Math. Ann. (2013) 356:1117–1162
DOI 10.1007/s00208-012-0874-1

Mathematische Annalen

Prime number races with three or more competitors

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Received: 7 January 2012 / Revised: 13 August 2012 / Published online: 25 November 2012

Abstract Let $q \geq 3$ and $2 \leq r \leq \phi(q)$ be positive integers, and $a_1, \ldots, a_r$ be distinct reduced residue classes modulo q. Rubinstein and Sarnak defined $\delta(q; a_1, \ldots, a_r)$ to be the logarithmic density of the set of real numbers x such that $\pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r)$. In this paper, we establish an asymptotic formula for $\delta(q; a_1, \ldots, a_r)$ when $r \geq 3$ is fixed and q is large. Several applications concerning these prime number races are then deduced. First, comparing with a recent work of Fiorilli and Martin on the case $r = 2$, we show that these densities behave differently when $r \geq 3$. Another surprising consequence of our results is that, unlike two-way races, biases do appear in races involving three or more squares (or non-squares) to large moduli. Furthermore, we establish a partial result towards a conjecture of Rubinstein and Sarnak on biased races, and disprove a recent conjecture of Feuerverger and Martin concerning bias factors. Lastly, we use our method to derive the Fiorilli and Martin asymptotic formula for the densities when $r = 2$.

Mathematics Subject Classification (2010) Primary 11N13; Secondary 11N69 · 11M26

The author is supported by a postdoctoral fellowship from the Natural Sciences and Engineering Research Council of Canada.

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Contents

1 Introduction

In 1853 Chebyshev observed that primes congruent to 3 modulo 4 seem to predominate over those congruent to 1 modulo 4. In general, if a is a non-square modulo q and b is a square modulo q then $\pi(x; q, a)$ has a strong tendency to be larger than $\pi(x; q, b)$, where $\pi(x; q, a)$ denotes the number of primes less than x that are congruent to a modulo q. This general phenomenon is known as "Chebyshev's bias". This bias might appear unexpected in view of the prime number theorem for arithmetic progressions which states that $\lim_{x\to\infty} \pi(x; q, a)/\pi(x; q, b) = 1$, for any a and b that are coprime to q. However, this asymptotic result does not give us any information on the difference $\pi(x; q, a) - \pi(x; q, b)$. In 1914, Littlewood [23] proved that the quantities $\pi(x; 4, 3) - \pi(x; 4, 1)$ and $\pi(x; 3, 2) - \pi(x; 3, 1)$ change sign infinitely often. Similar results to other moduli were subsequently derived by Knapowski and Turán [10–17], under some hypotheses on the zeros of Dirichlet L-functions.

A generalization of Chebyshev's question is the so-called "Shanks and Rényi prime number race" which is colorfully described by Knapowski and Turán in [10–17]. Let $q \geq 3$ and $2 \leq r \leq \phi(q)$ be positive integers, and denote by $\mathcal{A}_r(q)$ the set of ordered r-tuples of distinct residue classes $(a_1, a_2, \ldots, a_r)$ modulo q which are coprime to q. For $(a_1, a_2, \ldots, a_r) \in \mathcal{A}_r(q)$, consider a game with r players called "1" through "r", where at time x, the player "j" has a score of $\pi(x; q, a_j)$. Will all $r!$ orderings of the players occur for infinitely many integers x?

It is generally believed that the answer to this question is yes for all q and all $(a_1, a_2, \ldots, a_r) \in \mathcal{A}_r(q)$. This problem has been extensively studied by many authors, including Knapowski and Turán [10–17], Bays and Hudson [1] and [2], Kaczorowski [18, 19] and [20], Feuerverger and Martin [4], Martin [24], Ford and Konyagin [6] and [7], Fiorilli and Martin [5], and the author [22]. For a complete history of this subject, one can refer to the delightful articles of Granville and Martin [8], and Ford and Konyagin [7].

In their fundamental work of 1994, Rubinstein and Sarnak [25] solved the Shanks and Rényi prime race problem assuming the Generalized Riemann Hypothesis GRH and the Linear Independence Hypothesis LI (which is the assumption that the non-negative imaginary parts of the zeros of all Dirichlet L-functions attached to primitive characters modulo q are linearly independent over the rationals). For $(a_1, a_2, \ldots, a_r) \in \mathcal{A}_r(q)$, let $P_{q;a_1,\ldots,a_r}$ be the set of real numbers $x \geq 2$ such that

$$\pi(x;q,a_1) > \pi(x;q,a_2) > \cdots > \pi(x;q,a_r).$$

Rubinstein and Sarnak showed, assuming GRH and LI, that for any $(a_1,\ldots,a_r) \in \mathcal{A}_r(q)$ the logarithmic density of $P_{q;a_1,\ldots,a_r}$ defined by

$$\delta(q;a_1,\ldots,a_r) := \lim_{x\to\infty} \frac{1}{\log x} \int\limits_{t\in P_{q;a_1,\ldots,a_r}\cap[2,x]} \frac{dt}{t},$$

exists and is positive. In fact this is corollary of a stronger result they proved, that there exists an absolutely continuous measure $\mu_{q;a_1,\ldots,a_r}$ such that

$$\delta(q;a_1,\ldots,a_r) = \int\limits_{x_1>x_2>\cdots>x_r} d\mu_{q;a_1,\ldots,a_r}(x_1,\ldots,x_r). \tag{1.1}$$

All the results we obtain in this paper are conditional on the same two hypotheses (namely GRH and LI) as the work of Rubinstein and Sarnak. In [6], Ford and Konyagin showed that assumptions on the locations of the zeros of Dirichlet L-functions are indeed necessary in order to obtain results on prime number races with three or more competitors.

In the case of a race between two residue classes a and b modulo q, Rubinstein and Sarnak proved that $\delta(q;a,b) = \delta(q;b,a) = 1/2$ if a and b are both squares or both non-squares modulo q, and otherwise $\delta(q;a,b) > 1/2$ if a is a non-square and b is a square modulo q (note that $\delta(q;b,a) = 1-\delta(q;a,b)$). They also showed that $\delta(q;a,b) \to 1/2$ as $q\to\infty$, uniformly for all distinct reduced residue classes a, b modulo q. In fact, they proved that in general all biases disappear when $q\to\infty$. Let

$$\Delta_r(q) := \max_{(a_1,a_2,\ldots,a_r)\in\mathcal{A}_r(q)} \left|\delta(q;a_1,\ldots,a_r) - \frac{1}{r!}\right|.$$

Then for any fixed $r \geq 2$, Rubinstein and Sarnak showed, assuming GRH and LI, that

$$\Delta_r(q) \to 0 \quad \text{as } q\to\infty. \tag{1.2}$$

In the case $r = 2$, Fiorilli and Martin [5] have recently established an asymptotic expansion for $\delta(q;a,b) - 1/2$ when a is a non-square and b is a square modulo q, conditionally on GRH and LI. A corollary of their results is that for q large

$$\Delta_2(q) = \frac{1}{q^{1/2+o(1)}}.$$

A surprising consequence of our results is that $\Delta_r(q)$ behaves in a completely different way when $r \geq 3$.

Theorem 1.1 *Assume GRH and LI. Let $r \geq 3$ be a fixed integer. If q is large, then*

$$\Delta_r(q) \asymp_r \frac{1}{\log q}.$$

Recall that a bias occurs in a two-way race $\{q; a_1, a_2\}$ if and only if one of the residue classes a_1 and a_2 is a square and the other is a non-square modulo q. An interesting problem is then to determine when these biases appear for general races $\{q; a_1, \dots, a_r\}$ with $r \geq 3$. To make things clear we need to precisely define the notions of "biased" and "unbiased" races. Although Rubinstein and Sarnak called a race $\{q; a_1, \dots, a_r\}$ unbiased if the density function associated to the measure $\mu_{q;a_1,\dots,a_r}$ is symmetric, we believe that a more appropriate definition is the following

Definition 1.2 Let $(a_1, \dots, a_r) \in \mathcal{A}_r(q)$. The race $\{q; a_1, \dots, a_r\}$ is said to be *unbiased* if for every permutation σ of the set $\{1, 2, \dots, r\}$ we have

$$\delta(q; a_{\sigma(1)}, \dots, a_{\sigma(r)}) = \delta(q; a_1, \dots, a_r) = \frac{1}{r!}.$$

Furthermore, a race is said to be *biased* if this condition does not hold.

First, observe that if the race $\{q; a_1, \dots, a_r\}$ is unbiased then the races $\{q; a_{i_1}, \dots, a_{i_s}\}$ are unbiased for any subset $\{i_1, \dots, i_s\}$ of $\{1, \dots, r\}$. In view of the results of Rubinstein and Sarnak on two-way races, this clearly shows that a race $\{q; a_1, \dots, a_r\}$ is biased if there are $1 \leq i \neq j \leq r$ such that a_i is a square and a_j is a non-square modulo q. Furthermore, it is obvious from (1.1) that the race $\{q; a_1, \dots, a_r\}$ is unbiased if the density function of $\mu_{q;a_1,\dots,a_r}$ is symmetric. Rubinstein and Sarnak investigated the Fourier transform of $\mu_{q;a_1,\dots,a_r}$ for $r \geq 3$, and showed that the only case when this distribution is symmetric occurs when $r = 3$ and

$$a_2 \equiv a_1 \rho \bmod q, \quad a_3 \equiv a_1 \rho^2 \bmod q, \tag{1.3}$$

for some $\rho \neq 1$ with $\rho^3 \equiv 1 \bmod q$. However, this result still leaves open the possibility that unbiased races not satisfying assumption (1.3) might exist (since, for example, a function can be positive half of the time without being symmetric). Nonetheless, Rubinstein and Sarnak conjectured that the only case when a race involving three or more competitors is unbiased corresponds to (1.3).

Conjecture 1.3 (Rubinstein and Sarnak [25]) *When $r \geq 3$, the race $\{q; a_1, \dots, a_r\}$ is unbiased if and only if $r = 3$ and the residue classes a_1, a_2, and a_3 satisfy assumption* (1.3).

Feuerverger and Martin [4] were the first to exhibit explicit examples of biased races with three competitors, where the residue classes are either squares or non-squares not satisfying assumption (1.3). For example they showed, under GRH and LI, that the races $\{8; 3, 5, 7\}$ and $\{12; 5, 7, 11\}$ are biased. However, all the examples they considered satisfy $r \leq 4$ and $q \leq 12$, thus leaving open the problem of determining the existence of biased races of this type for any $q > 12$ and $3 \leq r \leq \phi(q)$. We answered this question for any fixed $r \geq 3$ if q is large enough. Indeed we show that unlike two-way races, biases do appear in races involving three or more squares (or non-squares) modulo q, if q is sufficiently large.

Theorem 1.4 *Assume GRH and LI. Given* $r \geq 3$*, there exists a positive number* $q_0(r)$ *such that for any* $q \geq q_0(r)$ *there are two* r*-tuples* $(a_1, \dots, a_r)$, $(b_1, \dots, b_r) \in \mathcal{A}_r(q)$*, with all of the* a_i *being squares and all of the* b_i *being non-squares modulo* q*, and such that both the races* $\{q; a_1, \dots, a_r\}$ *and* $\{q; b_1, \dots, b_r\}$ *are biased.*

For distinct non-zero integers $a_1, \dots, a_r$, we define $\mathcal{Q}_{a_1,\dots,a_r}$ to be the set of positive integers q such that $a_1, \dots, a_r$ are distinct modulo q, and $(q, a_i) = 1$ for all $1 \leq i \leq r$. When $r = 3$, assumption (1.3) implies that $a_1^2 \equiv a_2 a_3 \bmod q$, $a_2^2 \equiv a_1 a_3 \bmod q$, and $a_3^2 \equiv a_1 a_2 \bmod q$. Hence if $q > 2\max(|a_i|^2)$ then these congruences become identities. However, since the a_i are assumed to be distinct these equalities can not hold. This leads to a weak form of Conjecture 1.3:

Conjecture 1.5 *Let* $r \geq 3$ *and* $a_1, \dots, a_r$ *be distinct non-zero integers. Then for all positive integers* $q \in \mathcal{Q}_{a_1,\dots,a_r}$ *such that* $q > 2\max(|a_i|^2)$*, the race* $\{q; a_1, \dots, a_r\}$ *is biased.*

We prove the following partial result towards this conjecture, which follows from Theorem 2.6 below.

Theorem 1.6 *Assume GRH and LI. Let* $r \geq 3$ *and* $a_1, \dots, a_r$ *be distinct non-zero integers such that* a_j/a_k *equal* -1 *or a prime power, for some* $1 \leq j \neq k \leq r$*. Then for all but finitely many integers* $q \in \mathcal{Q}_{a_1,\dots,a_r}$*, the race* $\{q; a_1, \dots, a_r\}$ *is biased.*

To establish these results, we prove an asymptotic formula for $\delta(q; a_1, \dots, a_r)$ valid for large q, and then we investigate the behavior of its first few terms. Our approach is different from the one used by Fiorilli and Martin [5] in the case $r = 2$. Their method relies on using a certain symmetry of the measure $\mu_{q;a_1,a_2}$ (which is a measure on $\mathbb{R}^2$) to reduce the RHS of (1.1) to a one dimensional integral over a related measure ρ_q on $\mathbb{R}$. This symmetry could be used to reduce the measure $\mu_{q;a_1,\dots,a_r}$ in the case $r \geq 3$ to a measure on $\mathbb{R}^{r-1}$, as is done in [4]. However, this does not help in this case, and one might as well deal with the original measure on $\mathbb{R}^r$. Instead, our approach relies on studying the measure $\mu_{q;a_1,\dots,a_r}$ by carefully analyzing its Fourier transform $\hat{\mu}_{q;a_1,\dots,a_r}$. In particular, we will exploit the fact that $\hat{\mu}_{q;a_1,\dots,a_r}$ can be approximated by the Fourier transform of a multivariate Gaussian at small arguments, when $q \to \infty$.

In the next section we shall discuss these results in details. In particular we shall describe the asymptotic formula we prove for the densities $\delta(q; a_1, \dots, a_r)$ and deduce further consequences.

2 Detailed statement of results

For a non-principal Dirichlet character χ modulo q, we denote by $\{\gamma_\chi\}$ the multiset of imaginary parts of the non-trivial zeros of $L(s, \chi)$. Let χ_0 denote the principal character modulo q and define $S = \bigcup_{\chi \neq \chi_0 \bmod q} \{\gamma_\chi\}$. Moreover, let $\{U(\gamma_\chi)\}_{\gamma_\chi \in S}$ be a multiset of independent random variables uniformly distributed on the unit circle.

Rubinstein and Sarnak established, under GRH and LI, that the distribution $\mu_{q;a_1,\ldots,a_r}$ is the probability measure corresponding to the random vector

$$X_{q;a_1,\ldots,a_r} = (X(q,a_1),\ldots,X(q,a_r)),$$

where

$$X(q,a) = -C_q(a) + \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \sum_{\gamma_\chi>0} \frac{2\mathrm{Re}(\chi(a)U(\gamma_\chi))}{\sqrt{\frac{1}{4}+\gamma_\chi^2}},$$

and

$$C_q(a) := -1 + \sum_{\substack{b^2\equiv a \bmod q\\ 1\le b\le q}} 1. \tag{2.1}$$

Note that for $(a,q)=1$ the function $C_q(a)$ takes only two values: $C_q(a)=-1$ if a is a non-square modulo q, and $C_q(a)=C_q(1)$ if a is a square modulo q. Furthermore, an elementary argument shows that $C_q(a)\le d(q)\ll_\epsilon q^\epsilon$ for any $\epsilon>0$, where $d(q)=\sum_{m|q}1$ is the usual divisor function.

The covariance matrix of a random vector $\mathbf{Z}=(Z_1,\ldots,Z_n)$ is the $n\times n$ matrix whose (j,k) entry is $\mathrm{Cov}(Z_j,Z_k)=\mathbb{E}\Big(\big(Z_j-\mathbb{E}(Z_j)\big)\big(Z_k-\mathbb{E}(Z_k)\big)\Big)$, where $\mathbb{E}(Y)$ denotes the expectation of the random variable Y. Let $\mathrm{Cov}_{q;a_1,\ldots,a_r}$ be the covariance matrix of the random vector $X_{q;a_1,\ldots,a_r}$. A straightforward computation shows that the entries of $\mathrm{Cov}_{q;a_1,\ldots,a_r}$ are

$$\mathrm{Cov}_{q;a_1,\ldots,a_r}(j,k) = \begin{cases} N_q & \text{if } j=k\\ B_q(a_j,a_k) & \text{if } j\neq k,\end{cases}$$

where

$$N_q := 2\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \sum_{\gamma_\chi>0} \frac{1}{\frac{1}{4}+\gamma_\chi^2}, \quad \text{and} \quad B_q(a,b) := \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \sum_{\gamma_\chi>0} \frac{\chi\left(\frac{b}{a}\right)+\chi\left(\frac{a}{b}\right)}{\frac{1}{4}+\gamma_\chi^2}. \tag{2.2}$$

We shall later prove (see Lemma 3.1 and Corollary 5.4 below) that

$$N_q \sim \phi(q)\log q, \quad \text{and} \quad B_q(a,b) \ll \phi(q),$$

uniformly for all pairs (a,b) of distinct reduced residue classes modulo q.

In the case $r=2$, Fiorilli and Martin [5] established, under GRH and LI, that

$$\delta(q;a_1,a_2) = \frac{1}{2} - \frac{C_q(a_1)-C_q(a_2)}{\sqrt{2\pi V_q(a_1,a_2)}} + O\left(\frac{C_q(1)^3}{V_q(a_1,a_2)^{3/2}}\right), \tag{2.3}$$

where $V_q(a_1, a_2) = 2N_q - 2B_q(a_1, a_2)$. In Sect. 8 we shall derive this asymptotic using a slight modification of our method.

Before stating our results, let us define some notation which shall be used throughout this paper. Let

$$C_q = C_q(a_1, \ldots, a_r) := \max_{1\le j\le q} |C_q(a_j)|, \quad \text{and} \quad B_q = B_q(a_1, \ldots, a_r) := \max_{1\le j<k\le r} |B_q(a_j, a_k)|.$$

Moreover for $1 \le j \ne k \le r$, we define the following integrals which shall appear in the asymptotic formula of $\delta(q; a_1, \ldots, a_r)$

$$\alpha_j(r) := (2\pi)^{-r/2} \int\limits_{x_1>x_2>\ldots>x_r} x_j \exp\left(-\frac{x_1^2+\ldots+x_r^2}{2}\right) dx_1 \ldots dx_r,$$

$$\lambda_j(r) := (2\pi)^{-r/2} \int\limits_{x_1>x_2>\ldots>x_r} (x_j^2-1) \exp\left(-\frac{x_1^2+\ldots+x_r^2}{2}\right) dx_1 \ldots dx_r,$$

and

$$\beta_{j,k}(r) := (2\pi)^{-r/2} \int\limits_{x_1>x_2>\cdots>x_r} x_j x_k \exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right) dx_1 \ldots dx_r.$$

Theorem 2.1 *Assume GRH and LI. Fix an integer $r \ge 2$. If q is a large positive integer and $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$, then*

$$\begin{aligned}\delta(q; a_1, \ldots, a_r) = \frac{1}{r!} - \frac{1}{\sqrt{N_q}} \sum_{1\le j\le r} \alpha_j(r) C_q(a_j) + \frac{1}{N_q} \sum_{1\le j<k\le r} \beta_{j,k}(r) B_q(a_j, a_k) \\ + \frac{1}{2N_q}\left(\sum_{1\le j\le r} \lambda_j(r) C_q(a_j)^2 + 2 \sum_{1\le j<k\le r} \beta_{j,k}(r) C_q(a_j) C_q(a_k)\right) \\ + O_r\left(\frac{1}{N_q} + \frac{C_q B_q}{N_q^{3/2}} + \frac{B_q^2}{N_q^2}\right).\end{aligned}$$

As a corollary we obtain

Corollary 2.2 *Under the same assumptions of Theorem 2.1 we have*

$$\begin{aligned}\delta(q; a_1, \ldots, a_r) = \frac{1}{r!} - \frac{1}{\sqrt{N_q}} \sum_{1\le j\le r} \alpha_j(r) C_q(a_j) + \frac{1}{N_q} \sum_{1\le j<k\le r} \beta_{j,k}(r) B_q(a_j, a_k) \\ + O_r\left(\frac{C_q^2}{N_q} + \frac{B_q^2}{N_q^2}\right).\end{aligned}$$

In particular, we get for $r = 3$ that

Corollary 2.3 *Under the same assumptions of Theorem 2.1 we have*

$$\begin{aligned}\delta(q; a_1, a_2, a_3) = \frac{1}{6} &+ \frac{1}{4\sqrt{\pi N_q}}(C_q(a_3) - C_q(a_1)) \\ &+ \frac{1}{4\pi\sqrt{3}N_q}(B_q(a_1, a_2) + B_q(a_2, a_3) - 2B_q(a_1, a_3)) \\ &+ O\left(\frac{C_q^2}{N_q} + \frac{B_q^2}{N_q^2}\right).\end{aligned}$$

Remark The main difference between the cases $r = 2$ and $r \geq 3$ lies in the fact that $\beta_{1,2}(2) = 0$, which implies that the terms involving $B_q(a_j, a_k)$ are missing in the case $r = 2$. Indeed, we shall later prove that the contribution of these terms can be $\gg_r 1/\log q$. This explains the surprising behavior of $\Delta_r(q)$ when $r \geq 3$, since $C_q(a)/\sqrt{N_q} = q^{-1/2+o(1)}$. Note that our asymptotic formula is not accurate in the case $r = 2$ since the error term may exceed the main term. We shall slightly modify the argument of the proof to handle this case in Sect. 8.

Investigating the terms $B_q(a_j, a_k)$ and using the fact that $B_q \ll \phi(q)$, we prove the following result, which is a stronger form of Theorem 1.1.

Theorem 2.4 *Assume GRH and LI. Fix $r \geq 3$ and let q be large. Then for all $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$ we have*

$$\left|\delta(q; a_1, \ldots, a_r) - \frac{1}{r!}\right| \ll_r \frac{1}{\log q}.$$

Moreover there exist residue classes $(b_1, \ldots, b_r)$, $(d_1, \ldots, d_r) \in \mathcal{A}_r(q)$ such that

$$\delta(q; b_1, \ldots, b_r) > \frac{1}{r!} + \frac{c_1(r)}{\log q} \quad \text{and} \quad \delta(q; d_1, \ldots, d_r) < \frac{1}{r!} - \frac{c_1(r)}{\log q},$$

for some constant $c_1(r) > 0$ which depends only on r.

This result implies that for some residue classes $a_1, \ldots, a_r$ modulo q the distance $|\delta(q; a_1, \ldots, a_r) - 1/r!|$ can be $\gg_r 1/\log q$. An interesting question is then to investigate for which residue classes modulo q does this extreme bias occur. To this end let us make the following definition.

Definition 2.5 Fix $r \geq 3$ and let q be large. We call a race $\{q; a_1, \ldots, a_r\}$ "*q-extremely biased*" if for some permutation σ of the set $\{1, \ldots, r\}$ we have

$$\left|\delta(q; a_{\sigma(1)}, \ldots, a_{\sigma(r)}) - \frac{1}{r!}\right| \gg_r \frac{1}{\log q}.$$

We can completely characterize q-extremely biased races $\{q; a_1, \ldots, a_r\}$ when the residue classes $a_1, \ldots, a_r$ are bounded and q is large.

Theorem 2.6 *Assume GRH and LI. Fix $r \geq 3$ and let $A \geq 1$ be a real number. If $a_1, \ldots, a_r$ are distinct integers with $|a_i| \leq A$, and q is a large positive integer with $(q, a_i) = 1$, then the race $\{q; a_1, \ldots, a_r\}$ is extremely biased if and only if there exist $1 \leq j \neq k \leq r$ such that a_j/a_k equals -1 or a prime power.*

Moreover, if this condition does not hold, then for any permutation σ of the set $\{1, \ldots, r\}$

$$\delta(q; a_{\sigma(1)}, \ldots, a_{\sigma(r)}) - \frac{1}{r!}$$
$$\ll_{A,r} \begin{cases} \dfrac{\log q}{q} & \textit{if all the } a_i \textit{ are squares (or non-squares)} \bmod q, \\ q^{-1/2+o(1)} & \textit{otherwise.} \end{cases}$$

Since the functions $\sum_{j=1}^r x_j$, $\sum_{j=1}^r (x_j^2 - 1)$ and $\sum_{1\leq j<k\leq r} x_j x_k$ are symmetric in the variables $x_1, \ldots, x_r$, and $\int_{\mathbb{R}} x \exp(-x^2/2)dx = \int_{\mathbb{R}} (x^2 - 1)\exp(-x^2/2)dx = 0$, we deduce that

$$\sum_{j=1}^{r} \alpha_j(r) = \sum_{j=1}^{r} \lambda_j(r) = \sum_{1\leq j<k\leq r} \beta_{j,k}(r) = 0.$$

Therefore, in the case where all of the a_i are squares (or all of them are non-squares) modulo q we obtain the following corollary of Theorem 2.1:

Corollary 2.7 *Assume GRH and LI. Fix $r \geq 3$ and let q be large. Then, for any $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$ such that all of the a_i are squares (or all of them are non-squares) modulo q, we have*

$$\delta(q; a_1, \ldots, a_r) = \frac{1}{r!} + \frac{1}{N_q} \sum_{1\leq j<k\leq r} \beta_{j,k}(r) B_q(a_j, a_k) + O_r\left(\frac{1}{N_q} + \frac{C_q B_q}{N_q^{3/2}} + \frac{B_q^2}{N_q^2}\right).$$

Using this result along with an explicit construction of the residue classes $a_1, \ldots, a_r$ modulo q, we prove a strong from of Theorem 1.4.

Theorem 2.8 *Assume GRH and LI. Fix $r \geq 3$ and let q be large. Then there exist two r-tuples $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_r) \in \mathcal{A}_r(q)$, with all of the a_i being squares and all of the b_i being non-squares modulo q, and a permutation σ of the set $\{1, \ldots, r\}$, such that*

$$\delta(q; a_1, \ldots, a_r) = \delta(q; b_1, \ldots, b_r) < \frac{1}{r!} - \frac{c_2(r)}{\log^3 q}$$

and

$$\delta(q; a_{\sigma(1)}, \ldots, a_{\sigma(r)}) = \delta(q; b_{\sigma(1)}, \ldots, b_{\sigma(r)}) > \frac{1}{r!} + \frac{c_2(r)}{\log^3 q},$$

for some constant $c_2(r) > 0$ which depends only on r.

Remark In the proof of Theorem 2.8 we constructed the squares a_i by choosing $a_1 = 1$, $a_r = p_1^2$ and $a_j = (p_1 p_2)^{2j}$ for $2 \leq j \leq r-1$, where $p_1 \neq p_2$ are the smallest primes that are coprime with q. The non-squares b_i are simply constructed by taking $b_i = ba_i$ where b is any non-square modulo q. The primes p_1, p_2 can be as large as $\log q$, which explains why only the order of magnitude $1/(\log q)^3$ is obtained, rather than the $1/(\log q)$ that we derived in Theorem 2.4. Indeed if -1 is a square modulo q, or the primes p_1, p_2 are bounded, then we can modify our construction so that the term $c_2(r)/(\log q)^3$ can be replaced by $c_2(r)/\log q$ in the statement of Theorem 2.8.

It is clear from Theorem 2.1 that in order to understand the behavior of $\delta(q; a_1, \ldots, a_r)$, we have to investigate the size of $B_q(a,b)$ for $(a,b) \in \mathcal{A}_2(q)$. Recall that $B_q(a,b) \ll \phi(q)$. On the other hand we shall prove that this bound is attained if $a + b \equiv 0 \bmod q$ (this is a consequence of Proposition 5.1 below), so that

$$\max_{(a,b)\in\mathcal{A}_2(q)} |B_q(a,b)| \asymp \phi(q).$$

An interesting question is then to determine the order of magnitude of $|B_q(a,b)|$ for a generic pair $(a,b) \in \mathcal{A}_2(q)$. We prove that on average $|B_q(a,b)| \asymp \log q$.

Theorem 2.9 *Assume GRH. If q is large, then*

$$(1+o(1))\log q \leq \frac{1}{|\mathcal{A}_2(q)|} \sum_{(a,b)\in\mathcal{A}_2(q)} |B_q(a,b)| \leq (10+o(1))\log q.$$

In trying to quantify the biases for r-tuples $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$, Feuerverger and Martin [4] formulated the following conjecture:

Conjecture 2.10 (Feuerverger and Martin [4]) *Given $r \geq 2$, there exists a linear form $F(x_1, \ldots, x_r)$ on $\mathbb{R}^r$ such that*

$$\begin{aligned} &F(C_q(a_1), \ldots, C_q(a_r)) > F(C_q(b_1), \ldots, C_q(b_r)) \\ &\quad \Longrightarrow \delta(q; a_1, \ldots, a_r) > \delta(q; b_1, \ldots, b_r), \end{aligned}$$

for all races $\{q; a_1, \ldots, a_r\}$ and $\{q; b_1, \ldots, b_r\}$. In this case $F_{q;a_1,\ldots,a_r} = F(C_q(a_1), \ldots, C_q(a_r))$ is called "a bias factor".

This conjecture was motivated by some numerical computations along with the fact that the special case $r = 2$ holds, as shown by the work of Rubinstein and Sarnak (in this case one can check that $F_{q;a_1,a_2} = C_q(a_2) - C_q(a_1)$ is a bias factor). Using an explicit construction which involves Burgess's bound for the least quadratic non-residue modulo a prime (see Chapter 12 of [9]), we prove that this conjecture does not hold when $r \geq 3$.

Theorem 2.11 *Assume GRH and LI. Let $r \geq 3$ and $(\kappa_1, \ldots, \kappa_r) \in \mathbb{R}^r \backslash \{\mathbf{0}\}$. If q is large, then there exist two r-tuples $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_r) \in \mathcal{A}_r(q)$ such that*

$$\sum_{1\leq j\leq r} \kappa_j C_q(a_j) > \sum_{1\leq j\leq r} \kappa_j C_q(b_j) \text{ and } \delta(q; a_1, \ldots, a_r) < \delta(q; b_1, \ldots, b_r).$$

In the other direction, combining Theorems 2.1 and 2.9 we show that Conjecture 2.10 holds for almost all r-tuples $(a_1, \dots, a_r) \in \mathcal{A}_r(q)$.

Theorem 2.12 *Assume GRH and LI. Fix $r \geq 3$ and let q be large. Then there is a set $\Omega_r(q) \subset \mathcal{A}_r(q)$ with $|\Omega_r(q)| = o(|\mathcal{A}_r(q)|)$, such that for all r-tuples $(a_1, \dots, a_r)$, $(b_1, \dots, b_r) \in \mathcal{A}_r(q) \backslash \Omega_r(q)$ we have*

$$-\sum_{j=1}^{r} \alpha_j(r) C_q(a_j) > -\sum_{j=1}^{r} \alpha_j(r) C_q(b_j) \implies \delta(q; a_1, \dots, a_r) > \delta(q; b_1, \dots, b_r).$$

The plan of the paper is as follows. In the next section we study properties of the Fourier transform $\hat{\mu}_{q;a_1,\dots,a_r}$. These are then used to derive the asymptotic formula of Theorem 2.1 which is proved in Sect. 4. In Sect. 5 we study the behavior of $B_q(a,b)$ on average and prove Theorems 2.9 and 2.12. In Sect. 6 we describe the signs and extreme values of $B_q(a,b)$, and use these to explicitly construct biased races and prove Theorems 2.4, 2.8 and 2.11. In Sect. 7 we study q-extremely biased races and prove Theorem 2.6. Lastly, in Sect. 8 we derive the Fiorilli and Martin asymptotic formula for the densities in two-way races.

3 The Fourier transform $\hat{\mu}_{q;a_1,\dots,a_r}$

Assuming GRH and LI, Rubinstein and Sarnak obtained the following explicit formula for the Fourier transform of $\mu_{q;a_1,\dots,a_r}$

$$\hat{\mu}_{q;a_1,\dots,a_r}(t_1, \dots, t_r) = \exp\left(i \sum_{j=1}^{r} C_q(a_j) t_j\right) \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi > 0} J_0\left(\frac{2\left|\sum_{j=1}^{r} \chi(a_j) t_j\right|}{\sqrt{\frac{1}{4} + \gamma_\chi^2}}\right) \tag{3.1}$$

for $(t_1, \dots, t_r) \in \mathbb{R}^r$, where $J_0(z) = \sum_{m=0}^{\infty} (-1)^m (z/2)^{2m} / m!^2$ is the Bessel function of order 0.

For a non-trivial character χ modulo q, we let q_χ^* be the conductor of χ, and χ^* be the unique primitive character modulo q_χ^* which induces χ. First we record some standard formulas.

Lemma 3.1 *Assume GRH. Let χ be a non-trivial character modulo q. Then there exists an absolute constant b_0 such that*

$$\sum_{\gamma_\chi} \frac{1}{\frac{1}{4} + \gamma_\chi^2} = \log q_\chi^* + 2\mathrm{Re} \frac{L'(1, \chi^*)}{L(1, \chi^*)} - \chi(-1) \log 2 + b_0. \tag{3.2}$$

Moreover, we have

$$\sum_{\chi \bmod q} \chi(a) \log q_\chi^* = \begin{cases} \phi(q)\left(\log q - \displaystyle\sum_{p|q} \frac{\log p}{p-1}\right) & \textit{if } a \equiv 1 \bmod q, \\ -\phi(q)\dfrac{\Lambda(q/(q,a-1))}{\phi(q/(q,a-1))} & \textit{otherwise.} \end{cases}$$

and

$$N_q = \phi(q)\log q + O(\phi(q)\log\log q).$$

Proof The classical formula (3.2) can be derived from formulas (17) and (18) of chapter 12 in [3]. Indeed since GRH is assumed, these formulas imply that

$$\sum_{\gamma_\chi} \frac{1}{\frac{1}{4}+\gamma_\chi^2} = \log q_\chi^* + 2\mathrm{Re}\frac{L'(1,\chi^*)}{L(1,\chi^*)} + \mathrm{Re}\frac{\Gamma'\left(\frac{1}{2}+\frac{1}{2}a\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)},$$

where $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. Then (3.2) follows upon taking $b_0 = \Gamma'(1)/\Gamma(1) - \log 2$ and noting that

$$\Gamma'(1/2)/\Gamma(1/2) = \Gamma'(1)/\Gamma(1) - 2\log 2.$$

The second formula corresponds to Proposition 3.3 of [5]. Furthermore, recall that

$$N_q = 2\sum_{\chi\neq\chi_0}\sum_{\gamma_\chi>0}\frac{1}{\frac{1}{4}+\gamma_\chi^2} = \sum_{\chi\neq\chi_0}\sum_{\gamma_\chi}\frac{1}{\frac{1}{4}+\gamma_\chi^2},$$

since $\sum_{\gamma_\chi<0} 1/(\frac{1}{4}+\gamma_\chi^2) = \sum_{\gamma_{\overline{\chi}}>0} 1/(\frac{1}{4}+\gamma_{\overline{\chi}}^2)$, which is clear from the relation $\overline{L(\overline{s},\chi)} = L(s,\overline{\chi})$. On the other hand we have that

$$\sum_{p|q}\frac{\log p}{p-1} \le \sum_{p\le(\log q)^2}\frac{\log p}{p-1} + \frac{1}{\log q}\sum_{p|q} 1 \ll \log\log q,$$

using the trivial bound $\sum_{p|q} 1 \le \log q/\log 2$. Hence, the asymptotic for N_q follows upon combining this last estimate with the classical result of Littlewood [23] that $L'/L(1,\chi^*) = O(\log\log q)$, under GRH. □

For $t \in \mathbb{R}^n$ we shall use the notations $||t||$ and $|t|_\infty$ for the Euclidean norm and the maximum norm of t respectively. Rubinstein and Sarnak [25] noted that $\hat{\mu}_{q;a_1,\ldots,a_r}(t)$ is rapidly decreasing as $||t|| \to \infty$. The following result gives a quantitative statement of this decay. More precisely we establish an exponentially decreasing upper bound for $\hat{\mu}_{q;a_1,\ldots,a_r}(t)$ which depends on both t and q.

Proposition 3.2 *Assume GRH and LI. Fix an integer $r \geq 2$. Let q be a large positive integer, and let $0 < \epsilon < 1/2$ be a real number. Then, uniformly for all $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$ we have*

$$|\hat{\mu}_{q;a_1,\ldots,a_r}(t_1, \ldots, t_r)| \leq \exp(-c_3(r)\phi(q)||t||),$$

for $t = (t_1, \ldots, t_r) \in \mathbb{R}^r$ with $||t|| \geq 400$ and

$$|\hat{\mu}_{q;a_1,\ldots,a_r}(t_1, \ldots, t_r)| \leq \exp(-c_4(r)\epsilon^2\phi(q)\log q)$$

for $\epsilon \leq ||t|| \leq 400$, where $c_3(r)$ and $c_4(r)$ are positive constants that depend only on r.

Proof We begin by proving the first inequality. For any non-trivial character $\chi \bmod q$ we define

$$F(x, \chi) = \prod_{\gamma_\chi > 0} J_0\left(\frac{2x}{\sqrt{\frac{1}{4} + \gamma_\chi^2}}\right).$$

Then the explicit formula (3.1) implies that

$$|\hat{\mu}_{q;a_1,\ldots,a_r}(t_1, \ldots, t_r)| = \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \left| F\left(\left|\sum_{j=1}^{r} \chi(a_j)t_j\right|, \chi\right)\right|.$$

By Lemma 2.16 of [5] we know that there exists an absolute constant $c > 0$ such that

$$|F(x, \chi)F(x, \overline{\chi})| \leq e^{-cx} \tag{3.3}$$

for $x \geq 200$. On the other hand note that $|F(x, \chi)| \leq 1$ since $|J_0(x)| \leq 1$.

Let M_q be the set of non-trivial characters $\chi \bmod q$ such that $\left|\sum_{j=1}^{r} \chi(a_j)t_j\right| \geq ||t||/2$. We remark that $\chi \in M_q$ if and only if $\overline{\chi} \in M_q$. Moreover, if $\chi \in M_q$ and $||t|| \geq 400$ then $\left|\sum_{j=1}^{r} \chi(a_j)t_j\right| \geq 200$, which implies

$$\begin{aligned} |\hat{\mu}_{q;a_1,\ldots,a_r}(t_1, \ldots, t_r)|^2 &\leq \prod_{\chi \in M_q} \left| F\left(\left|\sum_{j=1}^{r} \chi(a_j)t_j\right|, \chi\right)\right|^2 \\ &= \prod_{\chi \in M_q} \left| F\left(\left|\sum_{j=1}^{r} \chi(a_j)t_j\right|, \chi\right) F\left(\left|\sum_{j=1}^{r} \chi(a_j)t_j\right|, \overline{\chi}\right)\right| \\ &\leq \exp\left(-c\sum_{\chi \in M_q}\left|\sum_{j=1}^{r} \chi(a_j)t_j\right|\right) \leq \exp\left(-\frac{c}{2}|M_q|||t||\right), \end{aligned} \tag{3.4}$$

using (3.3) along with the fact that every character in M_q appears once as χ and once as $\overline{\chi}$ in the product on the RHS of (3.4). Thus it remains only to prove a non-trivial lower bound for $|M_q|$. Let

$$\begin{aligned} S(t) &= \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \left|\sum_{j=1}^{r}\chi(a_j)t_j\right|^2 = \sum_{\chi \bmod q}\left|\sum_{j=1}^{r}\chi(a_j)t_j\right|^2 - \left(\sum_{j=1}^{r}t_j\right)^2\\ &= \sum_{j=1}^{r}\sum_{k=1}^{r}t_jt_k\sum_{\chi \bmod q}\chi(a_j)\overline{\chi(a_k)} - \left(\sum_{j=1}^{r}t_j\right)^2 = \phi(q)\sum_{j=1}^{r}t_j^2 - \left(\sum_{j=1}^{r}t_j\right)^2\\ &\geq (\phi(q)-r)\sum_{j=1}^{r}t_j^2, \end{aligned} \tag{3.5}$$

which follows from the Cauchy–Schwarz inequality. Therefore, using that $\left|\sum_{j=1}^{r}\chi(a_j)t_j\right|^2 \leq \left(\sum_{j=1}^{r}|t_j|\right)^2 \leq r||t||^2$, we deduce

$$S(t) = \sum_{\chi\in M_q}\left|\sum_{j=1}^{r}\chi(a_j)t_j\right|^2 + \sum_{\chi\notin M_q}\left|\sum_{j=1}^{r}\chi(a_j)t_j\right|^2 \leq r|M_q|||t||^2 + \frac{\phi(q)}{4}||t||^2.$$

Hence, combining this estimate with (3.5) we obtain $|M_q| \geq \phi(q)/(2r)$ if q is large enough. This together with (3.4) yields the first part of the proposition.

Now assume that $\epsilon \leq ||t|| \leq 400$. If $\chi \in M_q$ then $2\left|\sum_{j=1}^{r}\chi(a_j)t_j\right| \geq ||t|| \geq \epsilon$. We also note that $\epsilon\left(\frac{1}{4}+x^2\right)^{-1/2} \leq 1$, for any $x \in \mathbb{R}$. Hence, since J_0 is a positive decreasing function on $[0, 1]$ and $|J_0(z)| \leq J_0(1)$ for all $z \geq 1$, we get

$$\begin{aligned} |\hat{\mu}_{q;a_1,\ldots,a_r}(t_1,\ldots,t_r)| &\leq \prod_{\chi\in M_q}\prod_{\gamma_\chi>0}\left|J_0\left(\frac{2\left|\sum_{j=1}^{r}\chi(a_j)t_j\right|}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right)\right|\\ &\leq \prod_{\chi\in M_q}\prod_{\gamma_\chi>0}\left|J_0\left(\frac{\epsilon}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right)\right|. \end{aligned}$$

Moreover, it follows from Lemma 2.8 of [5] that for $|x| \leq 1$ we have $\log J_0(x) = -\sum_{n=1}^{\infty}u_{2n}x^{2n}$, where u_{2n} are positive real numbers with $u_2 = 1/4$. This implies that $|J_0(x)| \leq \exp(-x^2/4)$ for $|x| \leq 1$. Using this inequality we obtain

$$|\hat{\mu}_{q;a_1,\ldots,a_r}(t_1,\ldots,t_r)| \leq \exp\left(-\frac{\epsilon^2}{4}\sum_{\chi\in M_q}\sum_{\gamma_\chi>0}\frac{1}{\frac{1}{4}+\gamma_\chi^2}\right). \tag{3.6}$$

Since $L'/L(1,\chi^*) = O(\log\log q)$, then Lemma 3.1 gives

$$\sum_{\chi\in M_q}\sum_{\gamma_\chi>0}\frac{1}{\frac14+\gamma_\chi^2} = \frac12\sum_{\chi\in M_q}\sum_{\gamma_\chi}\frac{1}{\frac14+\gamma_\chi^2} = \frac12\sum_{\chi\in M_q}\log q_\chi^* + O(\phi(q)\log\log q). \tag{3.7}$$

Noting that $q_\chi^*\le q$, and using Lemma 3.1 we derive

$$\begin{aligned}\sum_{\chi\in M_q}\log q_\chi^* &\ge \sum_{\chi \bmod q}\log q_\chi^* - (\phi(q)-|M_q|)\log q\\ &\ge \frac{\phi(q)\log q}{2r} + O(\phi(q)\log\log q).\end{aligned}$$

The result then follows upon combining this estimate with Eqs. (3.6) and (3.7). □

Our next result (which is a crucial ingredient to the proof of Theorem 2.1) shows that $\hat{\mu}_{q;a_1,\ldots,a_r}$ can be approximated by a multivariate Gaussian in the range $||t|| \ll \phi(q)^{-1/2}$.

Proposition 3.3 *Assume GRH and LI. Fix an integer $r\ge 2$. Then, for any constant $A = A(r) > 0$ there exists $L(A) > 0$ such that for $L \ge L(A)$ and $t = (t_1,\ldots,t_r)\in\mathbb{R}^r$ with $||t|| \le A\sqrt{\log q}$, we have*

$$\begin{aligned}\hat{\mu}_{q;a_1,\ldots,a_r}\left(\frac{t_1}{\sqrt{N_q}},\ldots,\frac{t_r}{\sqrt{N_q}}\right) &= \exp\left(-\frac{t_1^2+\cdots+t_r^2}{2}\right)\Bigg(1+\frac{i}{\sqrt{N_q}}\sum_{j=1}^{r}C_q(a_j)t_j\\ &-\frac{1}{2N_q}\sum_{j=1}^{r}C_q(a_j)^2t_j^2 - \frac{1}{N_q}\left(\sum_{1\le j<k\le r}(B_q(a_j,a_k)+C_q(a_j)C_q(a_k))t_jt_k\right)\\ &+\frac{Q_4(t_1,\ldots,t_r)}{N_q} + \sum_{m=0}^{1}\sum_{s=0}^{2}\sum_{\substack{0\le l\le L\\ 2l\ge 3-2s-m}}\frac{C_q^mB_q^l}{N_q^{m/2+l+s}}P_{s,m,l}(t_1,\ldots,t_r)\\ &+O\left(\frac{r^{2L}B_q^L||t||^{2L}}{L!N_q^L}+\frac{C_q^3}{\phi(q)^{3/2}}\right)\Bigg),\end{aligned}$$

where Q_4 is a homogenous polynomial of degree 4 with bounded coefficients and $P_{s,m,l}$ are homogenous polynomials of degree $m+2l+4s$ whose coefficients are bounded uniformly by a function of l. Moreover the constant in the O is absolute.

Proof For simplicity let us write $\hat{\mu}_q = \hat{\mu}_{q;a_1,\ldots,a_r}$. From the explicit formula (3.1) we have

$$\log \hat{\mu}_q\left(\frac{t_1}{\sqrt{N_q}},\ldots,\frac{t_r}{\sqrt{N_q}}\right)=\frac{i}{\sqrt{N_q}}\sum_{j=1}^{r}C_q(a_j)t_j + \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\sum_{\gamma_\chi>0}\log J_0\left(\frac{2\left|\sum_{j=1}^r\chi(a_j)t_j\right|}{\sqrt{\frac14+\gamma_\chi^2}\sqrt{N_q}}\right).$$

For $|s|\leq 1$ Lemma 2.8 of [5] states that

$$\log J_0(s)=-\sum_{n=1}^{\infty}u_{2n}s^{2n},$$

where u_{2n} are positive real numbers with $u_2=1/4$ and $u_{2n}\ll(5/12)^{2n}$. This implies that for $t=(t_1,\ldots,t_r)$ with $||t||\leq A\sqrt{\log q}$ we have

$$\log \hat{\mu}_q\left(\frac{t_1}{\sqrt{N_q}},\ldots,\frac{t_r}{\sqrt{N_q}}\right)=\frac{i}{\sqrt{N_q}}\sum_{j=1}^{r}C_q(a_j)t_j - \sum_{n=1}^{\infty}\frac{u_{2n}2^{2n}}{N_q^n}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{\left|\sum_{j=1}^r\chi(a_j)t_j\right|^{2n}}{(\frac14+\gamma_\chi^2)^n}. \tag{3.8}$$

The contribution of the term $n=1$ to the RHS of (3.8) equals

$$\begin{aligned}&-\frac{1}{N_q}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{1}{\frac14+\gamma_\chi^2}\sum_{1\leq j,k\leq r}\chi(a_j)\overline{\chi(a_k)}t_jt_k\\ &=-\frac12(t_1^2+\cdots+t_r^2)-\frac{1}{N_q}\sum_{1\leq j<k\leq r}B_q(a_j,a_k)t_jt_k.\end{aligned} \tag{3.9}$$

The term $n=2$ contributes $Q_4(t_1,\ldots,t_r)/N_q$ where

$$\begin{aligned}Q_4(t_1,\ldots,t_r):&=-\frac{16u_4}{N_q}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{\left|\sum_{j=1}^r\chi(a_j)t_j\right|^{4}}{(\frac14+\gamma_\chi^2)^2}\\ &=-\frac{16u_4}{N_q}\sum_{1\leq j_1,j_2,j_3,j_4\leq r}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{\chi(a_{j_1})\chi(a_{j_2})\overline{\chi}(a_{j_3})\overline{\chi}(a_{j_4})}{(\frac14+\gamma_\chi^2)^2}t_{j_1}t_{j_2}t_{j_3}t_{j_4}.\end{aligned}$$

Since

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{1}{(\frac14+\gamma_\chi^2)^2}\le 2N_q,$$

then $Q_4(t_1,\dots,t_r)$ is a homogenous polynomial of degree 4 with bounded coefficients. Similarly, the contribution of the terms $n\ge 3$ to the RHS of (3.8) is $\ll ||t||^6/N_q^2 \ll_r (\log q)/\phi(q)^2$. Hence, we obtain

$$\log\hat{\mu}_q\left(\frac{t_1}{\sqrt{N_q}},\dots,\frac{t_r}{\sqrt{N_q}}\right)=-\frac12(t_1^2+\cdots+t_r^2)+\frac{i}{\sqrt{N_q}}\sum_{j=1}^r C_q(a_j)t_j \\ -\frac{1}{N_q}\sum_{1\le j<k\le r}B_q(a_j,a_k)t_jt_k+\frac{Q_4(t_1,\dots,t_r)}{N_q} \\ +O_r\left(\frac{\log q}{\phi(q)^2}\right). \tag{3.10}$$

Now, in our range of t we have

$$\exp\left(\frac{i}{\sqrt{N_q}}\sum_{j=1}^r C_q(a_j)t_j\right)=\sum_{m=0}^{2}\frac{1}{m!N_q^{m/2}}\left(i\sum_{j=1}^r C_q(a_j)t_j\right)^m+O_r\left(\frac{C_q^3}{\phi(q)^{3/2}}\right),$$

and $\exp(Q_4(t_1,\dots,t_r)/N_q)=1+Q_4(t_1,\dots,t_r)/N_q+O_r(\log^2 q/\phi(q)^2)$. Therefore, using that

$$\exp\left(-\frac{1}{N_q}\sum_{1\le j<k\le r}B_q(a_j,a_k)t_jt_k\right)=\sum_{l=0}^{\infty}\frac{\left(-\sum_{1\le j<k\le r}B_q(a_j,a_k)t_jt_k\right)^l}{l!N_q^l},$$

along with the previous estimates and Eq. (3.10) we deduce that the quotient of $\hat{\mu}_q\left(\frac{t_1}{\sqrt{N_q}},\dots,\frac{t_r}{\sqrt{N_q}}\right)$ and $\exp(-(t_1^2+\cdots+t_r^2)/2)$ equals

$$\sum_{s=0}^{1}\sum_{m=0}^{2}\sum_{l=0}^{\infty}\frac{Q_4(t_1,\dots,t_r)^s}{m!l!N_q^{m/2+l+s}}\left(i\sum_{j=1}^r C_q(a_j)t_j\right)^m\left(-\sum_{1\le j<k\le r}B_q(a_j,a_k)t_jt_k\right)^l \\ +O_r\left(\frac{C_q^3}{\phi(q)^{3/2}}\right). \tag{3.11}$$

We collect the summands above according to $D=m+2s+2l$ (which equals twice the power of N_q). Then, it is easy to check that the contribution of the terms $0\le D\le 2$

to the main term of (3.11) equals

$$1+\frac{i}{\sqrt{N_q}}\sum_{j=1}^{r}C_q(a_j)t_j-\frac{1}{2N_q}\left(\sum_{j=1}^{r}C_q(a_j)t_j\right)^2 \\ -\frac{1}{N_q}\sum_{1\le j<k\le r}B_q(a_j,a_k)t_jt_k+\frac{Q_4(t_1,\ldots,t_r)}{N_q}.$$

Let $P_{s,m,l}(t_1,\ldots,t_r)$ be the homogenous polynomial of degree $m+2l+4s$ defined by

$$P_{s,m,l}(t_1,\ldots,t_r)=\frac{1}{m!l!}C_q^{-m}B_q^{-l}Q_4(t_1,\ldots,t_r)^s\left(i\sum_{j=1}^{r}C_q(a_j)t_j\right)^m \\ \times\left(-\sum_{1\le j<k\le r}B_q(a_j,a_k)t_jt_k\right)^l.$$

Then the contribution of the terms with $D\ge 3$ to (3.11) equals

$$\sum_{m=0}^{1}\sum_{s=0}^{2}\sum_{\substack{l\ge 0\\ 2l\ge 3-2s-m}}\frac{C_q^mB_q^l}{N_q^{m/2+l+s}}P_{s,m,l}(t_1,\ldots,t_r).$$

Notice that the coefficients of $P_{s,m,l}$ are bounded uniformly by a function of l since r is fixed and $s,m\le 2$. On the other hand since $C_q=q^{o(1)}$ we get

$$\frac{C_q^mB_q^l}{N_q^{m/2+l+s}}P_{s,m,l}(t_1,\ldots,t_r)\ll\frac{r^{2l+m}C_q^mB_q^l||t||^{m+2l+4s}}{m!l!N_q^{m/2+l+s}}\ll\frac{r^{2l}B_q^l||t||^{2l}}{l!N_q^l}.$$

Now, Corollary 5.4 implies that $B_q\le c\phi(q)$ for some absolute constant $c>0$. Therefore, in our range of t, we have $r^2B_q||t||^2/(lN_q)\le 2r^2A/l$. This shows that for a suitably large constant $L(A)$ (which also depends on r) we have

$$\sum_{m=0}^{1}\sum_{s=0}^{2}\sum_{\substack{l>L\\ 2l\ge 3-2s-m}}\frac{C_q^mB_q^l}{N_q^{m/2+l+s}}P_{s,m,l}(t_1,\ldots,t_r)\ll\frac{r^{2L}B_q^L||t||^{2L}}{L!N_q^L},$$

for all $L\ge L(A)$, completing the proof. □

4 An asymptotic formula for the densities $\delta(q;a_1,\ldots,a_r)$

The first step to prove Theorem 2.1 is to truncate the integral on the RHS of (1.1). To this end we need to bound the tail of the distribution $\mu_{q;a_1,\ldots,a_r}$. Our idea consists in relating this tail to the Laplace transform of $\mu_{q;a_1,\dots,a_r}$ using Chernoff's bound. For $s=(s_1,s_2,\dots,s_r)\in\mathbb{R}^r$ we define

$$\mathcal{L}_{q;a_1,\dots,a_r}(s_1,s_2,\dots,s_r) := \int\limits_{x\in\mathbb{R}^r} e^{s_1x_1+\cdots+s_rx_r}\,d\mu_{q;a_1,\dots,a_r}(x_1,\dots,x_r),$$

if this integral converges. The same arguments as in the proof of Rubinstein and Sarnak for the explicit formula (3.1) of $\hat{\mu}_{q;a_1,\dots,a_r}$, show under GRH and LI, that $\mathcal{L}_{q;a_1,\dots,a_r}(s)$ exists for all $s\in\mathbb{R}^r$ and

$$\begin{aligned}\mathcal{L}_{q;a_1,\dots,a_r}(s_1,s_2,\dots,s_r) &= \exp\left(-\sum_{j=1}^r C_q(a_j)s_j\right)\\ &\quad\times\prod_{\substack{\chi\neq\chi_0\\ \chi\bmod q}}\prod_{\gamma_\chi>0} I_0\left(\frac{2\left|\sum_{j=1}^r\chi(a_i)s_i\right|}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right),\end{aligned}\tag{4.1}$$

where $I_0(t):=\sum_{n=0}^\infty (t/2)^{2n}/n!^2$ is the modified Bessel function of order 0. We prove

Proposition 4.1 *Assume GRH and LI. Fix an integer $r\geq 2$ and let q be a large positive integer. Then for $R\geq\sqrt{\phi(q)\log q}$ we have*

$$\mu_{q;a_1,\dots,a_r}(|x|_\infty > R)\leq\exp\left(-\frac{R^2}{2\phi(q)\log q}\left(1+O\left(\frac{\log\log q}{\log q}\right)\right)\right).$$

Proof First we note that

$$\mu_{q;a_1,\dots,a_r}(|x|_\infty > R)\leq\sum_{j=1}^r\mu_{q;a_1,\dots,a_r}(x_j>R)+\sum_{j=1}^r\mu_{q;a_1,\dots,a_r}(x_j<-R).$$

We shall bound only $\mu_{q;a_1,\dots,a_r}(x_j>R)$, since the corresponding bound for $\mu_{q;a_1,\dots,a_r}$ $(x_j<-R)$ can be obtained similarly. Let $s>0$. Then using (4.1) we get

$$\begin{aligned}\mu_{q;a_1,\dots,a_r}(x_j>R) &\leq e^{-sR}\int\limits_{(x_1,\dots,x_r)\in\mathbb{R}^r} e^{sx_j}\,d\mu_{q;a_1,\dots,a_r}(x_1,\dots,x_r)\\ &\leq e^{-sR-sC_q(a_j)}\prod_{\substack{\chi\neq\chi_0\\ \chi\bmod q}}\prod_{\gamma_\chi>0} I_0\left(\frac{2s}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right).\end{aligned}$$

Since $I_0(s) \leq \exp(s^2/4)$ for all $s \in \mathbb{R}$ we obtain

$$\prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi > 0} I_0\left(\frac{2s}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right) \leq \exp\left(s^2 \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{1}{\frac{1}{4}+\gamma_\chi^2}\right)$$
$$\leq \exp\left(\frac{s^2\phi(q)\log q}{2}\left(1+O\left(\frac{\log\log q}{\log q}\right)\right)\right),$$

by Lemma 3.1. The result follows by taking $s = R/(\phi(q)\log q)$ along with the fact that $C_q(a_j) \ll_\epsilon q^\epsilon$ for any $\epsilon > 0$. □

Let $\Phi(x) := e^{-x^2/2}$ and denote by $\Phi^{(n)}$ the n-th derivative of Φ. Then $\Phi^{(1)}(x) = -xe^{-x^2/2}$, $\Phi^{(2)}(x) = (x^2-1)e^{-x^2/2}$, and more generally we know that $\Phi^{(n)}(x) = (-1)^n H_n(x)e^{-x^2/2}$ where H_n is the n-th Hermite polynomial. The last ingredients we need in order to prove Theorem 2.1 are the following lemmas:

Lemma 4.2 *Let $n_1, \ldots, n_r$ be fixed non-negative integers, and M be a large positive number. Then for any $(x_1, \ldots, x_r) \in \mathbb{R}^r$, we have*

$$\int_{||\mathbf{t}||<M} e^{i(t_1x_1+\cdots+t_rx_r)} \prod_{j=1}^{r} t_j^{n_j}\Phi(t_j)d\mathbf{t} = (2\pi)^{r/2}\prod_{j=1}^{r} i^{n_j}H_{n_j}(x_j)e^{-x_j^2/2} + O\left(e^{-M^2/4}\right).$$

Proof First, notice that

$$\int_{\mathbf{t}\in\mathbb{R}^r} e^{i(t_1x_1+\cdots+t_rx_r)} \prod_{j=1}^{r} t_j^{n_j}\Phi(t_j)d\mathbf{t} = (2\pi)^{r/2}\prod_{j=1}^{r}\Psi_j(x_j),$$

where

$$\Psi_j(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iuv}\Phi(v)v^{n_j}dv.$$

Since the Fourier transform of $\Phi(u)/(2\pi)$ is $\Phi(v)/\sqrt{2\pi}$, then using standard properties of the Fourier transform (see Appendix 3 of [21]), we deduce that $\Phi(v)v^{n_j}/\sqrt{2\pi}$ is the Fourier transform of $\frac{(-i)^{n_j}}{2\pi}\Phi^{(n_j)}(v)$. Therefore the Fourier inversion formula gives

$$\Psi_j(u) = (-i)^{n_j}\Phi^{(n_j)}(u) = i^{n_j}H_{n_j}(u)e^{-u^2/2}.$$

Finally, note that

$$\int\limits_{||\mathbf{t}||>M}\left|\prod_{j=1}^{r} t_j^{n_j}\,\Phi(t_j)\right| d\mathbf{t} \ll \exp\left(-M^2/2\right) M^{n_1+\cdots+n_r} \ll \exp\left(-M^2/4\right),$$

if M is large enough, which completes the proof. □

Lemma 4.3 *Let $P_n(t_1,\ldots,t_r)$ be a homogeneous polynomial of degree n, whose coefficients are complex numbers uniformly bounded by a function of n. Let R be a large positive number and $M \geq \log R$ be a real number. Then we have*

$$\left|\int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty<R}}\ \int\limits_{||\mathbf{t}||\leq M} e^{i(t_1x_1+\cdots+t_rx_r)}\exp\left(-\frac{t_1^2+\cdots+t_r^2}{2}\right)P_n(t_1,\ldots,t_r)d\mathbf{t}d\mathbf{x}\right| \ll_{n,r} 1.$$

Proof Since the coefficients of $P_n(t_1,\ldots,t_r)$ are uniformly bounded by a function of n, it is sufficient to show that the statement holds when $P_n(t_1,\ldots,t_r) = t_1^{n_1}\ldots t_r^{n_r}$, where n_i are non-negative integers with $n_1+\cdots+n_r = n$. Using Lemma 4.2 we get

$$\begin{aligned}&\int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty<R}}\ \int\limits_{||\mathbf{t}||\leq M} e^{i(t_1x_1+\cdots+t_rx_r)}\exp\left(-\frac{t_1^2+\cdots+t_r^2}{2}\right)t_1^{n_1}\cdots t_r^{n_r}\,d\mathbf{t}d\mathbf{x}\\ &\quad= i^n(2\pi)^{r/2}\int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty<R}} H_{n_1}(x_1)\ldots H_{n_r}(x_r)\exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)d\mathbf{x}+o_R(1),\end{aligned} \tag{4.2}$$

since $\exp\left(-M^2/4\right)R^r \ll e^{-\frac{\log^2 R}{8}}$ by our hypothesis on M. The lemma then follows upon noting that

$$\begin{aligned}&\int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty>R}} H_{n_1}(x_1)\ldots H_{n_r}(x_r)\exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)dx_1\ldots dx_r\\ &\quad\ll_{n,r} R^n e^{-R^2/2} = o_R(1),\end{aligned}$$

and

$$\int\limits_{x_1>x_2>\cdots>x_r} H_{n_1}(x_1)\ldots H_{n_r}(x_r)\exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)dx_1\ldots dx_r \ll_{n,r} 1.$$

□

Proof of Theorem 2.1 Let $R := \sqrt{N_q}\log q$. To lighten the notation in this proof we write δ_q for $\delta(q; a_1, \ldots, a_r)$ and μ_q for $\mu_{q;a_1,\ldots,a_r}$. Then by Proposition 4.1 we obtain

$$\begin{aligned}\delta_q &= \int\limits_{y_1>y_2>\cdots>y_r} d\mu_q(y_1, \ldots, y_r)\\ &= \int\limits_{\substack{y_1>y_2>\cdots>y_r\\ |\mathbf{y}|_\infty \le R}} d\mu_q(y_1, \ldots, y_r) + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right).\end{aligned} \tag{4.3}$$

Next, we apply the Fourier inversion formula to the measure μ_q to get

$$\begin{aligned}\int\limits_{\substack{y_1>y_2>\cdots>y_r\\ |\mathbf{y}|_\infty \le R}} d\mu_q(y_1, \ldots, y_r) &= (2\pi)^{-r}\\ &\times \int\limits_{\substack{y_1>y_2>\cdots>y_r\\ |\mathbf{y}|_\infty \le R}} \int\limits_{\mathbf{s}\in\mathbb{R}^r} e^{i(s_1y_1+\cdots+s_ry_r)}\hat{\mu}_q(s_1, \ldots, s_r)d\mathbf{s}d\mathbf{y}.\end{aligned}$$

Let $A = A(r) \ge r$ be a suitably large constant. Then using Proposition 3.2 with $\epsilon := A(N_q)^{-1/2}\sqrt{\log q}$ we get

$$\begin{aligned}&\int\limits_{\mathbf{s}\in\mathbb{R}^r} e^{i(s_1y_1+\cdots+s_ry_r)}\hat{\mu}_q(s_1, \ldots, s_r)d\mathbf{s}\\ &\quad = \int\limits_{||\mathbf{s}||\le\epsilon} e^{i(s_1y_1+\cdots+s_ry_r)}\hat{\mu}_q(s_1, \ldots, s_r)d\mathbf{s} + O\left(\frac{1}{q^{2A}}\right).\end{aligned}$$

Hence we obtain

$$\delta_q = (2\pi)^{-r} \int\limits_{\substack{y_1>y_2>\ldots>y_r\\ |\mathbf{y}|_\infty \le R}} \int\limits_{||\mathbf{s}||\le\epsilon} e^{i(s_1y_1+\cdots+s_ry_r)}\hat{\mu}_q(s_1, \ldots, s_r)d\mathbf{s}d\mathbf{y} + O\left(\frac{1}{q^A}\right), \tag{4.4}$$

using that $R^r q^{-2A} \ll q^{-A}$. Upon making the change of variables

$$t_j := \sqrt{N_q}s_j, \quad \text{and} \quad x_j := \frac{y_j}{\sqrt{N_q}},$$

we infer from (4.4) that

$$\delta_q = (2\pi)^{-r} \int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty\le\log q}} \int\limits_{||\mathbf{t}||\le A\sqrt{\log q}} e^{i(t_1x_1+\cdots+t_rx_r)}\hat{\mu}_q\left(\frac{t_1}{\sqrt{N_q}},\ldots,\frac{t_r}{\sqrt{N_q}}\right)d\mathbf{t}d\mathbf{x} + O\left(\frac{1}{q^A}\right). \tag{4.5}$$

Now we use the asymptotic expansion of $\hat{\mu}_q\left(t_1N_q^{-1/2},\ldots,t_rN_q^{-1/2}\right)$ proved in Proposition 3.3. We take $L = L(A) \ge 2r$ to be a suitably large constant. Then, Lemma 4.3 shows that the contribution of the error term along with the terms corresponding to the polynomials Q_4 and $P_{s,m,l}$ (in the asymptotic expansion of Proposition 3.3) to the integral on the RHS of (4.5) is

$$\ll_r \frac{1}{N_q} + \sum_{m=0}^{1}\sum_{s=0}^{2}\sum_{\substack{0\le l\le L\\ 2l\ge 3-2s-m}} \frac{C_q^mB_q^l}{N_q^{m/2+l+s}} + \frac{(\log q)^rB_q^L}{N_q^L} \ll_r \frac{1}{N_q} + \frac{C_qB_q}{N_q^{3/2}} + \frac{B_q^2}{N_q^2}, \tag{4.6}$$

since $B_q \ll N_q/\log q$ by Corollary 5.4. Now we shall compute the contribution of the remaining terms in the asymptotic formula of $\hat{\mu}_q$ to the integral in (4.5). Appealing to Lemma 4.2 along with the fact that $\exp(-(x_1^2+\cdots+x_r^2)/2)$ is a continuous symmetric function in $x_1,\ldots,x_r$, we obtain

$$\begin{aligned}(2\pi)^{-r} &\int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty\le\log q}} \int\limits_{||\mathbf{t}||\le A\sqrt{\log q}} e^{i(t_1x_1+\cdots+t_rx_r)}\exp\left(-\frac{t_1^2+\cdots+t_r^2}{2}\right)d\mathbf{t}d\mathbf{x}\\ &= (2\pi)^{-r/2}\int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty\le\log q}} \exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)d\mathbf{x} + O\left(\frac{1}{q^A}\right)\\ &= \frac{1}{r!(2\pi)^{r/2}}\int\limits_{\mathbf{x}\in\mathbb{R}^r}\exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)d\mathbf{x} + O\left(\frac{1}{q^A}\right) = \frac{1}{r!} + O\left(\frac{1}{q^A}\right).\end{aligned} \tag{4.7}$$

Similarly, we infer from Lemma 4.2 that for $1 \le j \le r$, we have

$$(2\pi)^{-r} \int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty\le\log q}} \int\limits_{||\mathbf{t}||\le A\sqrt{\log q}} t_je^{i(t_1x_1+\cdots+t_rx_r)}\exp\left(-\frac{t_1^2+\cdots+t_r^2}{2}\right)d\mathbf{t}d\mathbf{x}$$

$$= i(2\pi)^{-r/2} \int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty\le\log q}} x_j \exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)d\mathbf{x}$$
$$+O\left(\frac{1}{q^A}\right) = i\alpha_j(r) + O\left(\frac{1}{q^A}\right), \tag{4.8}$$

and

$$(2\pi)^{-r} \int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty\le\log q}} \int\limits_{||\mathbf{t}||\le A\sqrt{\log q}} t_j^2 e^{i(t_1x_1+\cdots+t_rx_r)} \exp\left(-\frac{t_1^2+\cdots+t_r^2}{2}\right)d\mathbf{t}d\mathbf{x}$$
$$= -\lambda_j(r) + O\left(\frac{1}{q^A}\right). \tag{4.9}$$

For $1\le j<k\le r$ we analogously obtain

$$(2\pi)^{-r} \int\limits_{\substack{x_1>x_2>\cdots>x_r\\ |\mathbf{x}|_\infty\le\log q}} \int\limits_{||\mathbf{t}||\le A\sqrt{\log q}} t_j t_k e^{i(t_1x_1+\cdots+t_rx_r)} \exp\left(-\frac{t_1^2+\cdots+t_r^2}{2}\right)d\mathbf{t}d\mathbf{x}$$
$$= -\beta_{j,k}(r) + O\left(\frac{1}{q^A}\right). \tag{4.10}$$

The theorem now follows upon combining Proposition 3.3 with the estimates (4.5)–(4.10). □

In the remaining part of this section, we explicitly compute the constants $\alpha_j(r)$ and $\beta_{j,k}(r)$ for $r=3$. To simplify the computations we prove the following identities

Lemma 4.4 *Let $r\ge 2$. Then for any $1\le j<k\le r$ we have*

$$\alpha_j(r) = -\alpha_{r+1-j}(r) \text{ and } \beta_{j,k}(r) = \beta_{r+1-k,r+1-j}(r).$$

Proof We prove only the identity for the $\alpha_j(r)$ since the proof for the $\beta_{j,k}(r)$ is similar. Recall that

$$\alpha_j(r) = (2\pi)^{-r/2} \int\limits_{x_1>x_2>\cdots>x_r} x_j \exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)dx_1\dots dx_r.$$

Upon making the change of variables $y_k = -x_{r+1-k}$ for $1\le k\le r$, we deduce

$$\begin{aligned}\alpha_j(r) &= -(2\pi)^{-r/2}\int\limits_{y_1>y_2>\cdots>y_r} y_{r+1-j}\exp\left(-\frac{y_1^2+\cdots+y_r^2}{2}\right)dy_1\ldots dy_r\\ &= -\alpha_{r+1-j}(r).\end{aligned}$$

□

Lemma 4.5 *We have* $\beta_{1,2}(2)=0$. *Moreover, one has*

$$\alpha_1(3)=\frac{1}{4\sqrt{\pi}},\quad \alpha_2(3)=0,\quad \alpha_3(3)=-\frac{1}{4\sqrt{\pi}},$$

and

$$\beta_{1,2}(3)=\beta_{2,3}(3)=\frac{1}{4\pi\sqrt{3}},\quad \beta_{1,3}(3)=-\frac{1}{2\pi\sqrt{3}}.$$

Proof First we have

$$\beta_{1,2}(2)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{x_2}^{\infty} x_1x_2\exp\left(-\frac{x_1^2+x_2^2}{2}\right)dx_1dx_2=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} x_2e^{-x_2^2}dx_2=0.$$

Now we deal with the case $r=3$. Recall that

$$\sum_{j=1}^{3}\alpha_j(3)=\sum_{1\le j<k\le 3}\beta_{j,k}(3)=0. \tag{4.11}$$

We begin by computing

$$\alpha_1(3)=\frac{1}{(2\pi)^{3/2}}\int\limits_{x_1>x_2>x_3} x_1\exp\left(-\frac{x_1^2+x_2^2+x_3^2}{2}\right)dx_1dx_2dx_3.$$

To this end, we integrate with respect to x_1 first to get

$$\begin{aligned}\alpha_1(3)&=\frac{1}{(2\pi)^{3/2}}\int\limits_{x>y}\exp\left(-x^2-\frac{y^2}{2}\right)dxdy\\ &=\frac{1}{(2\pi)^{3/2}}\int\limits_{X<Y}\exp\left(-X^2-\frac{Y^2}{2}\right)dXdY,\end{aligned}$$

by making the change of variables $X=-x$ and $Y=-y$. Hence, we deduce that

$$\alpha_1(3)=\frac{1}{2(2\pi)^{3/2}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\exp\left(-x^2-\frac{y^2}{2}\right)dxdy=\frac{1}{4\sqrt{\pi}}.$$

On the other hand, Lemma 4.4 shows that $\alpha_3(3) = -\alpha_1(3)$, and this combined with equation (4.11) leads to $\alpha_2(3) = 0$. Furthermore, we have

$$\beta_{1,2}(3) = \frac{1}{(2\pi)^{3/2}} \int\limits_{x_1>x_2>x_3} x_1 x_2 \exp\left(-\frac{x_1^2+x_2^2+x_3^2}{2}\right) dx_1 dx_2 dx_3.$$

Performing the integration with respect to x_1 first, then with respect to x_2 gives us

$$\beta_{1,2}(3) = \frac{1}{2(2\pi)^{3/2}} \int\limits_{-\infty}^{\infty} e^{-\frac{3x_3^2}{2}} dx_3 = \frac{1}{4\pi\sqrt{3}}.$$

The remaining estimates follow upon using Lemma 4.4 to get $\beta_{2,3}(3) = \beta_{1,2}(3)$, and then applying Eq. (4.11) to deduce that $\beta_{1,3}(3) = -2\beta_{1,2}(3)$. □

5 The average order of $|B_q(a, b)|$

In this section we prove upper and lower bounds (of the same order of magnitude) for the first moment of $|B_q(a, b)|$ over pairs of residue classes $(a, b) \in \mathcal{A}_2(q)$. To this end, we begin by proving the following key proposition.

Proposition 5.1 *Assume GRH. Let q be a large integer, and $(a, b) \in \mathcal{A}_2(q)$. Put $x = (q \log q)^2$. Then we have*

$$\begin{aligned} B_q(a, b) = {} & 4\log q - \phi(q) l_q(a, b) \log 2 - \phi(q) \frac{\Lambda\left(\frac{q}{(q,a-b)}\right)}{\phi\left(\frac{q}{(q,a-b)}\right)} \\ & -\phi(q) \sum_{\substack{n \le 2x\log x \\ bn \equiv a \bmod q}} \frac{\Lambda(n)}{n} e^{-n/x} \\ & -\phi(q) \sum_{\substack{n \le 2x\log x \\ an \equiv b \bmod q}} \frac{\Lambda(n)}{n} e^{-n/x} - \phi(q) \sum_{p^\nu \| q} \sum_{\substack{1 \le e \le 2\log x \\ ap^e \equiv b \bmod q/p^\nu}} \frac{\log p}{p^{e+\nu-1}(p-1)} \\ & -\phi(q) \sum_{p^\nu \| q} \sum_{\substack{1 \le e \le 2\log x \\ bp^e \equiv a \bmod q/p^\nu}} \frac{\log p}{p^{e+\nu-1}(p-1)} + O(\log\log q), \end{aligned}$$

where $l_q(a, b) = 1$ if $a + b \equiv 0 \bmod q$ and 0 otherwise.

Remark 5.1 This result implies that $B_q(a, b) < 0$ if $|B_q(a, b)| > 5 \log q$.

Although the major part of this proposition is proved in [5] (see Theorems 1.4 and 1.7 there), we chose to include the details of the proof for the sake of completeness. The only new input is the following lemma which corresponds to the contribution of the principal character $\chi_0 \bmod q$.

Lemma 5.2 *Let q be a large positive integer and $y \geq q$ be a real number. Then*

$$\sum_{\substack{n\geq 1\\(n,q)=1}} \frac{\Lambda(n)}{n} e^{-n/y} = \log y + O(\log\log y).$$

Proof First note that

$$\sum_{\substack{n\geq 1\\(n,q)>1}} \frac{\Lambda(n)}{n} e^{-n/y} \leq \sum_{p|q}\sum_{k=1}^{\infty} \frac{\log p}{p^k} = \sum_{p|q} \frac{\log p}{p-1} \ll \log\log q.$$

Thus it suffices to evaluate

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} e^{-n/y}.$$

We split the above sum into three parts: $n > y\log^2 y$, $y \log\log y < n \leq y\log^2 y$ and finally $n \leq y\log\log y$. The contribution of the first part is

$$\sum_{n>y\log^2 y} \frac{\Lambda(n)}{n} e^{-n/y} \leq \sum_{n>y\log^2 y} \frac{1}{n^2} \leq \frac{1}{y},$$

which follows from the fact that $e^{-n/y} \leq n^{-2}$ for $n > y\log^2 y$. Now the contribution of the second part is

$$\sum_{y\log\log y<n\leq y\log^2 y} \frac{\Lambda(n)}{n} e^{-n/y} \leq \frac{1}{\log y} \sum_{n\leq y\log^2 y} \frac{\Lambda(n)}{n} \ll 1.$$

Finally using that $1 - e^{-t} \leq t$ for all $t > 0$, we deduce that the contribution of the last part equals

$$\begin{aligned}\sum_{n\leq y\log\log y} \frac{\Lambda(n)}{n} e^{-n/y} &= \sum_{n\leq y\log\log y} \frac{\Lambda(n)}{n} + O\left(\frac{1}{y}\sum_{n\leq y\log\log y} \Lambda(n)\right)\\ &= \log y + O(\log\log y),\end{aligned}$$

which follows from the prime number theorem. □

Proof of Proposition 5.1 Let $(a, b) \in \mathcal{A}_2(q)$. First we infer from Lemma 3.1 that

$$\begin{aligned} B_q(a,b) &= \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \sum_{\gamma_\chi>0} \frac{\chi\left(\frac{a}{b}\right)+\chi\left(\frac{b}{a}\right)}{\frac{1}{4}+\gamma_\chi^2} = \frac{1}{2}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \sum_{\gamma_\chi} \frac{\chi\left(\frac{a}{b}\right)+\chi\left(\frac{b}{a}\right)}{\frac{1}{4}+\gamma_\chi^2} \\ &= \frac{1}{2}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \left(\chi\left(\frac{a}{b}\right)+\chi\left(\frac{b}{a}\right)\right)\log q_\chi^* - \phi(q) l_q(a,b)\log 2 \\ &\quad + \sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \left(\chi\left(\frac{a}{b}\right)+\chi\left(\frac{b}{a}\right)\right)\mathrm{Re}\frac{L'(1,\chi^*)}{L(1,\chi^*)} + O(1), \end{aligned} \tag{5.1}$$

using the orthogonality relations for characters. In order to evaluate the first sum on the RHS of (5.1) we use Lemma 3.1 which gives

$$\frac{1}{2}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} (\chi(a/b)+\chi(b/a))\log q_\chi^* = -\phi(q)\frac{\Lambda\left(\frac{q}{(q,a-b)}\right)}{\phi\left(\frac{q}{(q,a-b)}\right)}. \tag{5.2}$$

Now we compute the sum over the L-values. First we record a standard approximation formula for $L'/L(1,\chi^*)$ under GRH, which corresponds to Proposition 3.10 of [5]:

$$\frac{L'(1,\chi^*)}{L(1,\chi^*)} = -\sum_{n=1}^{\infty}\frac{\chi^*(n)\Lambda(n)}{n}e^{-n/y} + O\left(\frac{\log q}{y^{1/2}}\right), \tag{5.3}$$

for all $y \geq 1$. Inserting this estimate into the second sum on the RHS of (5.1), we obtain

$$\begin{aligned} &\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \left(\chi\left(\frac{a}{b}\right)+\chi\left(\frac{b}{a}\right)\right)\mathrm{Re}\frac{L'(1,\chi^*)}{L(1,\chi^*)} = \mathrm{Re}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}} \left(\chi\left(\frac{a}{b}\right)+\chi\left(\frac{b}{a}\right)\right)\frac{L'(1,\chi^*)}{L(1,\chi^*)} \\ &= -\mathrm{Re}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}e^{-n/y}\sum_{\substack{\chi\neq\chi_0\\ \chi \bmod q}}(\chi(a/b)\chi^*(n)+\chi(b/a)\chi^*(n)) + O\left(\frac{\phi(q)\log q}{y^{1/2}}\right). \end{aligned} \tag{5.4}$$

Let p be a prime number and $e \geq 1$ a positive integer. To evaluate the inner sum over characters in the RHS of (5.4) we use Proposition 3.4 of [5] which states that

$$\sum_{\chi \bmod q} \chi(a/b)\,\chi^*(p^e) = \begin{cases} \phi(q) & \text{if } \ p \nmid q \text{ and } ap^e \equiv b \bmod q, \\ \phi(q/p^\nu) & \text{if } \ p^\nu \parallel q \text{ and } ap^e \equiv b \bmod q/p^\nu, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Note that the condition $ap^e \equiv b \bmod q$ implies that $p \nmid q$ since $(b, q) = 1$. Therefore, choosing $y = (q \log q)^2$ in (5.4), and adding the contribution of the principal character (which was evaluated in Lemma 5.2) we obtain from (5.5) that the RHS of (5.4) equals

$$\begin{aligned}
&-\phi(q) \sum_{\substack{n\geq 1\\ bn\equiv a \bmod q}} \frac{\Lambda(n)}{n} e^{-\frac{n}{y}} - \phi(q) \sum_{\substack{n\geq 1\\ an\equiv b \bmod q}} \frac{\Lambda(n)}{n} e^{-\frac{n}{y}}\\
&- \sum_{p^\nu \| q} \phi\left(\frac{q}{p^\nu}\right) \sum_{\substack{e\geq 1\\ ap^e\equiv b \bmod q/p^\nu}} \frac{\log p}{p^e} e^{-\frac{p^e}{y}}\\
&- \sum_{p^\nu \| q} \phi\left(\frac{q}{p^\nu}\right) \sum_{\substack{e\geq 1\\ bp^e\equiv a \bmod q/p^\nu}} \frac{\log p}{p^e} e^{-\frac{p^e}{y}} + 4\log q + O(\log\log q).
\end{aligned}$$

Moreover, if $n \geq 2y \log y$, then $e^{-n/y} \leq 1/n$. This implies that

$$\sum_{\substack{n\geq 2y\log y\\ bn\equiv a \bmod q}} \frac{\Lambda(n)}{n} e^{-\frac{n}{y}} + \sum_{p^\nu \| q} \sum_{\substack{e\geq 2\log y\\ bp^e\equiv a \bmod q/p^\nu}} \frac{\log p}{p^e} e^{-\frac{p^e}{y}} \ll \sum_{n\geq 2y\log y} \frac{\Lambda(n)}{n^2} \ll \frac{1}{q^2}.$$

Notice that when $p^\nu \parallel q$ we have $\phi(q/p^\nu) = \phi(q)/(p^{\nu-1}(p-1))$ since $(p^\nu, q/p^\nu) = 1$. Thus, using that $1 - e^{-t} \leq 2t$ for all $t > 0$, we obtain

$$\sum_{p^\nu \| q} \phi\left(\frac{q}{p^\nu}\right) \sum_{\substack{1\leq e\leq 2\log y\\ bp^e\equiv a \bmod q/p^\nu}} \frac{\log p}{p^e} \left(1 - e^{-\frac{p^e}{y}}\right) \ll \frac{1}{q\log q} \sum_{p|q} \frac{\log p}{p-1} \leq \frac{1}{q}.$$

The proposition follows upon collecting the above estimates. □

Next, we establish the following lemma which, when combined with Proposition 5.1, yields $B_q(a, b) \ll \phi(q)$.

Lemma 5.3 *Let q be a large positive integer, $(a, b) \in \mathcal{A}_2(q)$, and denote by s the least positive residue of $ab^{-1} \bmod q$. Put $x = (q \log q)^2$. Then*

$$\sum_{\substack{n\leq 2x\log x\\ bn\equiv a \bmod q}} \frac{\Lambda(n)}{n} e^{-n/x} = \frac{\Lambda(s)}{s} + O\left(\frac{\log^2 q}{q}\right).$$

Proof Since $1 - e^{-t} \leq 2t$ for all $t > 0$, then

$$\frac{\Lambda(s)}{s} e^{-s/x} = \frac{\Lambda(s)}{s} + O\left(\frac{1}{q^2}\right).$$

On the other hand if $n \neq s$ is a positive integer such that $n \equiv s \bmod q$, then $n = s + jq$ for some $j \geq 1$. Therefore, we have

$$\sum_{\substack{n \leq 2x \log x \\ n \equiv s \bmod q}} \frac{\Lambda(n)}{n} e^{-n/x} - \frac{\Lambda(s)}{s} \ll \log q \sum_{1 \leq j \leq q \log^4 q} \frac{1}{s + jq} + \frac{1}{q^2} \ll \frac{\log^2 q}{q}.$$

□

Corollary 5.4 *For any $(a, b) \in \mathcal{A}_2(q)$ we have*

$$|B_q(a, b)| \ll \phi(q).$$

Proof First we note that $\Lambda(s)/s \leq (\log s)/s$ which is a decreasing function for $s \geq 3$. Moreover, the term $\Lambda(q/(q, a-b))/\phi(q/(q, a-b))$ is non-zero only when $q/(q, a-b) = p^l$ for some prime $p \geq 2$ and $l \geq 1$. In this case

$$\frac{\Lambda(q/(q, a-b))}{\phi(q/(q, a-b))} = \frac{\log p}{p^{l-1}(p-1)} \leq \frac{\log p}{p-1} \leq \log 2.$$

Finally we have

$$\sum_{p^\nu \| q} \sum_{\substack{1 \leq e \leq 2 \log x \\ bp^e \equiv a \bmod q/p^\nu}} \frac{\log p}{p^{e+\nu-1}(p-1)} \leq \sum_{p|q} \frac{\log p}{(p-1)^2} \ll 1.$$

Thus by Lemma 5.3 and Proposition 5.1, the result follows. □

In the remaining part of this section, we prove Theorems 2.9 and 2.12.

Proof of Proposition 2.9 Surprisingly, the lower bound is much easier to establish than the upper bound. Indeed we use only the definition of $B_q(a, b)$ in this case.

The lower bound. Note that

$$\sum_{(a,b) \in \mathcal{A}_2(q)} B_q(a, b) = \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{1}{\frac{1}{4} + \gamma_\chi^2} \sum_{\substack{a \bmod q \\ (a,q)=1}} \sum_{\substack{b \neq a \bmod q \\ (b,q)=1}} (\chi(a/b) + \chi(b/a)).$$

Write $s \equiv ab^{-1} \bmod q$. When a is fixed and b varies over all reduced residue classes distinct from a, s runs over all reduced residue classes different from 1. Then, using the orthogonality relations for characters we obtain

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} \sum_{\substack{b \neq a \bmod q \\ (b,q)=1}} (\chi(a/b) + \chi(b/a)) = -2\phi(q).$$

Therefore, since $|\mathcal{A}_2(q)| = \phi(q)^2 - \phi(q)$, and $N_q = \phi(q)(\log q + O(\log\log q))$ we deduce that

$$\frac{1}{|\mathcal{A}_2(q)|}\sum_{(a,b)\in\mathcal{A}_2(q)} |B_q(a,b)| \geq -\frac{1}{|\mathcal{A}_2(q)|}\sum_{(a,b)\in\mathcal{A}_2(q)} B_q(a,b) = \log q + O(\log\log q).$$

The upper bound. We use Proposition 5.1. First, remark that $\sum_{(a,b)\in\mathcal{A}_2(q)} l_q(a,b) \leq \phi(q)$, which implies that the contribution of this sum to the upper bound in Theorem 2.9 is $\ll 1$. Take $1 \leq a, b \leq q-1$. Let $d = (q, a-b)$ and write $a - b = ds$. Then $-q/d \leq s \leq q/d$ and $(s, q/d) = 1$. On the other hand, for any choice of d and s satisfying these conditions there are at most $\phi(q)$ pairs (a,b) such that $1 \leq a \neq b \leq q-1$, a and b are coprime to q and $a - b = ds$. Thus we obtain

$$\begin{aligned}\sum_{(a,b)\in\mathcal{A}_2(q)} \frac{\Lambda\left(\frac{q}{(q,a-b)}\right)}{\phi\left(\frac{q}{(q,a-b)}\right)} &\leq \phi(q)\sum_{d|q}\frac{\Lambda(q/d)}{\phi(q/d)}\sum_{\substack{-q/d\leq s\leq q/d\\(s,q/d)=1}} 1\\ &= 2\phi(q)\sum_{d|q}\Lambda(q/d) = 2\phi(q)\log q. \qquad (5.6)\end{aligned}$$

Let $x = (q\log q)^2$. Then

$$\begin{aligned}\sum_{(a,b)\in\mathcal{A}_2(q)}\sum_{\substack{n\leq 2x\log x\\ n\equiv ab^{-1} \bmod q}}\frac{\Lambda(n)}{n}e^{-n/x} &= \sum_{\substack{n\leq 2x\log x\\(n,q)=1}}\frac{\Lambda(n)}{n}e^{-n/x}\sum_{\substack{(a,b)\in\mathcal{A}_2(q)\\ ab^{-1}\equiv n \bmod q}} 1\\ &\leq \phi(q)\sum_{\substack{n\leq 2x\log x\\(n,q)=1}}\frac{\Lambda(n)}{n}e^{-n/x}\\ &\leq 2\phi(q)\log q + O\left(\phi(q)\log\log q\right),\end{aligned}$$

which follows from Lemma 5.2. Finally, using an analogous argument we deduce that

$$\sum_{(a,b)\in\mathcal{A}_2(q)}\sum_{p^\nu\|q}\sum_{\substack{1\leq e\leq 2\log x\\ ap^e\equiv b \bmod q/p^\nu}}\frac{\log p}{p^{e+\nu-1}(p-1)} \leq \phi(q)\sum_{p|q}\sum_{e=1}^{\infty}\frac{\log p}{p^e(p-1)} \ll \phi(q),$$

which completes the proof. □

Proof of Proposition 2.12 First, notice that $|\mathcal{A}_r(q)| = \phi(q)^r + O_r\left(\phi(q)^{r-1}\right)$. Let S_q be the set of pairs $(a,b) \in \mathcal{A}_2(q)$ such that $|B_q(a,b)| \geq \sqrt{\phi(q)}$. Then Theorem 2.9 shows that

$$|S_q|\sqrt{\phi(q)} \leq \sum_{(a,b)\in\mathcal{A}_2(q)} |B_q(a,b)| \ll \phi(q)^2\log q,$$

which gives $|S_q| \ll \phi(q)^{3/2}\log q$. Now define $\Omega_r(q)$ to be the set of r-tuples $(a_1,\ldots,a_r)\in\mathcal{A}_r(q)$ such that $(a_i,a_j)\in S_q$ for some $1\le i\ne j\le r$. Then $|\Omega_r(q)|\ll_r \phi(q)^{r-1/2}\log q$. On the other hand, if $(a_1,\ldots,a_r)\in\mathcal{A}_r(q)\setminus\Omega_r(q)$ then $|B_q(a_i,a_j)|\le\sqrt{\phi(q)}$ for all $1\le i<j\le r$. Hence, in this case, we infer from Theorem 2.1 that

$$\delta(q;a_1,\ldots,a_r)=\frac{1}{r!}-\frac{1}{\sqrt{N_q}}\sum_{1\le j\le r}\alpha_j(r)C_q(a_j)+O\left(\frac{1}{\sqrt{N_q}\log q}\right).$$

Since the $C_q(a_j)$ are integers, the theorem follows upon noting that

$$\sum_{1\le j\le r}\alpha_j(r)C_q(a_j)\ne 0 \implies \left|\sum_{1\le j\le r}\alpha_j(r)C_q(a_j)\right|\gg_r 1.$$

□

6 Extreme values of $B_q(a,b)$ and explicit constructions

Throughout this section we take the residues a_i modulo q so that $|a_i|\le q/2$. The proofs of Theorems 2.4, 2.8 and 2.11 are based on explicit constructions of the a_i. Our strategy consists in choosing these residue classes in such a way to make exactly one of the terms $B_q(a_i,a_j)$ large (using Proposition 6.1 below) and all the others small. Moreover, since this term must be negative (see Remark 5.1), we use Lemma 6.3 below to control the sign of its contribution to the asymptotic formula of the densities $\delta(q;a_1,\ldots,a_r)$. When $|a|$ and $|b|$ are relatively small compared to q, we can precisely understand in which cases $B_q(a,b)$ gets large. Let us define the real valued function

$$\Lambda_0(x):=\begin{cases}\dfrac{\Lambda(x)}{x} & \text{if } x\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases} \tag{6.1}$$

Proposition 6.1 *Let q be a large integer and a,b be distinct integers coprime to q such that $1\le|a|,|b|<q/2$.*

(I) *If a and b have different signs, then*

$$B_q(a,b)=-\phi(q)l(a,b)\log 2+O\Big((|a|+|b|)\log^2 q\Big),$$

where $l(a,b)=1$ if $a=-b$, and equals 0 otherwise.

(II) *If a and b have the same sign, then*

$$B_q(a,b)=-\phi(q)\Lambda_0\left(\frac{\max(|a|,|b|)}{\min(|a|,|b|)}\right)+O\Big((|a|+|b|)\log^2 q\Big).$$

An important ingredient to the proof of this result is the following lemma.

Lemma 6.2 *Let q be a large integer and a, b be distinct integers coprime to q such that $1 \leq |a|, |b| < q/2$. Then*

$$\sum_{p^\nu \| q} \sum_{\substack{1 \leq e \leq 5 \log q \\ ap^e \equiv b \bmod q/p^\nu}} \frac{\log p}{p^{e+\nu-1}(p-1)} \ll \frac{(|a|+|b|)\log^2 q}{q}.$$

Proof First note that $ap^e - b$ can not vanish since $p|q$ and $(ab, q) = 1$. This implies that when q/p^ν divides $ap^e - b$, we must have $q/p^\nu \leq |a|p^e + |b|$, so that $p^{e+\nu} \geq q/(|a|+|b|)$. Therefore the sum we are seeking to bound is

$$\ll \frac{(|a|+|b|)\log q}{q} \sum_{p|q} \frac{p \log p}{p-1} \ll \frac{(|a|+|b|)\log^2 q}{q}.$$

□

Proof of Proposition 6.4 The proof relies on Proposition 5.1. Since $|a|, |b| < q/2$ then $a + b \equiv 0 \bmod q$ implies that $a = -b$. Moreover, notice that $(q, a-b) \leq |a| + |b|$, which gives

$$\frac{\Lambda\left(\frac{q}{(q,a-b)}\right)}{\phi\left(\frac{q}{(q,a-b)}\right)} \ll \frac{(|a|+|b|)\log^2 q}{q},$$

using the standard estimate $\phi(q) \gg q/\log q$. Combining this bound with Proposition 5.1 and Lemmas 5.3 and 6.2 we obtain

$$B_q(a,b) = -\phi(q)\left(l(a,b)\log 2 + \frac{\Lambda(s_1)}{s_1} + \frac{\Lambda(s_2)}{s_2}\right) + O\left((|a|+|b|)\log^2 q\right), \tag{6.2}$$

where s_1 and s_2 denote the least positive residues of ba^{-1} and ab^{-1} modulo q, respectively.

Let us first prove part I. Since a and b have different signs, then $s_1 a \neq b$ and $s_2 b \neq a$. On the other hand we have that q divides both $s_1 a - b$ and $s_2 b - a$. This implies that $q \leq s_i(|a|+|b|)$ for $i = 1, 2$, and thus $s_i \geq q/(|a|+|b|)$. Hence we get

$$\frac{\Lambda(s_1)}{s_1} + \frac{\Lambda(s_2)}{s_2} \ll \frac{(|a|+|b|)\log q}{q},$$

which, in view of Eq. (6.2), gives the first part of the Proposition.

Now, if a and b have the same sign, then $l(a,b) = 0$, and $|a| \neq |b|$. Without any loss of generality we may assume that $|a| < |b|$. Then $s_2 b \neq a$, which as before implies that $\Lambda(s_2)/s_2 \ll (|a|+|b|)(\log q)/q$. Furthermore, if $a|b$ then $s_1 = |b|/|a|$;

while if $a \nmid b$ then $s_1 \geq q/(|a|+|b|)$, and thus $\Lambda(s_1)/s_1 \ll (|a|+|b|)(\log q)/q$ in this case. Therefore, we obtain

$$\frac{\Lambda(s_1)}{s_1} = \Lambda_0\left(\frac{|b|}{|a|}\right) + O\left(\frac{(|a|+|b|)\log q}{q}\right).$$

Hence, part II follows upon combining these estimates with Eq. (6.2). □

Our next result determines the signs of some of the integrals $\beta_{j,k}(r)$.

Lemma 6.3 *For $r \geq 3$ we have $\beta_{1,r}(r) < 0$ and $\beta_{r-1,r}(r) > 0$.*

Proof First we have

$$\begin{aligned}
(2\pi)^{r/2}\beta_{r-1,r}(r) &= \int\limits_{x_1>x_2>\cdots>x_r} x_{r-1}x_r \exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)dx_1\ldots dx_r\\
&= \int\limits_{x_1>\cdots>x_{r-2}} \exp\left(-\frac{x_1^2+\cdots+x_{r-2}^2}{2}\right)\\
&\quad\times\int_{-\infty}^{x_{r-2}} x_{r-1}e^{-\frac{x_{r-1}^2}{2}}\int_{-\infty}^{x_{r-1}} x_r e^{-\frac{x_r^2}{2}}\,dx_r dx_{r-1}\ldots dx_1\\
&= -\int\limits_{x_1>\cdots>x_{r-2}} \exp\left(-\frac{x_1^2+\cdots+x_{r-2}^2}{2}\right)\\
&\quad\times\int_{-\infty}^{x_{r-2}} x_{r-1}e^{-x_{r-1}^2}dx_{r-1}\ldots dx_1\\
&= \frac{1}{2}\int\limits_{x_1>\cdots>x_{r-2}} \exp\left(-\frac{x_1^2+\cdots+x_{r-3}^2+3x_{r-2}^2}{2}\right)dx_{r-2}\ldots dx_1 > 0.
\end{aligned}$$

similarly we get

$$\begin{aligned}
(2\pi)^{r/2}\beta_{1,r}(r) &= \int\limits_{x_1>x_2>\cdots>x_r} x_1x_r \exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)dx_1\ldots dx_r\\
&= \int\limits_{x_2>\cdots>x_{r-1}} \exp\left(-\frac{x_2^2+\cdots+x_{r-1}^2}{2}\right)\int_{x_2}^{\infty} x_1 e^{-\frac{x_1^2}{2}}\\
&\quad\times\int_{-\infty}^{x_{r-1}} x_r e^{-\frac{x_r^2}{2}}\,dx_1dx_rdx_{r-1}\ldots dx_2
\end{aligned}$$

$$= - \int\limits_{x_1 > \cdots > x_{r-2}} \exp\left(- \frac{2x_2^2 + x_3^2 \cdots + x_{r-2}^2 + 2x_{r-1}^2}{2} \right) \times dx_{r-1} \ldots dx_2 < 0.$$

□

Before proving Theorems 2.4, 2.8 and 2.11, let us first define some notation. Let q be a large positive integer. Define p to be the largest prime divisor of q, and denote by p_0 the least non-quadratic residue modulo p (if $p = 2$ take $p_0 = 3$). Then Burgess's bound on short character sums (see [9]) implies that $p_0 \le p^{1/(4\sqrt{e})+\epsilon} \le q^{1/4}$. Moreover note that p_0 is a prime and is also a non-square modulo q. Furthermore we shall denote by $p_1 < p_2$ the smallest prime numbers such that $p_i \ne p_0$ for $i = 1, 2$, and $(p_1 p_2, q) = 1$. Then one has $p_1 < p_2 \le 2 \log q$, in view of the fact that $\prod_{p \le z} p = e^{z+o(z)}$ which follows from the prime number theorem.

Proof of Proposition 2.4 The first part that $|\delta(q; a_1, \ldots, a_r) - 1/r!| \ll_r 1/\log q$ follows from combining Theorem 2.1 with Corollary 5.4 and the fact that $|C_q| = q^{o(1)}$. Concerning the second part we first take $a_1 = 1, a_r = -1$ and $a_j = (p_1 p_2)^{2j}$ for $2 \le j \le r - 1$. Then $|a_j| \le (2 \log q)^{4(r-1)}$ for all $1 \le j \le r$. Using part II of Proposition 6.2 we obtain

$$B_q(a_j, a_k) \ll (\log q)^{4r}, \quad \text{for all } 1 \le j < k \le r - 1,$$

since $p_1 p_2 | a_k / a_j$ in this case. Furthermore, part I of the same proposition implies that

$$B_q(a_j, a_r) \ll (\log q)^{4r} \quad \text{for all } 2 \le j \le r - 1,$$

and

$$B_q(a_1, a_r) = -\phi(q) \log 2 + O\left(\log^2 q\right).$$

Therefore by Theorem 2.1 and Lemma 6.3 we deduce that

$$\begin{aligned} \delta(q; a_1, \ldots, a_r) &= \frac{1}{r!} + \frac{\beta_{1,r}(r) B_q(a_1, a_r)}{N_q} + O_\epsilon\left(\frac{1}{\phi(q)^{1/2-\epsilon}} \right) \\ &> \frac{1}{r!} + \frac{|\beta_{1,r}(r)| \log 2}{2 \log q}. \end{aligned}$$

Furthermore taking $b_1 = a_{r-1}, b_{r-1} = a_1$ and $b_j = a_j$ for all other values of j, we obtain by Lemma 6.3 that

$$\begin{aligned} \delta(q; b_1, \ldots, b_r) &= \frac{1}{r!} + \frac{\beta_{r-1,r}(r) B_q(b_{r-1}, b_r)}{N_q} + O_\epsilon\left(\frac{1}{\phi(q)^{1/2-\epsilon}} \right) \\ &< \frac{1}{r!} - \frac{|\beta_{r-1,r}(r)| \log 2}{2 \log q}, \end{aligned}$$

completing the proof. □

Proof of Proposition 2.8 We need only to construct the squares a_j modulo q, since in this case $\delta(q; ba_1, \ldots, ba_r) = \delta(q; a_1, \ldots, a_r)$ for any residue class b modulo q by Theorem 2 of Feuerverger and Martin [4]. Thus it suffices to take $b_j = ba_j$ for any non-square b modulo q, to get the analogous result for non-squares.

Let $a_1 = 1, a_r = p_1^2$ and $a_j = (p_1p_2)^{2j}$ for $2 \le j \le r-1$. Then $a_j \le (2\log q)^{4(r-1)}$ for all $1 \le j \le r$. Moreover for $1 \le j < k \le r-1$ notice that $p_1p_2|a_k/a_j$. Therefore part II of Proposition 6.1 gives that

$$B_q(a_j, a_k) \ll (\log q)^{4r}, \quad \text{for } 1 \le j < k \le r-1,$$

and

$$B_q(a_j, a_r) \ll (\log q)^{4r}, \quad \text{for } 2 \le j \le r-1,$$

since $p_1p_2|a_j/a_r$ in this case. Finally, since $a_r/a_1 = p_1^2$, we have

$$B_q(a_1, a_r) = -\phi(q)\frac{\log p_1}{p_1^2} + O\left((\log q)^{4r}\right).$$

Thus, combining these estimates with Corollary 2.7 and Lemma 6.3 we deduce

$$\delta(q; a_1, \ldots, a_r) = \frac{1}{r!} + \frac{\beta_{1,r}(r)B_q(a_1, a_r)}{N_q} + O\left(\frac{(\log q)^{4r}}{\phi(q)}\right) > \frac{1}{r!} + \frac{|\beta_{1,r}(r)|}{5\log^3 q}, \tag{6.3}$$

if q is sufficiently large. Furthermore, let σ be the permutation on the set $\{1, \ldots, r\}$ defined by $\sigma(1) = r-1$, $\sigma(r-1) = 1$, and $\sigma(j) = j$ for all other values of j. Then using Lemma 6.3 we obtain similarly to (6.3) that

$$\begin{aligned}\delta(q; a_{\sigma(1)}, \ldots, a_{\sigma(r)}) &= \frac{1}{r!} + \frac{\beta_{r-1,r}(r)B_q(1, p_1^2)}{N_q} + O\left(\frac{(\log q)^{4r}}{\phi(q)}\right)\\ &< \frac{1}{r!} - \frac{|\beta_{r-1,r}(r)|}{5\log^3 q},\end{aligned} \tag{6.4}$$

if q is sufficiently large, which completes the proof. □

Proof of Proposition 2.11 The main idea of the proof relies on the fact (proved in part II of Proposition 6.1) that when $a, b > 0$ and a, b are small compared to q, the quantity $B_q(a, b)$ is small unless $\max(a, b)/\min(a, b)$ equals a prime power. Since $(\kappa_1, \ldots, \kappa_r) \ne (0, \ldots, 0)$ then $\kappa_l \ne 0$ for some $l \in [1, r]$.

Case 1: $\kappa_r \ne 0$ or $\kappa_1 \ne 0$.
We handle only the case $\kappa_r \ne 0$, since the treatment of the case $\kappa_1 \ne 0$ follows simply by switching a_1 with a_r, and b_1 with b_r in every construction we make below. Assume first that $\kappa_r > 0$. In this case take $a_1 = 1$, $a_j = p_0(p_1p_2)^{2j}$ for $2 \le j \le r-1$ and $a_r = (p_1p_2)^2$. Then a_1 and a_r are squares and a_j is a

non-square modulo q for all $2 \le j \le r-1$. Moreover choose $b_j = a_j$ for all $1 \le j \le r-1$ and $b_r = p_0$. In this case b_1 is the only square among the b_j modulo q. Since $C_q(1) > -1$ we get that

$$\sum_{j=1}^{r} \kappa_j C_q(a_j) - \sum_{j=1}^{r} \kappa_j C_q(b_j) = \kappa_r C_q(a_r) - \kappa_r C_q(b_r) = \kappa_r (C_q(1)+1) > 0.$$

In the other direction, note that $|a_j| \le q^{1/4}(2\log q)^{4(r-1)}$ for all $1 \le j \le r$, and that $p_1 p_2$ divides $\max(a_j, a_k)/\min(a_j, a_k)$ for all $1 \le j < k \le r$. Therefore, upon using part II of Proposition 6.1 we deduce that

$$|B_q(a_j, a_k)| \ll q^{1/4}(\log q)^{4r} \quad \text{for all } 1 \le j < k \le r.$$

Hence by Theorem 2.1 we obtain

$$\delta(q; a_1, \ldots, a_r) = \frac{1}{r!} + O_\epsilon\left(\frac{1}{\phi(q)^{1/2-\epsilon}}\right). \tag{6.5}$$

Similarly, part II of Proposition 6.1 gives that $|B_q(b_j, b_k)| \ll q^{1/4}(\log q)^{4r}$ for all $\{j, k\} \ne \{1, r\}$ and

$$B_q(b_1, b_r) = -\phi(q)\frac{\log p_0}{p_0} + O\left(q^{1/4}(\log q)^{4r}\right).$$

Thus using Theorem 2.1 along with Lemma 6.3 and equation (6.5) we get

$$\begin{aligned}\delta(q; b_1, \ldots, b_r) &= \frac{1}{r!} + \frac{\beta_{1,r}(r) B_q(b_1, b_r)}{N_q} + O_\epsilon\left(\frac{1}{\phi(q)^{1/2-\epsilon}}\right)\\ &> \frac{1}{r!} + \frac{|\beta_{1,r}(r)| \log p_0}{2p_0 \log q} > \delta(q; a_1, \ldots, a_r).\end{aligned}$$

Now suppose that $\kappa_r < 0$. In this case we choose $a_1 = 1$ and $a_j = p_0(p_1p_2)^{2j}$ for all $2 \le j \le r$ (so that a_1 is the only square among the a_j); and $b_j = a_j$ for all $1 \le j \le r-1$, and $b_r = p_1^2$ (in this case both b_1 and b_r are squares modulo q). Then similarly to the case $k_r > 0$, one has

$$\sum_{j=1}^{r} \kappa_j C_q(a_j) - \sum_{j=1}^{r} \kappa_j C_q(b_j) = -\kappa_r (1 + C_q(1)) > 0,$$

$$\delta(q; a_1, \ldots, a_r) = \frac{1}{r!} + O_\epsilon\left(\frac{1}{\phi(q)^{1/2-\epsilon}}\right),$$

and

$$\begin{aligned}\delta(q; b_1, \ldots, b_r) &= \frac{1}{r!} + \frac{\beta_{1,r}(r) B_q(b_1, b_r)}{N_q} + O_\epsilon\left(\frac{1}{\phi(q)^{1/2-\epsilon}}\right)\\ &> \frac{1}{r!} + \frac{|\beta_{1,r}(r)| \log p_1}{2p_1^2 \log q} > \delta(q; a_1, \ldots, a_r),\end{aligned}$$

using Theorem 2.1, part II of Proposition 6.1 and Lemma 6.3.

Case 2: $\kappa_l \neq 0$ for some $2 \leq l \leq r-1$.

As before assume first that $k_l > 0$. For the a_i we choose $a_1 = 1, a_l = (p_1 p_2)^2$, and $a_j = p_0 (p_1 p_2)^{4j}$ for $2 \leq j \neq l \leq r$; and for the b_i we take $b_l = p_0 (p_1 p_2)^{4l}$, $b_r = p_0$ and $b_j = a_j$ for all other values of j. Then, an analogous argument to Case 1 gives that

$$\begin{aligned}\sum_{j=1}^{r} \kappa_j C_q(a_j) - \sum_{j=1}^{r} \kappa_j C_q(b_j) = \kappa_l (C_q(1) + 1) > 0, \text{ and } &\delta(q; b_1, \ldots, b_r)\\ &> \delta(q; a_1, \ldots, a_r),\end{aligned}$$

if q is large. Finally if $\kappa_l < 0$, we choose $a_1 = 1$, $a_r = (p_1 p_2)^4$ and $a_j = p_0 (p_1 p_2)^{4j}$ for $2 \leq j \leq r-1$; and $b_l = (p_1 p_2)^4$, $b_r = p_1^2$ and $b_j = a_j$ for all other values for j, to deduce the desired conclusion.

□

7 *q*-Extremely biased races

The idea behind the proof of Theorem 2.6 is to observe that when the a_i are small comparatively to q, the term $B_q(a_i, a_j)$ have a large contribution to the density $\delta(q; a_1, \ldots, a_r)$ if and only if $a_i = -a_j$ or a_i and a_j have the same sign and $\max(|a_i|, |a_j|)/\min(|a_i|, |a_j|)$ equals a prime power (this is proved in Proposition 6.1). The first step is to reduce to the case $r = 3$ (which is easier to deal with) using the following lemma.

Lemma 7.1 *Let $r \geq 3$ be a fixed integer, q be a large positive integer and $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$. If there exist $1 \leq i_1 < i_2 < i_3 \leq r$ such that the race $\{q; a_{i_1}, a_{i_2}, a_{i_3}\}$ is q-extremely biased, then the race $\{q; a_1, \ldots, a_r\}$ is q-extremely biased.*

Proof Suppose that there exist $1 \leq i_1 < i_2 < i_3 \leq r$ with the property that the race $\{q; a_{i_1}, a_{i_2}, a_{i_3}\}$ is q-extremely biased. Then, for some permutation ν of the set $\{i_1, i_2, i_3\}$ we have $|\delta(q; a_{\nu(i_1)}, a_{\nu(i_2)}, a_{\nu(i_3)}) - 1/6| \gg 1/\log q$. Let $j_l = \nu(i_l)$, and define S to be set of all permutations σ of $\{1, \ldots, r\}$ such that $\sigma(j_1) > \sigma(j_2) > \sigma(j_3)$. Note that under GRH and LI it follows from the work of Rubinstein and Sarnak [25] that the logarithmic density of the set of real numbers $x \geq 2$ with $\pi(x; q, a_i) = \pi(x; q, a_j)$ is 0. Using this fact along with the definition of the densities $\delta(q; a_1, \ldots, a_r)$ we obtain

$$\delta(q; a_{j_1}, a_{j_2}, a_{j_3}) = \sum_{\sigma \in S} \delta(q; a_{\sigma(1)}, \ldots, a_{\sigma(r)}). \tag{7.1}$$

Now, a simple combinatorial argument shows that $|S| = r!/3!$. Hence we obtain from (7.1) that

$$\frac{1}{\log q} \ll \left|\delta(q; a_{j_1}, a_{j_2}, a_{j_3}) - \frac{1}{6}\right| \le \sum_{\sigma\in S}\left|\delta(q; a_{\sigma(1)}, \ldots, a_{\sigma(r)}) - \frac{1}{r!}\right|$$
$$\ll_r \max_{\sigma\in S}\left|\delta(q; a_{\sigma(1)}, \ldots, a_{\sigma(r)}) - \frac{1}{r!}\right|,$$

which implies that the race $\{q; a_1, \ldots, a_r\}$ is q-extremely biased. □

The next step is to investigate the main contribution to $B_q(a, b)$ when $a, b > 0$ are relatively small compared to q and $\max(a, b)/\min(a, b)$ equals a prime power. To this end we establish some properties of the function $\Lambda_0(x)$ defined in (6.1).

Lemma 7.2 *The maximum of* $\Lambda_0(x)$ *over* $\mathbb{R}$ *equals* $(\log 3)/3$. *Moreover, if* n *is a positive integer with* $\Lambda_0(n) \neq 0$, *then* $\Lambda_0(m) = \Lambda_0(n)$ *implies that* $m = n$.

Proof We know that $\Lambda_0(x) \neq 0$ if and only if $x = p^l$ for some prime p and a positive integer l. In this case $\Lambda_0(x) = (\log p)/p^l \le \Lambda_0(p)$. The first part follows upon noting that the function $(\log x)/x$ is decreasing for $x \ge 3$ and $(\log 3)/3 > (\log 2)/2$.

If $\Lambda_0(m) = \Lambda_0(n) \neq 0$, then there exist primes p_1, p_2 and positive integers e_1, e_2 such that $n = p_1^{e_1}$, $m = p_2^{e_2}$ and $(\log p_1)/p_1^{e_1} = (\log p_2)/p_2^{e_2}$. This implies that $p_1^{p_2^{e_2}} = p_2^{p_1^{e_1}}$, from which one can deduce that $p_1 = p_2$ and thus $e_1 = e_2$. □

Lemma 7.3 *Let* a_1, a_2 *and* a_3 *be distinct positive real numbers. Define*

$$X_1 = \frac{\max(a_1, a_2)}{\min(a_1, a_2)}, \quad X_2 = \frac{\max(a_2, a_3)}{\min(a_2, a_3)}, \quad \textit{and} \quad X_3 = \frac{\max(a_1, a_3)}{\min(a_1, a_3)}.$$

If one of the values $\Lambda_0(X_1)$, $\Lambda_0(X_2)$ *and* $\Lambda_0(X_3)$ *is non-zero, then there exists a permutation* σ *of the set* $\{1, 2, 3\}$ *such that*

$$\Lambda_0(X_{\sigma(1)}) + \Lambda_0(X_{\sigma(2)}) - 2\Lambda_0(X_{\sigma(3)}) \neq 0.$$

Proof Assume without loss of generality that $a_1 < a_2 < a_3$. In this case $X_1 = a_2/a_1$, $X_2 = a_3/a_2$ and $X_3 = a_3/a_1$. Suppose that for all permutations σ of the set $\{1, 2, 3\}$ we have $\Lambda_0(X_{\sigma(1)}) + \Lambda_0(X_{\sigma(2)}) - 2\Lambda_0(X_{\sigma(3)}) = 0$. Then we must have $\Lambda_0(X_1) = \Lambda_0(X_2) = \Lambda_0(X_3)$. Furthermore since this value is non-zero we get by Lemma 7.2 that $X_1 = X_2 = X_3$. However this can not hold since $X_3 \neq X_1$ by our hypothesis on the a_i. □

Proof of Proposition 2.6 Assume First that neither i) nor ii) hold. In this case Proposition 6.1 implies that $B_q(a_j, a_k) = O_A\left(\log^2 q\right)$, for all $1 \le j < k \le r$. Inserting this estimate in Corollary 2.7 in the case where the a_i are all squares (or all non-squares) modulo q, gives $|\delta(q; a_1 \ldots, a_r) - 1/r!| \ll_{A,r} (\log q)/q$. Now if this is not the case then Theorem 2.1 implies that $|\delta(q; a_1 \ldots, a_r) - 1/r!| \ll_{\epsilon,r} q^{-1/2+\epsilon}$. Thus in both cases the race $\{q; a_1, \ldots, a_r\}$ is not q-extremely biased.

Next, let us consider the case where $a_j = -a_k = a$ for some $1 \leq j < k \leq r$. Since $r \geq 3$, then there exists $b \in \{a_1, \dots, a_r\}$ such that $b \neq a$ and $b \neq -a$. By Lemma 7.1 it suffices to prove that the race $\{q; a, -a, b\}$ is q-extremely biased. Without any loss of generality we may assume that a and b have the same sign (otherwise simply switch a and $-a$). Applying Proposition 6.1 we obtain $B_q(a, -a) = -\phi(q)\log 2 + O_A(\log^2 q)$, $B_q(b, -a) = O_A(\log^2 q)$ (since b and $-a$ have different signs and $b - a \neq 0$) and

$$B_q(a, b) = -\phi(q)\Lambda_0\left(\frac{\max(|a|, |b|)}{\min(|a|, |b|)}\right) + O_A(\log^2 q) \geq -\frac{\log 3}{3}\phi(q) + O_A(\log^2 q),$$

which follows from Lemma 7.2. Inserting these estimates in Corollary 2.3, and recalling that $N_q \sim \phi(q)\log q$ and $|C_q(a)| = q^{o(1)}$, we get

$$\delta(q; a, b, -a) \geq \frac{1}{6} + \frac{2\log 2 - (\log 3)/3}{8\pi\sqrt{3}}\frac{1}{\log q},$$

if q is large enough, so that the race $\{q; a, -a, b\}$ is q-extremely biased.

Now, suppose that $a_i \neq -a_j$ for all $1 \leq i < j \leq r$, and that there exist $b_1, b_2 \in \{a_1, \dots, a_r\}$ such that $b_1 = p^k b_2$ for some prime p, and a positive integer k. In this case part II of Proposition 6.1 yields

$$B_q(b_1, b_2) = -\phi(q)\frac{\log p}{p^k} + O_A(\log^2 q). \tag{7.2}$$

Since $r \geq 3$, then there exists $b_3 \in \{a_1, \dots, a_r\}$ with $b_3 \neq b_i$ for $i = 1, 2$. First if b_3 and b_1 have different signs, then part I of Proposition 6.1 implies that

$$B_q(b_1, b_3), B_q(b_2, b_3) \ll_A \log^2 q. \tag{7.3}$$

Therefore, inserting the estimates (7.2) and (7.3) in Corollary 2.3 gives

$$\delta(q; b_1, b_2, b_3) = \frac{1}{6} - \frac{\log p}{4\pi\sqrt{3}p^k(\log q)}(1 + o(1)),$$

and thus the race $\{q; b_1, b_2, b_3\}$ is q-extremely biased. Hence, it remains only to handle the case where all the b_i have the same sign. Let us denote by S_3 the set of all permutations of $\{1, 2, 3\}$. Since $|b_1|$, $|b_2|$ and $|b_3|$ are distinct by our hypothesis, and $\Lambda_0(|b_1|/|b_2|) \neq 0$, then Lemma 7.3 shows that there exists $\sigma \in S_3$ such that

$$\Lambda_0(X_{\sigma(1)}) + \Lambda_0(X_{\sigma(2)}) - 2\Lambda_0(X_{\sigma(3)}) \neq 0,$$

where

$$X_1 = \frac{|b_1|}{|b_2|} = p^k, \quad X_2 = \frac{\max(|b_2|, |b_3|)}{\min(|b_2|, |b_3|)}, \quad \text{and} \quad X_3 = \frac{\max(|b_1|, |b_3|)}{\min(|b_1|, |b_3|)}.$$

Therefore, upon using part II of Proposition 6.1 along with Corollary 2.3, we deduce that

$$\max_{\nu\in S_3}\left|\delta(q;b_{\nu(1)},b_{\nu(2)},b_{\nu(3)})-\frac{1}{6}\right|\gg\frac{|\Lambda_0(X_{\sigma(1)})+\Lambda_0(X_{\sigma(2)})-2\Lambda_0(X_{\sigma(3)})|}{\log q},$$

which implies that the race $\{q;b_1,b_2,b_3\}$ is q-extremely biased. Thus, appealing to Lemma 7.1 the result follows. □

8 Another proof for the asymptotic in two-way races

In this section we derive Fiorilli and Martin [5] asymptotic formula for the densities in the case $r=2$, using a slight modification of the method used to establish Theorem 2.1. In the version presented below, our main concern is to obtain the main term of (2.3) without giving much attention to the error term, in order to keep the exposition simple. Nonetheless, our approach would give an asymptotic expansion for $\delta(q;a_1,a_2)$ with little extra work, if one allows more terms in the asymptotic series of the Fourier transform $\hat{\mu}_{q;a_1,a_2}$ in Lemma 8.1 below. Indeed we shall establish that

$$\delta(q;a_1,a_2)=\frac{1}{2}-\frac{C_q(a_1)-C_q(a_2)}{\sqrt{2\pi V_q(a_1,a_2)}}+O\left(\frac{C_q(1)^2\log^2 q}{V_q(a_1,a_2)}\right), \tag{8.1}$$

for $(a_1,a_2)\in\mathcal{A}_2(q)$. We begin by proving the analogue of Proposition 3.3

Lemma 8.1 *For $t=(t_1,t_2)\in\mathbb{R}^2$ with $||t||\le N_q^{1/4}$ we have*

$$\hat{\mu}_{q;a_1,a_2}\left(\frac{t_1}{\sqrt{N_q}},\frac{t_2}{\sqrt{N_q}}\right)=\exp\left(-\frac{t_1^2+t_2^2}{2}-\frac{B_q(a_1,a_2)}{N_q}t_1t_2\right)F_{q;a_1,a_2}(t_1,t_2),$$

where

$$F_{q;a_1,a_2}(t_1,t_2)=1+\frac{i}{\sqrt{N_q}}(C_q(a_1)t_1+C_q(a_2)t_2)+O\left(\frac{||t||^4}{N_q}+\frac{||t||^2C_q(1)^2}{N_q}\right).$$

Proof We follow closely the proof of Proposition 3.3. Indeed, for $||t||\le N_q^{1/4}$ the explicit formula (3.1) implies that $\log\hat{\mu}_{q;a_1,a_2}\left(t_1N_q^{-1/2},t_2N_q^{-1/2}\right)$ equals

$$\frac{i}{\sqrt{N_q}}(C_q(a_1)t_1+C_q(a_2)t_2)-\frac{1}{N_q}\sum_{\substack{\chi\ne\chi_0\\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{|\chi(a_1)t_1+\chi(a_2)t_2|^2}{\frac{1}{4}+\gamma_\chi^2}+O\left(\frac{||t||^4}{N_q}\right)$$
$$=\frac{i}{\sqrt{N_q}}(C_q(a_1)t_1+C_q(a_2)t_2)-\frac{t_1^2+t_2^2}{2}-\frac{B_q(a_1,a_2)}{N_q}t_1t_2+O\left(\frac{||t||^4}{N_q}\right).$$

Thus, the lemma follows upon noting that

$$\exp\left(\frac{i}{\sqrt{N_q}}(C_q(a_1)t_1+C_q(a_2)t_2)\right)$$
$$=1+\frac{i}{\sqrt{N_q}}(C_q(a_1)t_1+C_q(a_2)t_2)+O\left(\frac{||t||^2C_q(1)^2}{N_q}\right).$$

□

Our next result is an analogue of Lemma 4.2 in the case of a bivariate normal distribution.

Lemma 8.2 *Let ρ be a real number such that $|\rho|\le 1/2$, n_1, n_2 are fixed non-negative integers, and M a large positive number. Then*

$$\int\limits_{||t||\le M} e^{i(t_1x_1+t_2x_2)}t_1^{n_1}t_2^{n_2}\exp\left(-\frac{t_1^2+t_2^2+2\rho t_1t_2}{2}\right)dt_1dt_2$$
$$=\frac{1}{i^{n_1+n_2}}\frac{\partial^{n_1+n_2}\Phi_\rho(x_1,x_2)}{\partial x_1^{n_1}\partial x_2^{n_2}}+O\left(\exp\left(-\frac{M^2}{8}\right)\right),$$

where

$$\Phi_\rho(x_1,x_2)=\frac{2\pi}{\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x_1^2+x_2^2-2\rho x_1x_2)\right).$$

Proof First, notice that $t_1^2+t_2^2+2\rho t_1t_2\ge(t_1^2+t_2^2)/2$ which follows from the fact that $|t_1t_2|\le(t_1^2+t_2^2)/2$. This implies that the integral we are seeking to estimate equals

$$\int\limits_{t\in\mathbb{R}^2} e^{i(t_1x_1+t_2x_2)}t_1^{n_1}t_2^{n_2}\exp\left(-\frac{t_1^2+t_2^2+2\rho t_1t_2}{2}\right)dt_1dt_2+O\left(\exp\left(-\frac{M^2}{8}\right)\right).$$

Moreover, since the last integral is absolutely and uniformly convergent for $(x_1,x_2)\in\mathbb{R}^2$, we get that

$$\int\limits_{t\in\mathbb{R}^2} e^{i(t_1x_1+t_2x_2)}t_1^{n_1}t_2^{n_2}\exp\left(-\frac{t_1^2+t_2^2+2\rho t_1t_2}{2}\right)dt_1dt_2=\frac{1}{i^{n_1+n_2}}\frac{\partial^{n_1+n_2}\Phi_\rho(x_1,x_2)}{\partial x_1^{n_1}\partial x_2^{n_2}},$$

where

$$\Phi_\rho(x_1,x_2)=\int\limits_{t\in\mathbb{R}^2} e^{i(t_1x_1+t_2x_2)}\exp\left(-\frac{t_1^2+t_2^2+2\rho t_1t_2}{2}\right)dt_1dt_2.$$

On the other hand, note that $\frac{\sqrt{1-\rho^2}}{2\pi}\Phi_\rho(x_1, x_2)$ is the characteristic function of the bivariate normal distribution whose density is

$$f(t_1, t_2) = \frac{\sqrt{1-\rho^2}}{2\pi} \exp\left(-\frac{t_1^2 + t_2^2 + 2\rho t_1 t_2}{2}\right).$$

Therefore, we obtain that

$$\frac{\sqrt{1-\rho^2}}{2\pi}\Phi_\rho(x_1, x_2) = \exp\left(-\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2)\right),$$

which completes the proof. □

We are now ready to establish (8.1). We begin by following the proof of Theorem 2.1. Write $\mu_q = \mu_{q;a_1,a_2}$ and let $R = \sqrt{N_q}\log q$. Then Proposition 4.1 yields

$$\delta(q; a_1, a_2) = \int\limits_{-R<y_2<y_1<R} d\mu_q(y_1, y_2) + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right).$$

Applying the Fourier inversion formula to the measure μ_q gives that

$$\delta(q; a_1, a_2) = \frac{1}{(2\pi)^2} \int\limits_{-R<y_2<y_1<R} \int\limits_{s\in\mathbb{R}^2} e^{i(s_1y_1+s_2y_2)}\hat{\mu}_q(s_1, s_2) d\mathbf{s} d\mathbf{y} + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right). \tag{8.2}$$

Moreover, using Proposition 3.2 with $\epsilon = \log q N_q^{-1/2}$ gives

$$\int\limits_{s\in\mathbb{R}^2} e^{i(s_1y_1+s_2y_2)}\hat{\mu}_q(s_1, s_2) d\mathbf{s} = \int\limits_{||s||\le\epsilon} e^{i(s_1y_1+s_2y_2)}\hat{\mu}_q(s_1, s_2) d\mathbf{s} + O\left(\exp\left(-c\log^2 q\right)\right),$$

for some constant $c > 0$. Inserting this estimate in (8.2), and making the change of variables $t_j = \sqrt{N_q}s_j$ and $x_j = y_j/\sqrt{N_q}$ for $j = 1, 2$, we infer from Lemma 8.1 that

$$\begin{aligned}\delta(q; a_1, a_2) &= \frac{1}{(2\pi)^2} \int\limits_{-\log q<x_2<x_1<\log q} \int\limits_{||t||<\log q} e^{i(t_1x_1+t_2x_2)}\hat{\mu}_q\left(\frac{t_1}{\sqrt{N_q}}, \frac{t_2}{\sqrt{N_q}}\right) d\mathbf{t} d\mathbf{x} \\ &\quad + O\left(\exp\left(-\log^{3/2} q\right)\right). \\ &= I_0 + \frac{iC_q(a_1)}{\sqrt{N_q}} I_1 + \frac{iC_q(a_2)}{\sqrt{N_q}} I_2 + O\left(\frac{C_q(1)^2\log^2 q}{N_q}\right), \end{aligned} \tag{8.3}$$

where

$$I_0 = \frac{1}{(2\pi)^2} \int\limits_{-\log q < x_2 < x_1 < \log q} \int\limits_{||t|| < \log q} e^{i(t_1x_1 + t_2x_2)} \times \exp\left(-\frac{t_1^2 + t_2^2}{2} - \frac{B_q(a_1, a_2)}{N_q} t_1 t_2\right) d\mathbf{t} d\mathbf{x},$$

and

$$I_j = \frac{1}{(2\pi)^2} \int\limits_{-\log q < x_2 < x_1 < \log q} \int\limits_{||t|| < \log q} e^{i(t_1x_1 + t_2x_2)} t_j \times \exp\left(-\frac{t_1^2 + t_2^2}{2} - \frac{B_q(a_1, a_2)}{N_q} t_1 t_2\right) d\mathbf{t} d\mathbf{x},$$

for $j = 1, 2$. We shall first evaluate I_0. Let $\rho = B_q(a_1, a_2)/N_q$. Then Corollary 5.4 implies that $|\rho| \le 1/2$ for q large. Hence Lemma 8.2 yields

$$I_0 = \frac{1}{2\pi\sqrt{1-\rho^2}} \int\limits_{-\log q < x_2 < x_1 < \log q} \exp\left(-\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2)\right) dx_1 dx_2 + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right).$$

Now the integral on the RHS of the last estimate equals

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \int\limits_{x_1 > x_2} \exp\left(-\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2)\right) dx_1 dx_2 + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right).$$

Therefore, using that the integrand is symmetric in x_1 and x_2, along with the fact that

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2)\right) dx_1 dx_2 = 1,$$

we deduce that

$$I_0 = \frac{1}{2} + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right). \tag{8.4}$$

Using similar ideas along with Lemma 8.2 gives

$$\begin{aligned} I_1 &= \frac{1}{(2\pi)^2 i} \int\limits_{x_1 > x_2} \frac{\partial \Phi_\rho(x_1, x_2)}{\partial x_1} dx_1 dx_2 + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right) \\ &= -\frac{1}{(2\pi)^2 i} \int\limits_{-\infty}^{\infty} \Phi_\rho(x_2, x_2) dx_2 + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right). \end{aligned}$$

Furthermore, one has

$$\int\limits_{-\infty}^{\infty} \Phi_\rho(y, y) dy = \frac{2\pi}{\sqrt{1-\rho^2}} \int\limits_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\left(\frac{2}{1+\rho}\right)\right) dy = \frac{2\pi^{3/2}}{\sqrt{1-\rho}}.$$

Note that $2(1-\rho) = V_q(a_1, a_2)/N_q$. Thus, upon combining the above estimates we get

$$I_1 = -\frac{\sqrt{N_q}}{i\sqrt{2\pi V_q(a_1, a_2)}} + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right). \tag{8.5}$$

Similarly one obtains

$$I_2 = \frac{\sqrt{N_q}}{i\sqrt{2\pi V_q(a_1, a_2)}} + O\left(\exp\left(-\frac{\log^2 q}{10}\right)\right). \tag{8.6}$$

Finally, inserting the estimates (8.4)–(8.6) into Eq. (8.3), and using the fact that $V_q(a_1, a_2) \sim 2N_q$ give the desired result.

Acknowledgments I would like to thank Andrew Granville for introducing me to this delightful subject and for many comments and suggestions. I also thank Kevin Ford for several valuable discussions on the results of this paper.

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