,s)$. We will show that the existence of highly biased elliptic curve prime number races is very strong; under the following assumption, it implies the Riemann Hypothesis for an infinite family of L-functions as well as the existence of elliptic curves over $\mathbb{Q}$ of arbitrarily large analytic rank.

Hypothesis LI(E) (Linear Independence). The function $L(E,s)$ has at least one zero on the line $\Re(s) = \beta_E := \sup\{\Re(\rho) : L(E,\rho) = 0\}$. Moreover, the multiset $Z(E) := \{\Im(\rho) \geq 0 : L(E,\rho) = 0, \Re(\rho) = \beta_E, \rho \neq 1\}$ is linearly independent over $\mathbb{Q}$. □

Remark 1.4. If the Riemann Hypothesis holds for $L(E,s)$, that is, $\beta_E = \sup\{\Re(\rho) : L(E,\rho) = 0\} = 1$, then Hypothesis LI(E) implies that the multiset of all positive imaginary parts of zeros of $L(E,s)$ is linearly independent over $\mathbb{Q}$. The reason why only positive imaginary parts are considered is that the zeros of $L(E,s)$ come in conjugate pairs, since this L-function is self-dual (because $a_p(E)\in\mathbb{R}$). Also we can potentially have $L(E,1)=0$ (in the case $\beta_E=1$), and hence it is important not to include this zero in the multiset $Z(E)$.

If $L(E,s)$ has a nontrivial zero outside the critical line $\Re(s)=1$, that is, $\beta_E>1$, then one should be careful with the additional symmetry of the set of zeros created by the functional equation. However, if ρ is a zero of $L(E,s)$, then the set $Z(E)=\{\Im(\rho)\geq 0 : L(E,\rho)=0, \Re(\rho)=\beta_E, \rho\neq 1\}$ contains at most one of the numbers $\{\rho,\bar{\rho}, 2-\rho, 2-\bar{\rho}\}$.

Note that Hypothesis LI(E) is a hypothesis on the zeros of $L(E,s)$ lying on the line $\Re(s)=\beta_E$. In particular, if the Riemann Hypothesis does not hold for this L-function, then nothing is assumed on the zeros lying on the critical line.

Finally, note that if $\beta_E>1$, then LI(E) implies that $L(E,\beta_E)\neq 0$, since a set containing zero is linearly dependent over $\mathbb{Q}$. This is similar to Chowla's Conjecture, which states that Dirichlet L-functions do not vanish for $s\in(0,1]$. □

Theorem 1.5 (Arbitrarily biased elliptic curve prime number races $\Rightarrow$ unbounded rank)**.** Assume that there exists a sequence of elliptic curves E_n over $\mathbb{Q}$ whose conductor tends to infinity with n, for which $LI(E_n)$ holds for $n\geq 1$ and whose associated prime number race is arbitrarily biased, that is, as $n\to\infty$,

$$\bar{\delta}(E_n)\to 1.$$

Then, there exist elliptic curves over $\mathbb{Q}$ of arbitrarily large analytic rank. More precisely, Conjecture 1.1 holds. □

Remark 1.6. In the proof of Theorem 1.5, we actually show that under LI(E), the bias in $S(t)$ implies the Riemann Hypothesis for $L(E,s)$. The next theorem is a precise statement of this implication. □

Theorem 1.7 (Biased elliptic curve prime number race $\Rightarrow$ Riemann Hypothesis)**.** Assume that LI(E) holds and that either $\underline{\delta}(E)\neq\frac{1}{2}$ or $\bar{\delta}(E)\neq\frac{1}{2}$. Then the Riemann Hypothesis holds for $L(E,s)$. □

Theorems 1.5 and 1.7 provide a method to simultaneously probe the Riemann Hypothesis and the unboundedness of the rank of elliptic curves over $\mathbb{Q}$. By computing the local points on an elliptic curve E, one can plot the prime number race $S(t)$ and if this graph is very biased toward positive values, then this gives evidence toward these two outstanding conjectures. A very strong bias is already present in the quantity $-T(t)$ associated to the rank 3 curve $E: y^2 + y = x^3 - 7x + 6$ appearing in Figure 2.5 of [18], whose graph is reproduced in Figure 1.

Remark 1.8. One can weaken the second hypothesis of Theorem 1.5 to

$$\limsup_{N_E \to \infty} \sqrt{\log N_E} \left(\bar{\delta}(E) - \frac{1}{2} \right) = \infty,$$

and still deduce the unboundedness of the analytic rank of elliptic curves over $\mathbb{Q}$. □

2 Proof of the Necessary Condition (Theorem 1.2)

We start with an outline of the proof of Theorem 1.2. Our strategy is to show that under ECRH, the quantity

$$E(e^y) := -\frac{y}{e^{y/2}} \sum_{p \le e^y} \frac{a_p(E)}{\sqrt{p}}$$

has a limiting distribution which is the same as the distribution of a certain random variable X_E. While we do not use this explicitly, one can see that

$$X_E = 2r_{\mathrm{an}}(E) - 1 + \sum_{\gamma > 0} \frac{2\Re(Z_\gamma)}{\frac{1}{4} + \gamma^2},$$

where γ runs over the imaginary parts of the nontrivial zeros of $L(E,s)$ and the Z_γ are identically distributed random variables, uniform on the unit circle in $\mathbb{C}$. We will then compute the first two moments of X_E. While the Z_γ are not necessarily independent (unless we assume a linear independence hypothesis), one can show that they have no covariance: if $\lambda > \gamma > 0$, then $\mathrm{Cov}(Z_\gamma, Z_\lambda) = 0$. This explains the simple formula for the variance appearing in Lemma 2.6. Finally, we will see that under ECRH and BM, if $r_{\mathrm{an}}(E)$ is significantly larger than $\sqrt{\log N_E}$, then the mean of X_E is significantly larger than its standard deviation, resulting in a very large bias by Chebyshev's inequality.

The fundamental tool we will use is the explicit formula for $L(E,s)$. We start with a technical estimate for the tail of a sum over zeros of $L(E, s + \frac{1}{2})$.

Lemma 2.1. We have, for $x, T \ge 2$, that

$$\sum_{|\Im(\rho)| > T} \frac{x^\rho}{\rho} \ll \log x + \frac{x}{T} \left((\log x)^2 + \frac{\log(TN_E)}{\log x} \right),$$

where ρ runs over the nontrivial zeros of $L(E, s + \frac{1}{2})$. □

Proof. We first write

$$L\left(E, s+\frac{1}{2}\right) = \prod_{p\mid N_E}\left(1-\frac{a_p(E)}{p^{s+\frac{1}{2}}}\right)^{-1}\prod_{p\nmid N_E}\left(1-\frac{a_p(E)}{p^{s+\frac{1}{2}}}+\frac{1}{p^{2s}}\right)^{-1}$$
$$= \prod_{p\mid N_E}\left(1-\frac{a_p(E)}{p^{s+\frac{1}{2}}}\right)^{-1}\prod_{p\nmid N_E}\left(1-\frac{\alpha_p}{p^s}\right)^{-1}\left(1-\frac{\beta_p}{p^s}\right)^{-1},$$

where $\beta_p = \bar{\alpha}_p$, $|\alpha_p| = |\beta_p| = 1$ and $\alpha_p + \beta_p = a_p(E)/\sqrt{p}$. The profound works of Wiles [30], Taylor and Wiles [27] and Breuil et al. [4] shows that the function $L(E, s)$ is a modular L-function, and hence it can be extended to an entire function. Following the proof of Theorem 6.9 of [20] (see also [21]), we obtain using the truncated Perron Formula that, for $x, T \geq 2$,

$$\sum_{\substack{p^e\leq x\\ e\geq 1\\ p\nmid N_E}}(\alpha_p^e+\beta_p^e)\log p + \sum_{\substack{p^e\leq x\\ e\geq 1\\ p\mid N_E}}\frac{a_p(E)^e}{p^{e/2}}\log p = -\frac{1}{2\pi i}\int_{\substack{\Re(s)=1+\frac{1}{\log x}\\ |\Im(s)|\leq T}}\frac{L'(E,s+\frac{1}{2})}{L(E,s+\frac{1}{2})}x^s\frac{ds}{s}$$
$$+ O\left(\log x + \frac{x}{T}(\log x)^2\right).$$

In particular, we have that

$$\sum_{\substack{p^e\leq x\\ e\geq 1\\ p\nmid N_E}}(\alpha_p^e+\beta_p^e)\log p + \sum_{\substack{p^e\leq x\\ e\geq 1\\ p\mid N_E}}\frac{a_p(E)^e}{p^{e/2}}\log p = -\frac{1}{2\pi i}\int_{\Re(s)=1+\frac{1}{\log x}}\frac{L'(E,s+\frac{1}{2})}{L(E,s+\frac{1}{2})}x^s\frac{ds}{s} + O(\log x),$$

and hence subtracting these two estimates, we obtain

$$\int_{\substack{\Re(s)=1+\frac{1}{\log x}\\ |\Im(s)|>T}}\frac{L'(E,s+\frac{1}{2})}{L(E,s+\frac{1}{2})}x^s\frac{ds}{s} \ll \log x + \frac{x}{T}(\log x)^2. \tag{3}$$

Now, if s is at a distance $\gg (\log N_E)^{-1}$ from the zeros of $L(E, s+\frac{1}{2})$, then (5.27) and (5.28) of [15] give the bound

$$\frac{L'(E,s+\frac{1}{2})}{L(E,s+\frac{1}{2})} \ll \log(N_E(|\Im(s)|+2)).$$

Using this bound, (3) becomes

$$\sum_{|\Im(\rho)|>T}\frac{x^\rho}{\rho} + \frac{1}{2\pi i}\int_{\substack{-\infty<\Re(s)\leq 1+\frac{1}{\log x}\\ |\Im(s)|=T}}\frac{L'(E,s+\frac{1}{2})}{L(E,s+\frac{1}{2})}x^s\frac{ds}{s} \ll \log x + \frac{x}{T}(\log x)^2,$$

where the contour of integration of the last integral should be slightly perturbed to a contour C which is at a distance $\gg (\log(N_E T))^{-1}$ from each zero of $L(E,s)$ (this is possible owing to the Riemann–von Mangoldt Formula and the zero-free region of $L(E,s)$), and thus

$$\sum_{|\Im(\rho)|>T} \frac{x^{\rho}}{\rho} \ll \int_C \log(N_E T) x^{\Re(s)} \frac{|\mathrm{d}s|}{|s|} + \log x + \frac{x}{T}(\log x)^2 \ll \log x + \frac{x}{T}\left(\frac{\log(N_E T)}{\log x} + (\log x)^2\right). \blacksquare$$

The main tool we will use is the explicit formula (see the corresponding (13) of [25]).

Lemma 2.2. Assume the Riemann Hypothesis for $L(E,s)$. Then we have, for $x, T \geq 2$, that

$$E(x) := -\frac{\log x}{\sqrt{x}} \sum_{p \leq x} \frac{a_p(E)}{\sqrt{p}} = 2r_{\mathrm{an}}(E) - 1 + \sum_{\gamma_E \neq 0} \frac{\mathrm{e}^{\mathrm{i}\gamma_E \log x}}{\frac{1}{2} + \mathrm{i}\gamma_E} + o_E(1) \tag{4}$$

$$= 2r_{\mathrm{an}}(E) - 1 + \sum_{0<|\gamma_E| \leq T} \frac{\mathrm{e}^{\mathrm{i}\gamma_E \log x}}{\frac{1}{2} + \mathrm{i}\gamma_E} + O\left(\frac{\sqrt{x}}{T}(\log(xTN_E))^2\right) + o_{x\to\infty}(1), \tag{5}$$

where γ_E runs over the imaginary parts of the nontrivial zeros of $L(E,s)$. □

Proof. We start with the explicit formula for the summatory function of the coefficients of the logarithmic derivative of

$$L\left(E, s+\frac{1}{2}\right) = \prod_{p \mid N_E} \left(1 - \frac{a_p(E)}{p^{s+\frac{1}{2}}}\right)^{-1} \prod_{p \nmid N_E} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1},$$

where, as before, $\beta_p = \overline{\alpha_p}$, $|\alpha_p| = |\beta_p| = 1$ and $\alpha_p + \beta_p = a_p(E)/\sqrt{p}$. Taking $T = x$ in (5.53) of [15] and bounding the rest of the sum over zeros using Lemma 2.1, we obtain the estimate

$$\sum_{\substack{p^e \leq x \\ e \geq 1 \\ p \nmid N_E}} (\alpha_p^e + \beta_p^e) \log p = -\sum_{\gamma_E} \frac{x^{\frac{1}{2} + \mathrm{i}\gamma_E}}{\frac{1}{2} + \mathrm{i}\gamma_E} + O(\log x \log(xN_E)). \tag{6}$$

Using the trivial bound on the terms on the left-hand side with $e \geq 3$, this becomes

$$-x^{-\frac{1}{2}} \sum_{p \leq x} \frac{a_p(E)}{\sqrt{p}} \log p = x^{-\frac{1}{2}} \sum_{p \leq \sqrt{x}} (\alpha_p^2 + \beta_p^2) \log p + 2r_{\mathrm{an}}(E) + \sum_{\gamma_E \neq 0} \frac{\mathrm{e}^{\mathrm{i}\gamma_E \log x}}{\frac{1}{2} + \mathrm{i}\gamma_E} + O_E(x^{-\frac{1}{6}}). \tag{7}$$

Now, $L'(\mathrm{Sym}^2 E, s+1)/L(\mathrm{Sym}^2 E, s+1)$ is holomorphic at $s=1$, and a Tauberian argument shows that

$$\sum_{p\le\sqrt{x}} (\alpha_p^2+\alpha_p\beta_p+\beta_p^2)\log p = o_E(\sqrt{x}),$$

which, combined with (7), the fact that $\alpha_p\beta_p=1$, and the Prime Number Theorem, gives

$$-x^{-\frac{1}{2}}\sum_{p\le x}\frac{a_p(E)}{\sqrt{p}}\log p = 2r_{\mathrm{an}}(E)-1+\sum_{\gamma_E\neq 0}\frac{\mathrm{e}^{\mathrm{i}\gamma_E\log x}}{\frac{1}{2}+\mathrm{i}\gamma_E}+o_E(1).$$

The estimate (4) follows by a summation by parts as in Lemma 2.1 of [24], and (5) follows by applying Lemma 2.1. ∎

Lemma 2.3. Assume the Riemann Hypothesis for $L(E,s)$. Then the quantity $E(x)$ defined in Lemma 2.2 has a limiting logarithmic distribution, that is there exists a Borel measure μ_E on $\mathbb{R}$ such that, for any bounded Lipschitz continuous function $f:\mathbb{R}\to\mathbb{R}$, we have

$$\lim_{Y\to\infty}\frac{1}{Y}\int_2^Y f(E(e^y))\,\mathrm{d}y = \int_{\mathbb{R}} f(t)\,\mathrm{d}\mu_E(t). \tag{8}$$

□

Proof. The proof follows from [1]. ∎

Remark 2.4. By taking f to be identically one in (8) we deduce that $\mu_E(\mathbb{R})=1$, that is μ_E is a probability measure. □

Let X_E be the random variable associated to μ_E. We will show that the moments of $E(e^y)$ agree with those of X_E.

Lemma 2.5. Assume the Riemann Hypothesis for $L(E,s)$. We have, for $k\ge 1$, that

$$\lim_{Y\to\infty}\frac{1}{Y}\int_2^Y E(e^y)^k\,\mathrm{d}y = \int_{\mathbb{R}} t^k\,\mathrm{d}\mu_E(t).$$

□

Proof. We will only prove the $k=1$ case since the general result follows along the same lines. Let $S\ge 1$ and define the bounded Lipschitz continuous function

$$H_S(t) := \begin{cases} 0 & \text{if } |t|\le S, \\ |t|-S & \text{if } S<|t|\le S+1, \\ 1 & \text{if } |t|>S+1. \end{cases}$$

By Lemma 2.3 we have that

$$\lim_{Y\to\infty}\frac{1}{Y}\int_2^Y H_S(E(e^y))\,\mathrm{d}y=\int_{\mathbb{R}} H_S(t)\,\mathrm{d}\mu_E(t)\leq \mu_E((-\infty,-S]\cup[S,\infty)).$$

In a similar way to Theorem 1.2 of [24], one can show that μ_E has exponentially small tails:

$$\mu_E((-\infty,-S]\cup[S,\infty))\ll_E \exp(-c_E\sqrt{S}),$$

from which we obtain that

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_{\substack{2\leq y\leq Y\\ |E(e^y)|>S+1}}\mathrm{d}y\leq \lim_{Y\to\infty}\frac{1}{Y}\int_2^Y H_S(E(e^y))\,\mathrm{d}y\ll_E \exp(-c_E\sqrt{S}).$$

By using dyadic intervals, we easily show that this bound implies

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_{\substack{2\leq y\leq Y\\ |E(e^y)|\geq S}}|E(e^y)|\,\mathrm{d}y\ll_E \exp(-c'_E\sqrt{S}). \tag{9}$$

Therefore, defining the bounded Lipschitz continuous function

$$G_S(t):=\begin{cases} t & \text{if } |t|\leq S,\\ S(S+1-|t|) & \text{if } S<|t|\leq S+1,\\ 0 & \text{if } |t|>S+1,\end{cases}$$

we obtain using (9) and Lemma 2.3 that

$$\begin{aligned}\limsup_{Y\to\infty}\frac{1}{Y}\int_2^Y E(e^y)\,\mathrm{d}y&=\limsup_{Y\to\infty}\frac{1}{Y}\int_2^Y G_S(E(e^y))\,\mathrm{d}y+O_E(\exp(-c''_E\sqrt{S}))\\ &=\int_{\mathbb{R}} G_S(t)\,\mathrm{d}\mu_E(t)+O_E(\exp(-c''_E\sqrt{S}))\\ &=\int_{\mathbb{R}} t\,\mathrm{d}\mu_E(t)+O_E(\exp(-c'''_E\sqrt{S})),\end{aligned}$$

and so, taking $S\to\infty$, we obtain that

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_2^Y E(e^y)\,\mathrm{d}y=\int_{\mathbb{R}} t\,\mathrm{d}\mu_E(t).$$

The same argument works for the lim inf, and hence the assertion is proved. ■

We now explicitly compute the first two moments of X_E, the random variable associated to the measure μ_E. This is analogous to Schlage-Puchta's result [23].

Lemma 2.6. Assume the Riemann Hypothesis for $L(E,s)$. Then

$$\mathbb{E}[X_E] = 2r_{\mathrm{an}}(E) - 1, \quad \mathrm{Var}[X_E] = \sideset{}{^*}\sum_{\gamma_E \neq 0} \frac{m(\gamma_E)^2}{\frac{1}{4} + \gamma_E^2},$$

where the last sum runs over the imaginary parts of the nontrivial zeros of $L(E,s)$, the star indicating that we count the zeros without multiplicity, and $m(\gamma_E)$ denotes the multiplicity of the zero $\rho_E = 1 + \mathrm{i}\gamma_E$. □

Proof. By Lemma 2.2, we have that

$$\int_2^Y E(e^y)\,\mathrm{d}y = (2r_{\mathrm{an}}(E) - 1)(Y-2) + \sum_{\gamma_E \neq 0} \frac{1}{\frac{1}{2} + \mathrm{i}\gamma_E} \int_2^Y \mathrm{e}^{\mathrm{i}\gamma_E y}\,\mathrm{d}y + o(Y)$$

$$= (2r_{\mathrm{an}}(E) - 1)(Y-2) + O_E(1) + o(Y),$$

since the sum $\sum_{\gamma_E \neq 0} \frac{1}{|\frac{1}{2} + \mathrm{i}\gamma_E|\gamma_E}$ converges. Taking $Y \to \infty$ and applying Lemma 2.5 gives that

$$\mathbb{E}[X_E] = \lim_{Y \to \infty} \frac{1}{Y} \int_2^Y E(e^y)\,\mathrm{d}y = 2r_{\mathrm{an}}(E) - 1.$$

As for the second assertion, it follows from Plancherel's identity for Besicovitch B^2 almost-periodic functions. In an effort to be more self-contained, we include a proof which follows [23]. We use Lemma 2.2 again. Letting γ and λ run through the ordinates of the nontrivial zeros of $L(E,s)$, we have

$$\int_2^Y \left| \sum_{0<|\gamma_E|\leq T} \frac{\mathrm{e}^{\mathrm{i}\gamma_E y}}{\frac{1}{2} + \mathrm{i}\gamma_E} \right|^2 \mathrm{d}y = \sideset{}{^*}\sum_{0<|\gamma|,|\lambda|\leq T} \frac{m(\gamma)m(\lambda)}{(\frac{1}{2} + \mathrm{i}\gamma)(\frac{1}{2} - \mathrm{i}\lambda)} \int_2^Y \mathrm{e}^{\mathrm{i}y(\gamma-\lambda)}\,\mathrm{d}y$$

$$= (Y-2) \sideset{}{^*}\sum_{0<|\gamma_E|\leq T} \frac{m(\gamma_E)^2}{\frac{1}{4} + \gamma_E^2} + O\left(\sum_{\substack{0<|\gamma|,|\lambda|\leq T \\ \gamma \neq \lambda}} \frac{\min\{Y, |\gamma - \lambda|^{-1}\}}{(1+|\gamma|)(1+|\lambda|)} \right).$$

(Note that we have removed the star in the sum in the error term, which explains why the multiplicities disappeared.) The first sum converges absolutely, since the Riemann–von

Mangoldt formula (see [15, Theorem 5.8])

$$N(T,E) := \#\{\gamma_E : |\gamma_E| \leq T\} = \frac{T}{\pi}\log\left(\frac{N_E T^2}{(2\pi e)^2}\right) + O(\log(N_E(T+2)))$$

implies that $m(\gamma_E) \ll \log(N_E(3+|\gamma_E|))$. Introducing a parameter $1 \leq U < T$, the sum appearing in the error term is at most

$$\sum_{\substack{0<|\gamma|,|\lambda|\leq T\\ |\gamma-\lambda|\geq 1}} \frac{|\gamma-\lambda|^{-1}}{(1+|\gamma|)(1+|\lambda|)} + \sum_{\substack{U<|\gamma|,|\lambda|\leq T\\ |\gamma-\lambda|\leq 1}} \frac{Y}{(1+|\gamma|)(1+|\lambda|)} + \sum_{\substack{0<|\gamma|,|\lambda|\leq U\\ 0<|\gamma-\lambda|\leq 1}} \frac{|\gamma-\lambda|^{-1}}{(1+|\gamma|)(1+|\lambda|)}$$

$$\ll_E \sum_{\substack{\gamma,\lambda\\ |\gamma-\lambda|\geq 1}} \frac{|\gamma-\lambda|^{-1}}{(1+|\gamma|)(1+|\lambda|)} + Y\sum_{U\leq|\gamma|\leq T} \frac{\log|\gamma|}{(1+|\gamma|)^2} + S(U)$$

$$\ll_E 1 + Y\frac{(\log U)^2}{U} + S(U), \tag{10}$$

since the integral $\iint_{|x-y|\geq 1} \frac{|x-y|^{-1}\log x \log y}{(|x|+1)(|y|+1)}\,\mathrm{d}x\,\mathrm{d}y$ converges. Here,

$$S(U) := \sum_{\substack{0<|\gamma|,|\lambda|\leq U\\ 0<|\gamma-\lambda|\leq 1}} \frac{|\gamma-\lambda|^{-1}}{(1+|\gamma|)(1+|\lambda|)}.$$

Define $Y_U \geq U^2$ to be an increasing function of U such that, for each $U \geq 1$, $US(U) \leq Y_U$ (this is ineffective). Inverting this process, we find an increasing function $U(Y) \leq \sqrt{Y}$ such that $U(Y) \to \infty$ as $Y \to \infty$, and such that, for Y large enough, $U(Y)S(U(Y)) \leq Y$. This shows that (10) is

$$\ll 1 + Y\frac{(\log U(Y))^2}{U(Y)} + \frac{Y}{U(Y)} = o_{Y\to\infty}(Y).$$

That is, we have shown that

$$\int_2^Y \left|\sum_{0<|\gamma_E|\leq T} \frac{\mathrm{e}^{\mathrm{i}\gamma_E y}}{\frac{1}{2}+\mathrm{i}\gamma_E}\right|^2 \mathrm{d}y = (Y-2)\sum_{0<|\gamma_E|\leq T}^{*} \frac{m(\gamma_E)^2}{\frac{1}{4}+\gamma_E^2} + o_{Y\to\infty}(Y).$$

Therefore, by Lemma 2.2 we obtain, by taking $T = e^{2Y}$, that

$$\int_2^Y |E(e^y) - \mathbb{E}[X_E]|^2\,\mathrm{d}y = \int_2^Y \left|\sum_{0<|\gamma_E|\leq e^{2Y}} \frac{\mathrm{e}^{\mathrm{i}\gamma_E y}}{\frac{1}{2}+\mathrm{i}\gamma_E}\right|^2 \mathrm{d}y$$

$$+ O_E\left(\int_2^Y \left|\sum_{0<|\gamma_E|\leq e^{2Y}} \frac{\mathrm{e}^{\mathrm{i}\gamma_E y}}{\frac{1}{2}+\mathrm{i}\gamma_E}\right| o_{Y\to\infty}(1)\,\mathrm{d}y + o_{Y\to\infty}(Y)\right)$$

$$= (Y-2) \sum_{0<|\gamma_E|\le e^{2Y}}^{*} \frac{m(\gamma_E)^2}{\frac{1}{4}+\gamma_E^2} + o_{Y\to\infty}(Y),$$

by the Cauchy–Schwartz inequality. The result follows by taking $Y \to \infty$ and applying Lemma 2.5. ■

Lemma 2.7. Assume the Riemann Hypothesis for $L(E,s)$. If

$$B(E) := \frac{\mathbb{E}[X_E]}{\sqrt{\mathrm{Var}[X_E]}}$$

is large enough, then

$$\underline{\delta}(E) \ge 1 - 2\frac{\mathrm{Var}[X_E]}{\mathbb{E}[X_E]^2}.$$ □

Proof. It is clear from Lemma 2.6 and the Riemann–von Mangoldt formula that $\mathrm{Var}[X_E] \gg \log N_E$, and therefore our assumption that $B(E)$ is large enough implies that $\mathbb{E}[X_E]$ is also large enough, say at least 4. Let now

$$H(x) := \begin{cases} 1 & \text{if } x \ge 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad f(x) := \begin{cases} 1 & \text{if } x \ge 1, \\ x & \text{if } 0 \le x \le 1, \\ 0 & \text{if } x < 0. \end{cases}$$

Clearly, $f(x)$ is bounded Lipschitz continuous and $f(x) \le H(x)$. Therefore,

$$\underline{\delta}(E) = \liminf_{Y\to\infty} \frac{1}{Y}\int_2^Y H(E(e^y))\,\mathrm{d}y \ge \liminf_{Y\to\infty} \frac{1}{Y}\int_2^Y f(E(e^y))\,\mathrm{d}y,$$

which, by Lemma 2.3, is equal to

$$\begin{aligned}\int_{\mathbb{R}} f(t)\,\mathrm{d}\mu_E(t) &= 1 - \int_{\mathbb{R}} (1-f(t))\,\mathrm{d}\mu_E(t) \\ &= 1 - \int_{-\infty}^{1} (1-f(t))\,\mathrm{d}\mu_E(t) \ge 1 - \mu_E(-\infty,1].\end{aligned}$$

We now apply Chebyshev's inequality:

$$\begin{aligned}\mu_E(-\infty,1] &= \mathrm{Prob}[X_E \le 1] = \mathrm{Prob}[X_E - \mathbb{E}[X_E] \le 1 - \mathbb{E}[X_E]] \\ &\le \mathrm{Prob}[|X_E - \mathbb{E}[X_E]| \ge \mathbb{E}[X_E]-1] \le \frac{\mathrm{Var}[X_E]}{(\mathbb{E}[X_E]-1)^2} \le 2\frac{\mathrm{Var}[X_E]}{\mathbb{E}[X_E]^2}\end{aligned}$$

since $\mathbb{E}[X_E] \geq 4$, and therefore

$$\underline{\delta}(E) \geq 1 - 2\frac{\mathrm{Var}[X_E]}{\mathbb{E}[X_E]^2}. \qquad \blacksquare$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let X_E be the random variable associated to the measure μ_E. By Lemma 2.6, its mean is equal to $\mathbb{E}[X_E] = 2r_{\mathrm{an}}(E) - 1$, and by our assumption that the nonreal zeros of $L(E,s)$ have bounded multiplicity we have

$$\sum_{\gamma_E \neq 0} \frac{1}{\frac{1}{4} + \gamma_E^2} \leq \mathrm{Var}[X_E] = \sum_{\gamma_E \neq 0}^{*} \frac{m(\gamma_E)^2}{\frac{1}{4} + \gamma_E^2} \ll \sum_{\gamma_E \neq 0} \frac{1}{\frac{1}{4} + \gamma_E^2},$$

and thus, by the Riemann–von Mangoldt formula,

$$\mathrm{Var}[X_E] \asymp \log N_E.$$

The condition

$$\limsup_{N_E \to \infty} \frac{r_{\mathrm{an}}(E)}{\sqrt{\log N_E}} = \infty$$

then implies that

$$\limsup_{N_E \to \infty} \frac{\mathbb{E}[X_E]}{\sqrt{\mathrm{Var}[X_E]}} = \infty.$$

Combining this with Lemma 2.7 shows that

$$\sup_E \underline{\delta}(E) = 1.$$

The last thing to show is that $\bar{\delta}(E) < 1$ for all elliptic curves; however, this follows from an analysis as in Lemma 2.7 combined with a lower bound on $\mu_E(-\infty, -1]$ similar to that in Theorem 1.2 of [24], which can be derived using similar techniques. $\blacksquare$

3 Proof of the Sufficient Condition (Theorem 1.5)

The first step will be to show that, under the first assumption in Theorem 1.5, the quantity

$$E_n(x) := -\frac{\log x}{x^{\beta_n - \frac{1}{2}}} \sum_{p \leq x} \frac{a_p(E_n)}{\sqrt{p}}$$

has a limiting logarithmic distribution as $x \to \infty$.

Lemma 3.1. Fix $T \geq 1$ and assume that $L(E,s)$ has at least one zero on the line $\Re(s) = \beta_0 := \sup\{\Re(z) : L(E,z) = 0\}$. Then letting ρ run over the nontrivial zeros of $L(E, s+\frac{1}{2})$, we have that the quantity

$$F_T(x) := x^{-\beta_0+\frac{1}{2}} \sum_{\substack{\rho \\ |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho} - x^{1-\beta_0}$$

has a limiting logarithmic distribution as $x \to \infty$, that is there exists a Borel measure $\mu_E^{(T)}$ such that, for every bounded Lipschitz continuous function f, we have

$$\lim_{Y\to\infty} \frac{1}{Y} \int_2^Y f(F_T(e^y))\,\mathrm{d}y = \int_{\mathbb{R}} f(t)\,\mathrm{d}\mu_E^{(T)}(t). \qquad \square$$

Remark 3.2. Since the nontrivial zeros of $L(E,s)$ are symmetric about the line $\Re(s) = 1$, we always have $\beta_0 \geq 1$. The Riemann Hypothesis for $L(E,s)$ states that $\beta_0 = 1$. $\square$

Proof of Lemma 3.1. Define $\beta_T := \sup\{\Re(z) : L(E,z) = 0, |\Im(z)| \leq T, \Re(z) < \beta_0\}$, which is strictly less than β_0 since $L(E,s)$ has only finitely many zeros of height at most T. We have, by the Riemann–von Mangoldt Formula, that

$$F_T(x) = \sum_{\substack{\rho \\ \Re(\rho)=\beta_0-\frac{1}{2} \\ |\Im(\rho)| \leq T}} \frac{x^{\mathrm{i}\gamma}}{\rho} - x^{1-\beta_0} + O_E(x^{-\delta_T}(\log T)^2), \tag{11}$$

where $\delta_T = \beta_0 - \beta_T > 0$ and $\rho = \eta + \mathrm{i}\gamma$ runs over the nontrivial zeros of $L(E, s+\frac{1}{2})$. Hence, the limiting logarithmic distribution of $F_T(x)$ coincides with the limiting distribution of

$$G_T(y) := -\epsilon(\beta_0) + \sum_{\substack{\rho \\ \Re(\rho)=\beta_0-\frac{1}{2} \\ |\Im(\rho)| \leq T}} \frac{e^{\mathrm{i}\gamma y}}{\rho},$$

which exists by arguments analogous to Lemma 2.3 of [24]. Here,

$$\epsilon(\beta_0) := \begin{cases} 1 & \text{if } \beta_0 = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

We now adapt Lemma 2.2 of [24].

Lemma 3.3. Let

$$\epsilon(x;T) := x^{-\beta_0+\frac{1}{2}} \sum_{\substack{\rho \\ |\Im(\rho)|>T}} \frac{x^{\rho}}{\rho},$$

where $\beta_0 = \sup\{\Re(z) : L(E,z)=0\}$ and ρ runs over the nontrivial zeros of $L(E, s+\frac{1}{2})$. Then we have, for $T \geq 1$ and $Y \geq 2$, that

$$\int_2^Y |\epsilon(e^y;T)|^2 \,\mathrm{d}y \ll_E Y\frac{(\log T)^2}{T} + \frac{(\log T)^3}{T}. \qquad \square$$

Proof. We compute

$$\int_2^Y |\epsilon(e^y;T)|^2 \,\mathrm{d}y = \sum_{\substack{\rho_1,\rho_2 \\ |\Im(\rho_1)|,|\Im(\rho_2)|>T}} \int_2^Y \frac{e^{y(\rho_1+\bar{\rho}_2-2\beta_0+1)}}{\rho_1\bar{\rho}_2} \,\mathrm{d}y$$

$$\ll_E \sum_{\substack{\rho_1,\rho_2 \\ |\Im(\rho_1)|,|\Im(\rho_2)|>T}} \frac{1}{|\Im(\rho_1)||\Im(\rho_2)|} \min(Y, |\rho_1+\bar{\rho}_2-2\beta_0+1|^{-1}),$$

since one can easily show that, for any $s \in \mathbb{C}$ with $\Re(s) \leq 0$,

$$\left| \int_2^Y e^{sy} \,\mathrm{d}y \right| \leq \min(2|s|^{-1}, Y).$$

The proof follows as in Lemma 2.2 of [24] since $|\rho_1+\bar{\rho}_2-2\beta_0+1|^{-1} \leq |\Im(\rho_1) - \Im(\rho_2)|^{-1}$. ■

Lemma 3.4. Assume that $L(E,s)$ has at least one zero on the line $\Re(s) = \beta_0 := \sup\{\Re(z) : L(E,z)=0\} \geq 1$. Then the quantity

$$E(x) := -\frac{\log x}{x^{\beta_0-\frac{1}{2}}} \sum_{p\leq x} \frac{a_p(E)}{\sqrt{p}}$$

has a limiting logarithmic distribution μ_E as $x \to \infty$. $\square$

Proof. We argue as in Lemma 2.2. Defining α_p and β_p as we did in this lemma, we have by (5.53) of [15] that, for $1 \leq U \leq x$ (see the corresponding (13) of [25]),

$$\sum_{\substack{p^e\leq x \\ e\geq 1 \\ p\nmid N_E}} (\alpha_p^e + \beta_p^e)\log p = -\sum_{|\Im(\rho)|\leq U} \frac{x^{\rho}}{\rho} + O\left(\frac{x}{U}\log x \log(N_E x)\right),$$

where ρ runs over the nontrivial zeros of $L(E, s+\frac{1}{2})$, and hence

$$-x^{-\beta_0+\frac{1}{2}}\sum_{p\le x}\frac{a_p(E)}{\sqrt{p}}\log p = x^{-\beta_0+\frac{1}{2}}\sum_{p\le\sqrt{x}}(\alpha_p^2+\beta_p^2)\log p + x^{-\beta_0+\frac{1}{2}}\sum_{|\Im(\rho)|\le U}\frac{x^{\rho}}{\rho} + O_E\left(\frac{x^{\frac{3}{2}-\beta_0}}{U}\log x\log(N_E x) + x^{\frac{5}{6}-\beta_0}\right).$$

As in Lemma 2.2, combining this with a Tauberian argument on $L'(\mathrm{Sym}^2 E, s+1)/L(\mathrm{Sym}^2 E, s+1)$ and a summation by parts gives that

$$E(x) = x^{-\beta_0+\frac{1}{2}}\sum_{|\Im(\rho)|\le U}\frac{x^{\rho}}{\rho} - x^{1-\beta_0} + O_E\left(\frac{x^{\frac{3}{2}-\beta_0}}{U}(\log x)^2\right) + o_{x\to\infty}(1),$$

and so taking $U = x$, using that $\beta_0\ge 1$, and applying Lemma 2.1, we obtain that, for any $T\ge 1$,

$$E(x) = x^{-\beta_0+\frac{1}{2}}\sum_{\rho}\frac{x^{\rho}}{\rho} - x^{1-\beta_0} + o_{x\to\infty}(1) = F_T(x) + \epsilon(x;T) + o_{x\to\infty}(1),$$

where $F_T(x)$ and $\epsilon(x;T)$ are defined in Lemmas 3.1 and 3.3, respectively.

Let now f be a bounded Lipschitz continuous function. We have as in Section 2.1 of [24] that

$$\begin{aligned}\frac{1}{Y}\int_2^Y f(E(e^y))\,\mathrm{d}y &= \frac{1}{Y}\int_2^Y f(F_T(e^y))\,\mathrm{d}y + O_f\left(\frac{1}{Y}\int_2^Y|\epsilon(e^y;T)|\,\mathrm{d}y\right) + o_{Y\to\infty}(1)\\ &= \frac{1}{Y}\int_2^Y f(F_T(e^y))\,\mathrm{d}y + O_f\left(\frac{1}{\sqrt{Y}}\left(\int_2^Y|\epsilon(e^y;T)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\right) + o_{Y\to\infty}(1)\\ &= \frac{1}{Y}\int_2^Y f(F_T(e^y))\,\mathrm{d}y + O_f\left(\frac{\log T}{\sqrt{T}} + \frac{(\log T)^{\frac{3}{2}}}{\sqrt{TY}}\right) + o_{Y\to\infty}(1)\end{aligned}$$

by the Cauchy–Schwartz inequality and Lemma 3.3. Taking $Y\to\infty$ and using that $F_T(e^y)$ has a limiting distribution (Lemma 3.1), we obtain

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu_E^{(T)}(x) - O\left(\frac{\log T}{\sqrt{T}}\right) &\le \liminf_{Y\to\infty}\frac{1}{Y}\int_2^Y f(E(e^y))\,\mathrm{d}y\\ &\le \limsup_{Y\to\infty}\frac{1}{Y}\int_2^Y f(E(e^y))\,\mathrm{d}y\\ &\le \int_{\mathbb{R}} f(x)\,\mathrm{d}\mu_E^{(T)}(x) + O\left(\frac{\log T}{\sqrt{T}}\right).\end{aligned} \tag{12}$$

As in [1] (see also Section 5.1 of [22]), we apply Helly's Theorem to the sequence of probability measures $\{\mu_E^{(T)}\}_{T\geq 1}$; this ensures the existence of a subsequence $\{\mu_E^{(T_k)}\}_{k\geq 1}$ which converges weakly to a limiting probability measure μ_E (since $\mu_E^{(T_k)}(\mathbb{R})=1$). The estimate (12) then shows that $\mu_E^{(T)}$ converges weakly to μ_E as $T\to\infty$, and thus

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_2^Y f(E(e^y))\,\mathrm{d}y=\liminf_{Y\to\infty}\frac{1}{Y}\int_2^Y f(E(e^y))\,\mathrm{d}y=\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu_E(x). \qquad \blacksquare$$

Remark 3.5. Alternatively, we could have concluded the existence of a limiting distribution by applying [10, Lemma 1.11], which asserts that if $\hat{\mu}_E^{(T)}(\xi)$ converges to a function uniformly in all compact subsets of $\mathbb{R}$, then $\mu_E^{(T)}$ converges weakly to a probability measure. We will see in Lemma 3.5 that (see also (21) of [25])

$$\hat{\mu}_E^{(T)}(\xi)=\mathrm{e}^{\mathrm{i}\xi(2r_{\mathrm{an}}(E)-1)\epsilon(\beta_0)}\prod_{\substack{\rho\\ \Re(\rho)=\beta_0-\frac{1}{2}\\ 0<\Im(\rho)\leq T}} J_0\left(\frac{2\xi}{|\rho|}\right),$$

which converges absolutely and uniformly in any compact subset of $\mathbb{R}$ to the function on the right-hand side of (13). $\square$

In the next lemma, we give an explicit description of the Fourier Transform of μ_E, which corresponds to (21) of [25].

Lemma 3.6. Assume that $L(E,s)$ has at least one zero on the line $\Re(s)=\beta_0:=\sup\{\Re(z):L(E,z)=0\}$ and assume that the set $\{\Im(z)\geq 0: L(E,z)=0, \Re(z)=\beta_0, z\neq 1\}$ is linearly independent over $\mathbb{Q}$. Then the Fourier Transform of μ_E is given by

$$\hat{\mu}_E(\xi)=\mathrm{e}^{\mathrm{i}\xi(2r_{\mathrm{an}}(E)-1)\epsilon(\beta_0)}\prod_{\substack{\rho\\ \Re(\rho)=\beta_0-\frac{1}{2}\\ \Im(\rho)>0}} J_0\left(\frac{2\xi}{|\rho|}\right), \tag{13}$$

where ρ runs over the nontrivial zeros of $L(E,s+\frac{1}{2})$ and

$$\epsilon(\beta_0):=\begin{cases}1 & \text{if } \beta_0=1,\\ 0 & \text{otherwise.}\end{cases} \qquad \square$$

Proof. We first compute the Fourier Transform of $\mu_E^{(T)}$. Note that the assumption that the set $\{\Im(z)\geq 0 : L(E,z)=0,\ \Re(z)=\beta_0,\ z\neq 1\}$ is linearly independent over $\mathbb{Q}$ implies that if $\beta_0>1$, then $L(E,\beta_0)\neq 0$. Hence, (11) becomes

$$F_T(x)=(2r_{\mathrm{an}}(E)-1)\epsilon(\beta_0)+\sum_{\substack{\rho \\ \Re(\rho)=\beta_0-\frac{1}{2} \\ 0<|\Im(\rho)|\leq T}}\frac{x^{\mathrm{i}\gamma}}{\rho}+o_{x\to\infty}(1)+O_E(x^{-\delta_T}(\log T)^2),$$

where ρ runs over the nontrivial zeros of $L(E,s+\frac{1}{2})$. Now, since $\mu_E^{(T)}$ is the limiting distribution of $F_T(e^y)$ by Lemma 3.1, we deduce by classical arguments (see, for instance, the proof of Proposition 2.13 of [12]) that

$$\hat{\mu}_E^{(T)}(\xi)=\mathrm{e}^{\mathrm{i}\xi(2r_{\mathrm{an}}(E)-1)\epsilon(\beta_0)}\prod_{\substack{\rho \\ \Re(\rho)=\beta_0-\frac{1}{2} \\ 0<\Im(\rho)\leq T}} J_0\left(\frac{2\xi}{|\rho|}\right).$$

The proof follows from the fact that the measures $\mu_E^{(T)}$ converge weakly to μ_E (this was established in the proof of Lemma 3.4), and thus Lévy's criterion implies that $\hat{\mu}_E^{(T)}(\xi)\to\hat{\mu}_E(\xi)$ pointwise (see [10, Lemma 1.11]). ■

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let $\{E_n\}_{n\geq 1}$ be a sequence of elliptic curves over $\mathbb{Q}$ for which $\mathrm{LI}(E_n)$ holds and for which, as $n\to\infty$,

$$\bar{\delta}(E_n)\to 1.$$

Assume also that the Riemann Hypothesis does not hold for $L(E_n,s)$, that is $\beta_n:=\sup\{\Re(z) : L(E_n,z)=0\}>1$, for arbitrarily large values of n. By Lemmas 3.4 and 3.6, the quantity

$$E_n(x):=-\frac{\log x}{x^{\beta_n-\frac{1}{2}}}\sum_{p\leq x}\frac{a_p(E_n)}{\sqrt{p}}$$

has a limiting logarithmic distribution μ_{E_n} whose Fourier transform is given, for the values of n for which $\beta_n>1$, by

$$\hat{\mu}_{E_n}(\xi)=\prod_{\substack{\rho \\ \Re(\rho)=\beta_n-\frac{1}{2} \\ \Im(\rho)>0}} J_0\left(\frac{2\xi}{|\rho|}\right), \tag{14}$$

where ρ runs over the nontrivial zeros of $L(E_n, s+\frac{1}{2})$. This implies that the limiting distribution of $E_n(e^y)$ is continuous, that is, μ_{E_n} is absolutely continuous with respect to Lebesgue measure. Indeed, if $L(E_n, s+\frac{1}{2})$ has a finite number of zeros ρ on the line $\Re(\rho)=\beta_n-\frac{1}{2}$, then the bound $|J_0(x)|\leq \min(1, (\pi|x|/2)^{-\frac{1}{2}})$ (see [24, (4.5)]) implies that, for $\xi \gg_n 1$,

$$|\hat{\mu}_{E_n}(\xi)| \leq \prod_{\substack{\rho \\ \Re(\rho)=\beta_n-\frac{1}{2} \\ \Im(\rho)>0}} (\pi|\xi|/|\rho|)^{-\frac{1}{2}} \ll_n |\xi|^{-\frac{1}{2}},$$

and thus combining this with the bound $|\hat{\mu}_{E_n}(\xi)|\leq 1$, the absolute continuity of μ_{E_n} follows by applying [10, Lemma 1.23]. As for the case where $L(E_n, s+\frac{1}{2})$ has infinitely many zeros ρ on the line $\Re(\rho)=\beta_n-\frac{1}{2}$, we have by adapting the proof of Lemma 2.16 of [12] that, for $\xi \gg_n 1$,

$$|\hat{\mu}_{E_n}(\xi)| \leq \prod_{\substack{\rho \\ \Re(\rho)=\beta_n-\frac{1}{2} \\ 0<\Im(\rho)<|\xi|/2-1}} (\pi|\xi|/|\rho|)^{-\frac{1}{2}} \leq 2^{-\#\{\rho:\Re(\rho)=\beta_n-\frac{1}{2}, 0<\Im(\rho)<|\xi|/2-1\}/2} = o_{|\xi|\to\infty}(1),$$

and once more the absolute continuity of μ_{E_n} follows by applying [10, Lemma 1.23].

Now, $J_0(t)$ is an even function, and thus so is $\hat{\mu}_{E_n}(\xi)$ by (14). Since $\hat{\mu}_{E_n}(\xi)$ is also real for real ξ, this implies that μ_{E_n} is symmetrical, and hence, for arbitrarily large values of n, we have by the absolute continuity of μ_{E_n} that

$$\bar{\delta}(E_n) = \underline{\delta}(E_n) = \delta(E_n) = \mu_{E_n}([0,\infty)) = \frac{1}{2},$$

contradicting our assumption that $\delta(E_n)\to 1$ as n tends to infinity. Having reached a contradiction, we deduce that, for all large enough values of n, the Riemann Hypothesis holds for $L(E_n, s)$.

We now show the existence of elliptic curves of arbitrarily large rank. Assume that Conjecture 1.1 is false, that is, for all elliptic curves E over $\mathbb{Q}$,

$$r_{\text{an}}(E) \ll \sqrt{\log N_E}.$$

We know that the Riemann Hypothesis holds for $L(E_n, s)$ for large enough n, and hence all of the lemmas of Section 2 hold. In particular, taking E to be any of these curves and denoting by X_E the random variable associated with μ_E, we have, by Lemma 2.6, that

$$\mathbb{E}[X_E] = 2r_{\text{an}}(E)-1, \quad \text{Var}[X_E] = \sum_{\gamma_E\neq 0} \frac{1}{\frac{1}{4}+\gamma_E^2} \asymp \log N_E, \tag{15}$$

(see also [25, (27)–(29)]) since the assumption LI(E) implies that the nonreal zeros of $L(E,s)$ are simple. Moreover, the Riemann hypothesis for $L(E,s)$ implies that we have $\rho = \frac{1}{2} + \mathrm{i}\gamma$ in Lemma 3.6, that is,

$$\hat{\mu}_E(\xi) = \mathrm{e}^{\mathrm{i}\xi(2r_{\mathrm{an}}(E)-1)} \prod_{\substack{\rho \\ \Im(\rho)>0}} J_0\left(\frac{2\xi}{\sqrt{\frac{1}{4}+\Im(\rho)^2}}\right).$$

Defining

$$Y_E := \frac{X_E - \mathbb{E}[X_E]}{\sqrt{\mathrm{Var}[X_E]}},$$

where X_E is the random variable associated with the measure μ_E, we have by the analyticity of $\log J_0(z)$ in the disk $|z| \le 12/5$ (The function $\log J_0(z)$ is holomorphic in the disk $|z| < x_0$, where $x_0 = 2.4048\ldots$ is the first zero of $J_0(z)$), that taking Taylor series in the range $|t| \le \sqrt{\mathrm{Var}[X_E]}$,

$$\begin{aligned}
\log \hat{Y}_E(t) &= \sum_{\substack{\rho \\ \Im(\rho)>0}} \log J_0\left(\frac{2t}{\sqrt{\mathrm{Var}[X_E]}\sqrt{\frac{1}{4}+\Im(\rho)^2}}\right) \\
&= -\frac{(2t)^2}{4\mathrm{Var}[X_E]} \sum_{\substack{\rho \\ \Im(\rho)>0}} \frac{1}{\frac{1}{4}+\Im(\rho)^2} + O\left(\frac{t^4}{(\log N_E)^2} \sum_{\substack{\rho \\ \Im(\rho)>0}} \frac{1}{\left(\frac{1}{4}+\Im(\rho)^2\right)^2}\right) \\
&= -\frac{t^2}{2} + O\left(\frac{t^4}{\log N_E}\right)
\end{aligned}$$

by the Riemann–von Mangoldt formula. Hence, $\hat{Y}_E(t) \to \mathrm{e}^{-t^2/2}$ pointwise as $N_E \to \infty$, and thus Lévy's criterion (see [10, Lemma 1.11]) implies that Y_E converges weakly to a Gaussian distribution. By the absolute continuity of μ_E, we have that $\delta(E)$ exists, and

$$\delta(E) = \mathrm{Prob}[X_E > 0] = \mathrm{Prob}[Y_E > -\mathbb{E}[X_E]/\sqrt{\mathrm{Var}[X_E]}].$$

Therefore, the assumption that $r_{\mathrm{an}}(E) \ll \sqrt{\log N_E}$ and (15) imply that this last quantity is

$$\le \mathrm{Prob}[Y_E > -C],$$

for some absolute constant C. By the central limit theorem we just proved, this quantity tends to

$$\frac{1}{\sqrt{2\pi}}\int_{-C}^{\infty} e^{-\frac{t^2}{2}}\,dt < 1$$

as $N_E \to \infty$. Therefore, we obtain the bound

$$\limsup_{N_E\to\infty} \delta(E) \le \frac{1}{\sqrt{2\pi}}\int_{-C}^{\infty} e^{-\frac{t^2}{2}}\,dt < 1,$$

which contradicts our assumption that $\delta(E_n) = \bar{\delta}(E_n) \to 1$ as $n \to \infty$. Having reached a contradiction, we conclude that Conjecture 1.1 holds. ■

Proof of Theorem 1.7. Following the steps of the proof of Theorem 1.5, one sees that if LI(E) holds and $\beta_E > 1$, then by the symmetry and absolute continuity of μ_E we have

$$\bar{\delta}(E_n) = \underline{\delta}(E_n) = \delta(E_n) = \mu_{E_n}([0,\infty)) = \frac{1}{2}.$$

Hence, we have proved the contrapositive Theorem 1.7. ■

Acknowledgement

I would like to thank Enrico Bombieri, Andrew Granville, Jeffrey Lagarias, and Peter Sarnak for their support and valuable advice. I also thank Barry Mazur for motivating me to weaken the Linear Independence Hypothesis in the necessary condition. I thank Jeffrey Lagarias for suggesting to me to provide an equivalence rather than a one-sided implication. Finally, I thank Byungchul Cha, Peter Humphries, Kartik Prasanna, Jan-Christoph Schlage-Puchta, and Anders Södergren for their comments and for fruitful conversations, and the referee for his comments.

Funding

This work was supported by an NSERC Postdoctoral Fellowship, as well as NSF grant DMS-0635607, and was accomplished at the Institute for Advanced Study and at the University of Michigan.

Appendix 1. Comparison of Conjectures on Large Ranks of Elliptic Curves

We conclude the paper with a numerical study of Conjecture 1.1, comparing the conjectures of Ulmer and Farmer, Gonek, and Hughes. In this section, we assume the

Birch and Swinnerton-Dyer Conjecture, that is, $r_{\mathrm{an}}(E) = r_{\mathrm{al}}(E)$ for every elliptic curve E over $\mathbb{Q}$.

Mestre [19] has shown that

$$r_{\mathrm{an}}(E) \ll \log N_E,$$

and under ECRH,

$$r_{\mathrm{an}}(E) \ll \frac{\log N_E}{\log\log N_E}. \tag{A.1}$$

For elliptic curves over function fields, Ulmer has shown that the analog of (A.1) is best possible, and thus he conjectured that (A.1) is also best possible for elliptic curves over $\mathbb{Q}$ [28, 29]. Elkies and Watkins [9] have given numerical evidence for Ulmer's conjecture by finding elliptic curves having large rank and moderate conductor. They mention that numerical data show that the statement

$$0 < \limsup_{N_E \to \infty} \frac{r_{\mathrm{an}}(E)}{\log N_E / \log\log N_E} < \infty \tag{A.2}$$

is quite likely to be true. A few years later, Farmer et al. [11] constructed a random matrix model that suggests the conjecture

$$0 < \limsup_{N_E \to \infty} \frac{r_{\mathrm{an}}(E)}{\sqrt{\log N_E \log\log N_E}} < \infty. \tag{A.3}$$

Interestingly, Elkies and Watson's numerical data support both (A.2) and (A.3). The reason for this is that the quotient between the two conjectures, that is

$$f(N_E) := \frac{(\log N_E)^{\frac{1}{2}}}{(\log\log N_E)^{\frac{3}{2}}},$$

is contained in the interval [0.86, 1.5] for all conductors $25 \le N_E \le 10^{250}$, and hence it is impossible to decide which of (A.2) or (A.3) is more likely to be true with the current data. Let us compare these conjectures with the elliptic curves appearing in [9]:

$r_{\mathrm{al}}(E)$	N_E	$\log N_E/\log\log N_E$	$\sqrt{\log N_E \log\log N_E}$
5	$1.9 \cdot 10^{7}$	6.874	5.946
6	$5.2 \cdot 10^{9}$	8.338	7.198
7	$3.8 \cdot 10^{11}$	9.358	8.122
8	$2.5 \cdot 10^{14}$	10.773	9.469
9	$3.2 \cdot 10^{16}$	11.759	10.448
10	$1.0 \cdot 10^{18}$	12.861	11.582
11	$1.8 \cdot 10^{22}$	14.203	13.018

and those appearing in [19] (The first column of this table is actually a lower bound on the rank, which can be shown to equal the rank under standard hypotheses.):

$r_{\mathrm{al}}(E)$	N_E	$\log N_E/\log\log N_E$	$\sqrt{\log N_E \log\log N_E}$
3	$5.1 \cdot 10^{3}$	4.277	3.980
4	$5.4 \cdot 10^{5}$	5.839	5.118
5	$1.7 \cdot 10^{8}$	7.482	6.455
6	$5.1 \cdot 10^{10}$	8.893	7.696
7	$3.2 \cdot 10^{12}$	9.841	8.574
8	$1.8 \cdot 10^{15}$	11.181	9.870
9	$7.0 \cdot 10^{19}$	13.215	11.956
10	$5.2 \cdot 10^{22}$	14.386	13.217
11	$1.8 \cdot 10^{24}$	14.989	13.884
12	$2.7 \cdot 10^{29}$	16.903	16.073
13	$2.1 \cdot 10^{38}$	19.885	19.699
14	$3.6 \cdot 10^{37}$	19.640	19.391

Both of these tables show that (A.2) and (A.3) are equally likely to be true.

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