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# THE PRIME NUMBER RACE AND ZEROS OF DIRICHLET $L$-FUNCTIONS OFF THE CRITICAL LINE: PART III

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**Abstract**

We show, for any $q \geqslant 3$ and distinct reduced residues $a, b \pmod q$, that the existence of certain hypothetical sets of zeros of Dirichlet $L$-functions lying off the critical line implies that $\pi(x; q, a) < \pi(x; q, b)$ for a set of real $x$ of asymptotic density 1.

## 1. Introduction

For $(a, q) = 1$, let $\pi(x; q, a)$ denote the number of primes $p \leqslant x$ with $p \equiv a \pmod q$. The study of the relative magnitudes of the functions $\pi(x; q, a)$ for a fixed $q$ and varying $a$ is known colloquially as the 'prime race problem' or 'Shanks–Rényi prime race problem'. For a survey of problems and results on prime races, the reader may consult the papers [3, 5]. One basic problem is the study of $P_{q;a_1,\dots,a_r}$, the set of real numbers $x \geqslant 2$ such that $\pi(x; q, a_1) > \cdots > \pi(x; q, a_r)$. It is generally believed that all sets $P_{q;a_1,\dots,a_r}$ are unbounded. Assuming the generalized Riemann hypothesis for Dirichlet $L$-functions modulo $q$ ($\mathrm{GRH}_q$) and that the non-negative imaginary parts of zeros of these $L$-functions are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown, for any $r$-tuple of reduced residue classes $a_1, \dots, a_r$ modulo $q$, that $P_{q;a_1,\dots,q_r}$ has a positive logarithmic density (although it may be quite small in some cases). We recall that the logarithmic density of a set $E \subset (0, +\infty)$ is defined as

$$\delta(E) = \lim_{X\to\infty} \frac{1}{\log X} \int_{[2,X]\cap E} \frac{\mathrm{d}t}{t},$$

provided that the limit exists.

In [2, 4], Ford and Konyagin investigated how possible violations of the GRH would affect prime number races. In [2], they proved that the existence of certain sets of zeros off the critical line would imply that some of the sets $P_{q;a_1,a_2,a_3}$ are bounded, giving a negative answer to the prime race problem with $r = 3$. Paper [4] was devoted to similar questions for $r$-way prime races with $r > 3$. One result from [4] states that, for any $q$, $r \leqslant \phi(q)$ and set $\{a_1, \dots, a_r\}$ of reduced residues modulo $q$, the

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existence of certain hypothetical sets of zeros of Dirichlet $L$-functions modulo $q$ implies that at most $r(r-1)$ of the sets $P_{q;\sigma(a_1),\dots,\sigma(a_r)}$ are unbounded, $\sigma$ running over all permutations of $\{a_1,\dots,a_r\}$.

In this paper, we investigate the effect of zeros of $L$-functions lying off the critical line for two-way prime races. This case is harder, since it is unconditionally proved that, for certain races $\{q;a,b\}$, the set $P_{q;a,b}$ is unbounded. For example, Littlewood [11] proved that $P_{4;3,1}$, $P_{4;1,3}$, $P_{3;1,2}$ and $P_{3;2,1}$ are unbounded. Later Knapowski and Turán [9, 10] proved, for many $q,a,b$ that $\pi(x;q,b)-\pi(x;q,a)$ changes sign infinitely often and more recently Sneed [13] showed that $P_{q;a,b}$ is unbounded for every $q \leqslant 100$ and all possible pairs $(a,b)$.

Nevertheless, we prove that the existence of certain zeros off the critical line would imply that the set $P_{q;a,b}$ has asymptotic density zero, in contrast to a conditional result of Kaczorowski [7] on GRH, which asserts that $P_{q;1,b}$ and $P_{q;b,1}$ have positive lower densities for all $(b,q)=1$.

Let $q \geqslant 3$ be a positive integer and $a,b$ be distinct reduced residues modulo $q$. Moreover, for any set $\mathcal{S}$ of real numbers we define $\mathcal{S}(X)=\mathcal{S}\cap[2,X]$.

Theorem 1.1 *Let $q \geqslant 3$ and suppose that $a$ and $b$ are distinct reduced residues modulo $q$. Let $\chi$ be a non-principal Dirichlet character with $\chi(a)\neq\chi(b)$, and put $\xi=\arg(\chi(a)-\chi(b))\in[0,2\pi)$. Suppose $\frac{1}{2}<\sigma<1$, $0<\delta<\sigma-\frac{1}{2}$, $A>0$ and $\mathcal{B}=\mathcal{B}(\xi,\sigma,\delta,A)$ is a multiset of complex numbers satisfying the conditions listed in Section 2. If $L(\rho,\chi)=0$, for all $\rho\in\mathcal{B}$, $L(s,\chi)$, has no other zeros in the region $\{s:\mathrm{Re}(s)\geqslant\sigma-\delta, \mathrm{Im}(s)\geqslant 0\}$, and for all other non-principal characters $\chi'$ modulo $q$, $L(s,\chi')\neq 0$ in the region $\{s:\mathrm{Re}(s)\geqslant\sigma-\delta, \mathrm{Im}(s)\geqslant 0\}$, then*

$$\lim_{X\to\infty}\frac{\mathrm{meas}(P_{q;a,b}(X))}{X}=0.$$

Remarks A character $\chi$ with $\chi(a)\neq\chi(b)$ exists whenever $a$ and $b$ are distinct modulo $q$. The sets $\mathcal{B}$ have the property that any $\rho\in\mathcal{B}$ has real part in $[\sigma-\delta,\sigma]$, imaginary part greater than $A$ and multiplicity $O((\log\mathrm{Im}(\rho))^{3/4})$ (that is, the multiplicities are much smaller than known bounds on the multiplicity of zeros of Dirichlet $L$-functions). The number of elements of $\mathcal{B}$ (counted with multiplicity) with imaginary part less than $T$ is $O((\log T)^{5/4})$, and thus $\mathcal{B}$ is quite a 'thin' set. Also, we note that if $L(\beta+i\gamma,\chi)=0$, then $L(\beta-i\gamma,\bar{\chi})=0$, which is a consequence of the functional equation for Dirichlet $L$-functions (see, for example, [1, Chapter 9]). The point of Theorem 1.1 is that proving

$$\limsup_{X\to\infty}\frac{\mathrm{meas}(P_{q;a,b}(X))}{X}>0$$

requires showing that the multiset of zeros of $L(s,\chi)$ cannot contain any of the multisets $\mathcal{B}$. This is beyond what is possible with existing technology (see, for example, [6] for the best known estimates for multiplicities of zeros). In other words, Theorem 1.1 claims that under certain suppositions the set $P_{q;a,b}(X)$ has the zero asymptotic density. This implies that its logarithmic density is also zero, in contrast to conditional results from [12].

Our method works as well for the difference $\pi(x)-\mathrm{li}(x)$, the error term in the prime number theorem. Littlewood [11] established that this quantity changes sign infinitely often. Let $P_1$ be the set of real numbers $x\geqslant 2$ such that $\pi(x)>\mathrm{li}(x)$. In [8], Kaczorowski proved, assuming the Riemann Hypothesis, that both $P_1$ and $\bar{P}_1$ have positive lower densities. Assuming the Riemann Hypothesis and that the non-negative imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$ are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown that $P_1$ has a positive logarithmic density $\delta_1\approx 0.00000026$. In contrast to these results, we prove that the existence of certain zeros of $\zeta(s)$ off the critical line would imply that the set $P_1$ has asymptotic density zero (or asymptotic density 1).

THEOREM 1.2 *Suppose* $\frac{1}{2} < \sigma < 1$, $0 < \delta < \sigma - \frac{1}{2}$ *and* $A > 0$. (i) *If* $\xi = 0$, $\mathcal{B} = \mathcal{B}(\xi, \sigma, \delta, A)$ *satisfies the conditions of Section* 2, $\zeta(\rho) = 0$ *for all* $\rho \in \mathcal{B}$, *and* $\zeta(s)$ *has no other zeros in the region* $\{s : \mathrm{Re}(s) \geqslant \sigma - \delta, \mathrm{Im}(s) \geqslant 0\}$, *then*

$$\lim_{X\to\infty} \frac{\mathrm{meas}(P_1(X))}{X} = 0.$$

(ii) *If* $\xi = \pi$, $\mathcal{B}$ *satisfies the conditions of Section* 2, $\zeta(\rho) = 0$ *for all* $\rho \in \mathcal{B}$, *and* $\zeta(s)$ *has no other zeros in the region* $\{s : \mathrm{Re}(s) \geqslant \sigma - \delta, \mathrm{Im}(s) \geqslant 0\}$, *then*

$$\lim_{X\to\infty} \frac{\mathrm{meas}(P_1(X))}{X} = 1.$$

We omit the proof of Theorem 1.2, as it is nearly identical to the proof of Theorem 1.1 in the case $q = 4$.

## 2. The construction of $\mathcal{B}$

For $j \geqslant 1$, we fix any real numbers $\gamma_j$, $\delta_j$ and $\theta_j$ satisfying

$$\begin{aligned} &\exp(j^8) \leqslant \gamma_j \leqslant 2\exp(j^8), \quad \left|\delta_j - \frac{1}{j^8}\right| \leqslant \frac{1}{j^9} \\ &\text{and} \quad \left|\theta_j - \frac{\xi - \pi/2}{j^{16}}\right| \leqslant \frac{1}{j^{17}}. \end{aligned} \tag{1}$$

We choose $j_0$ so large that, for all $j \geqslant j_0$, $\gamma_j > A$ and $\sigma - \delta \leqslant \sigma - \delta_j$. Then we take $\mathcal{B}$ to be the union, over $j \geqslant j_0$ and $1 \leqslant k \leqslant j^3$, of $m(k, j) = k(j^3 + 1 - k)$ copies of $\rho_{j,k}$, where

$$\rho_{j,k} = \sigma - \delta_j + \mathrm{i}(k\gamma_j + \theta_j).$$

## 3. Preliminary results

The following classical-type explicit formula was established in [2, Lemma 1.1] when $x' = x$. The slightly more general result below, which is more convenient for us, is proved in exactly the same way.

LEMMA 3.1 *Let* $\beta \geqslant \frac{1}{2}$ *and for each non-principal character* $\chi$ mod $q$, *let* $B(\chi)$ *be the sequence of zeros* (*duplicates allowed*) *of* $L(s, \chi)$ *with* $\mathrm{Re}(s) > \beta$ *and* $\mathrm{Im}(s) > 0$. *Suppose further that all* $L(s, \chi)$ *are zero-free on the real segment* $\beta < s < 1$. *If* $(a, q) = (b, q) = 1$, $x$ *is sufficiently large and* $x' \geqslant x$, *then*

$$\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = -2\,\mathrm{Re}\left(\sum_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{\rho\in B(\chi) \\ |\mathrm{Im}(\rho)|\leqslant x'}} f(\rho)\right) + O(x^\beta \log^2 x),$$

*where*

$$f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho}\int_2^x \frac{t^\rho}{t \log^2 t}\, \mathrm{d}t = \frac{x^\rho}{\rho \log x} + O\left(\frac{x^{\mathrm{Re}(\rho)}}{|\rho|^2 \log^2 x}\right).$$

REMARK For Theorem 1.2, we use a similar explicit formula for $\pi(x)$ in terms of the zeros $B(\zeta)$ of the Riemann zeta function which satisfy $\Re\rho > \beta$ and $\Im\rho > 0$:

$$\pi(x) = \mathrm{li}(x) - 2\Re \sum_{\substack{\rho\in\mathcal{B}(\zeta)\\ |\Im\rho|\leqslant x'}} f(\rho) + O(x^\beta \log^2 x).$$

Using properties of the Fejér kernel, we prove the following key proposition.

PROPOSITION 3.2 *Let* $\gamma \geqslant 1$, $L \geqslant 4$ *and* $X \geqslant 2$. *Define*

$$F_{\gamma,L}(x) = \sum_{k=1}^{L-1}(L-k)\cos(k\gamma \log x).$$

*Then*

$$\mathrm{meas}\left\{x \in [1, X] : F_{\gamma,L}(x) \geqslant -\frac{L}{4}\right\} \ll \frac{X}{\sqrt{L}}.$$

*Proof*. The Fejér kernel satisfies the following identity:

$$\frac{1}{L}\left(\frac{\sin(L\theta/2)}{\sin(\theta/2)}\right)^2 = 1 + 2\sum_{k=1}^{L-1}\left(1 - \frac{k}{L}\right)\cos(k\theta).$$

This yields

$$F_{\gamma,L}(x) = \frac{\sin^2(L\gamma \log x/2)}{2\sin^2(\gamma \log x/2)} - \frac{L}{2}.$$

Therefore, if $F_{\gamma,L}(x) \geqslant -L/4$, then

$$\sin^2\left(\frac{\gamma \log x}{2}\right) \leqslant \frac{2}{L}\sin^2\left(\frac{L\gamma \log x}{2}\right) \leqslant \frac{2}{L}.$$

Hence,

$$\left\|\frac{\gamma \log x}{2\pi}\right\| \leqslant \varepsilon := \frac{1}{\sqrt{2L}},$$

where $\|t\|$ denotes the distance to the nearest integer. We observe that the condition $\|\gamma \log x/2\pi\| \leqslant \varepsilon$ means that, for some integer $k$, we have

$$k - \varepsilon \leqslant \frac{\gamma \log x}{2\pi} \leqslant k + \varepsilon,$$

or equivalently $e^{2\pi(k-\varepsilon)/\gamma} \leqslant x \leqslant e^{2\pi(k+\varepsilon)/\gamma}$. Thus,

$$\begin{aligned}\operatorname{meas}\left\{x \in [1, X] : F_{\gamma,L}(x) \geqslant -\frac{L}{4}\right\} &\leqslant \operatorname{meas}\left\{x \in [1, X] : \quad \left\|\frac{\gamma \log x}{2\pi}\right\| \leqslant \varepsilon\right\} \\ &\leqslant \sum_{0 \leqslant k \leqslant \gamma \log X/2\pi + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} - e^{2\pi(k-\varepsilon)/\gamma} \\ &\ll \frac{\varepsilon}{\gamma} \sum_{0 \leqslant k \leqslant \gamma \log X/2\pi + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} \ll \varepsilon X.\end{aligned}$$

□

## 4. Proof of Theorem 1.1

Suppose that $X$ is large and $\sqrt{X} \leqslant x \leqslant X$. For brevity, let

$$\Delta = \phi(q)(\pi(x; q, a) - \pi(x; q, b)).$$

It follows from Lemma 3.1 with $x' = \max(x, \max\{j^3\gamma_j : \gamma_j \leqslant x\})$ that

$$\begin{aligned}\Delta &= -\frac{2}{\log x}\operatorname{Re}\left((\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\gamma_j \leqslant x} \sum_{k=1}^{j^3} \frac{x^{\sigma - \delta_j + i(k\gamma_j + \theta_j)} m(k, j)}{\sigma - \delta_j + i(k\gamma_j + \theta_j)}\right) \\ &\quad + O\left(\frac{x^\sigma}{\log^2 x} \sum_{\gamma_j \leqslant x} \frac{x^{-\delta_j}}{\gamma_j^2} \sum_{k=1}^{j^3} \frac{m(k, j)}{k^2} + x^{\sigma-\delta} \log^2 x\right) \\ &= \frac{2x^\sigma}{\log x}\operatorname{Re}\left(i(\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\gamma_j \leqslant x} \frac{x^{-\delta_j}}{\gamma_j} \sum_{k=1}^{j^3} x^{i(k\gamma_j + \theta_j)}(j^3 + 1 - k)\right) \\ &\quad + O\left(\frac{x^\sigma}{\log x} \sum_{\gamma_j \leqslant x} \frac{j^4 x^{-\delta_j}}{\gamma_j^2} + x^{\sigma-\delta} \log^2 x\right). \end{aligned} \tag{2}$$

Note that

$$\frac{x^{-\delta_j}}{\gamma_j} = \exp\left(-\frac{\log x}{j^8}\left(1 + O\left(\frac{1}{j}\right)\right) - j^8 + O(1)\right).$$

The maximum of this function over $j$ occurs around $J = J(x) := [(\log x)^{1/16}]$. In this case, we have $\log x = J^{16}(1 + O(1/J))$ so that

$$\frac{x^{-\delta_J}}{\gamma_J} = \exp(-2J^8 + O(J^7)) = \exp(-2(\log x)^{1/2} + O((\log x)^{7/16})). \tag{3}$$

We will prove that most of the contribution to the main term on the right-hand side of (2) comes for the $j$s in the range $J - J^{3/4} \leqslant j \leqslant J + J^{3/4}$. First, if $j \geqslant 3J/2$ or $j \leqslant J/2$, then

$$\frac{x^{-\delta_j}}{\gamma_j} \ll \exp(-4J^8) \ll \exp(-(\log x)^{1/2})\frac{x^{-\delta_J}}{\gamma_J}.$$

Now suppose that $J/2 < j < J - J^{3/4}$ or $J + J^{3/4} < j < 3J/2$. Write $j = J + r$ with $J^{3/4} < |r| < J/2$. For $x > 0$, $x + 1/x = 2 + (x-1)^2/x$, hence

$$\left(1+\frac{r}{J}\right)^8 + \left(1+\frac{r}{J}\right)^{-8} \geqslant \left(1+\left|\frac{r}{J}\right|\right)^8 + \left(1+\left|\frac{r}{J}\right|\right)^{-8} \geqslant 2 + \frac{(8r/J)^2}{1+8r/J} \geqslant 2 + 12(r/J)^2.$$

We infer from (3) that

$$\begin{aligned}\frac{x^{-\delta_j}}{\gamma_j} &= \exp\left(-\frac{J^{16}}{j^8}\left(1+O\left(\frac{1}{J}\right)\right) - j^8\right)\\ &= \exp\left(-J^8\left(\left(1+\frac{r}{J}\right)^8 + \left(1+\frac{r}{J}\right)^{-8}\right) + O(J^7)\right)\\ &\leqslant \exp\left(-2J^8\left(1+\frac{6}{\sqrt{J}}\right) + O(J^7)\right)\\ &\ll \exp(-2(\log x)^{1/3})\frac{x^{-\delta_J}}{\gamma_J}.\end{aligned}$$

Since $\gamma_j \leqslant x$ implies that $j \ll (\log x)^{1/8}$, the contribution of the terms $1 \leqslant j < J - J^{3/4}$ or $J + J^{3/4} < j$ to the main term of (2) is

$$\ll \exp(-2(\log x)^{1/3})\frac{x^{\sigma-\delta_J}}{\gamma_J}\sum_{j\leqslant(\log x)^{1/4}}\sum_{k=1}^{j^3}(j^3+1-k) \ll \exp(-(\log x)^{1/3})\frac{x^{\sigma-\delta_J}}{\gamma_J}. \tag{4}$$

Similarly, we have

$$\begin{aligned}\frac{x^{-\delta_j}}{\gamma_j^2} &= \exp\left(-\frac{\log x}{j^8}\left(1+O\left(\frac{1}{j}\right)\right) - 2j^8 + O(1)\right)\\ &\ll \exp(-2\sqrt{2}(\log x)^{1/2}(1+o(1)))\\ &\ll \exp(-2(\log x)^{1/3})\frac{x^{-\delta_J}}{\gamma_J},\end{aligned}$$

which follows from (3) along with the fact that the maximum of $f(t) = -\log x/t^8 - 2t^8$ occurs at $t = (\log x/2)^{1/16}$. Hence, using (3), the contribution of the error term of (2) is

$$\ll \exp(-2(\log x)^{1/3})\frac{x^{\sigma-\delta_J}}{\gamma_J}\sum_{j\leqslant(\log x)^{1/4}} j^4 + x^{\sigma-\delta}\log^2 x \ll \exp(-(\log x)^{1/3})\frac{x^{\sigma-\delta_J}}{\gamma_J}. \tag{5}$$

Therefore, inserting the bounds (4) and (5) in (2), we deduce that

$$\begin{aligned}\Delta = \frac{2x^\sigma}{\log x}\mathrm{Re}&\left(\mathrm{i}(\bar\chi(a)-\bar\chi(b))\sum_{|j-J|\leqslant J^{3/4}}\frac{x^{-\delta_j}}{\gamma_j}\sum_{k=1}^{j^3}\exp(\mathrm{i}(k\gamma_j+\theta_j)\log x)(j^3+1-k)\right)\\ &+ O\left(\exp(-(\log x)^{1/3})\frac{x^{\sigma-\delta_J}}{\gamma_J}\right).\end{aligned} \tag{6}$$

Let $J - J^{3/4} \leqslant j \leqslant J + J^{3/4}$. Then $j^{16} = J^{16}(1 + O(J^{-1/4}))$. Hence, we get

$$\theta_j \log x = \left(\arg(\chi(a) - \chi(b)) - \frac{\pi}{2}\right) \frac{\log x}{j^{16}} + O\left(\frac{\log x}{j^{17}}\right)$$
$$= \left(\arg(\chi(a) - \chi(b)) - \frac{\pi}{2}\right) + O\left(\frac{1}{J^{1/4}}\right).$$

This implies

$$\mathrm{i}(\bar{\chi}(a) - \bar{\chi}(b)) \exp(\mathrm{i}\theta_j \log x) = |\chi(a) - \chi(b)| \left(1 + O\left(\frac{1}{J^{1/4}}\right)\right),$$

since $\mathrm{e}^{\mathrm{i}\arg z} = z/|z|$. Inserting this estimate in (6), we obtain

$$\Delta = \left(1 + O\left(\frac{1}{\log^{1/64} x}\right)\right) 2|\chi(a) - \chi(b)| \sum_{|j-J| \leqslant J^{3/4}} \frac{x^{\sigma - \delta_j}}{\gamma_j \log x} F_{\gamma_j, j^3+1}(x)$$
$$+ O\left(\exp(-(\log x)^{1/3}) \frac{x^{\sigma - \delta_J}}{\gamma_J}\right). \tag{7}$$

For $x \in [\sqrt{X}, X]$, we have $\frac{1}{4}(\log X)^{1/16} \leqslant J - J^{3/4}$ and $J + J^{3/4} \leqslant 4(\log X)^{1/16}$ if $X$ is sufficiently large, since $J = (\log x)^{1/16} + O(1)$. We define

$$\Omega := \left\{x \in [\sqrt{X}, X] : F_{\gamma_j, j^3}(x) \leqslant -\frac{j^3}{4} \text{ for all } \frac{1}{4}(\log X)^{1/16} \leqslant j \leqslant 4(\log X)^{1/16}\right\}.$$

Then it follows from Proposition 3.2 that

$$\operatorname{meas} \Omega = X + O\left(X \sum_{\frac{1}{4}(\log X)^{1/16} \leqslant j \leqslant 4(\log X)^{1/16}} \frac{1}{j^{3/2}} + \sqrt{X}\right)$$
$$= X(1 + O((\log X)^{-1/32})). \tag{8}$$

Furthermore, if $x \in \Omega$, then we infer from (7) that

$$\Delta \leqslant -\frac{1}{3}|\chi(a) - \chi(b)| \sum_{|j-J| \leqslant J^{3/4}} \frac{j^3 x^{\sigma - \delta_j}}{\gamma_j \log x} + O\left(\exp(-(\log x)^{1/3}) \frac{x^{\sigma - \delta_J}}{\gamma_J}\right).$$
$$\leqslant -\frac{1}{3}|\chi(a) - \chi(b)| \frac{J^3 x^{\sigma - \delta_J}}{\gamma_J \log x}(1 + o(1)) \leqslant -x^{\sigma} / \exp((2 + o(1))\sqrt{x}) < 0$$

as $X \to \infty$, which completes the proof.

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## References

1. H. Davenport, *Multiplicative Number Theory*, 3rd edn, Graduate Texts in Mathematics 74, Springer, New York, 2000.
2. K. Ford and S. Konyagin, The prime number race and zeros of $L$-functions off the critical line, *Duke Math. J.* **113** (2002), 313–330.
3. K. Ford and S. Konyagin, *Chebyshev's conjecture and the prime number race*, IV International Conference 'Modern Problems of Number Theory and its Applications': Current Problems, Part II (Russian) (Tula, 2001), 67–91, Mosk. Gos. Univ. im. Lomonosova, Mekh.-Mat. Fak., Moscow, 2002. Also available on the first author's web page: http://www.math.uiuc.edu/∼ford/papers/html.
4. K. Ford and S. Konyagin, *The prime number race and zeros of $L$-functions off the critical line, II*, Proceedings of the session in analytic number theory and Diophantine equations (Bonn, January–June 2002), Bonner Mathematische Schiften, Nr. 360 (Eds. D. R. Heath-Brown and B. Z. Moroz), Bonn, Germany, 2003, 40 pp. Also available on the first author's homepage http://www.math.uiuc.edu/∼ford/papers.html.
5. A. Granville and G. Martin, Prime number races, *Amer. Math. Monthly* **113** (2006), 1–33.
6. A. Ivić, On the multiplicity of zeros of the zeta-function, *Bull. Cl. Sci. Math. Nat. Sci. Math.* **24** (1999), 119–132.
7. J. Kaczorowski, A contribution to the Shanks–Rényi race problem, *Quart. J. Math. Oxford Ser.* (2) **44** (1993), 451–458.
8. J. Kaczorowski, Results on the distribution of primes, *J. Reine Angew. Math.* **446** (1994), 89–113.
9. S. Knapowski and P. Turán, Comparative prime number theory Part I, *Acta. Math. Sci. Hungar.* **13** (1962), 299–314; Part II. **13** (1962), 315–342; Part III. **13** (1962), 343–364; Part IV. **14** (1963), 31–42; Part V. **14** (1963), 43–63; Part VI. **14** (1963), 65–78; Part VII. **14** (1963), 241–250; Part VIII. **14** (1963), 251–268.
10. S. Knapowski and P. Turán, Further developments in the comparative prime-number theory. Part I, *Acta Arith.* **9** (1964), 23–40; Part II. **10** (1964), 293–313; Part III. **11** (1965), 115–127; Part IV. **11** (1965), 147–161; Part V. **11** (1965), 193–202; Part VI. **12** (1966), 85–96; Part VII. **21** (1972), 193–201.
11. J. E. Littlewood, Sur la distribution des nombres premiers, *C. R. Akad. Sci. Paris* **158** (1974), 1869–1872.
12. M. Rubinstein and P. Sarnak, Chebyshev's bias, *J. Exper. Math.* **3** (1994), 173–197.
13. J. P. Sneed, Lead changes in the prime number race, *Math. Comp.*, to appear.