THE PRIME NUMBER RACE AND ZEROS OF DIRICHLET
L-FUNCTIONS OFF THE CRITICAL LINE: PART III

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Abstract

We show, for any \( q \geq 3 \) and distinct reduced residues \( a, b \pmod{q} \), that the existence of certain hypothetical sets of zeros of Dirichlet \( L \)-functions lying off the critical line implies that \( \pi(x; q, a) < \pi(x; q, b) \) for a set of real \( x \) of asymptotic density 1.

1. Introduction

For \((a, q) = 1\), let \( \pi(x; q, a) \) denote the number of primes \( p \leq x \) with \( p \equiv a \pmod{q} \). The study of the relative magnitudes of the functions \( \pi(x; q, a) \) for a fixed \( q \) and varying \( a \) is known colloquially as the ‘prime race problem’ or ‘Shanks–Rényi prime race problem’. For a survey of problems and results on prime races, the reader may consult the papers [3, 5]. One basic problem is the study of \( P_{q; a_1,\ldots,a_r} \), the set of real numbers \( x \geq 2 \) such that \( \pi(x; q, a_1) > \cdots > \pi(x; q, a_r) \). It is generally believed that all sets \( P_{q; a_1,\ldots,a_r} \) are unbounded. Assuming the generalized Riemann hypothesis for Dirichlet \( L \)-functions modulo \( q \) (GRH\(_q\)) and that the non-negative imaginary parts of zeros of these \( L \)-functions are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown, for any \( r \)-tuple of reduced residue classes \( a_1,\ldots,a_r \) modulo \( q \), that \( P_{q; a_1,\ldots,a_r} \) has a positive logarithmic density (although it may be quite small in some cases). We recall that the logarithmic density of a set \( E \subset (0, +\infty) \) is defined as

\[
\delta(E) = \lim_{X \to \infty} \frac{1}{\log X} \int_{[2,X]\cap E} \frac{dt}{t},
\]

provided that the limit exists.

In [2, 4], Ford and Konyagin investigated how possible violations of the GRH would affect prime number races. In [2], they proved that the existence of certain sets of zeros off the critical line would imply that some of the sets \( P_{q; a_1, a_2, a_3} \) are bounded, giving a negative answer to the prime race problem with \( r = 3 \). Paper [4] was devoted to similar questions for \( r \)-way prime races with \( r > 3 \). One result from [4] states that, for any \( q, r \leq \phi(q) \) and set \( \{a_1,\ldots,a_r\} \) of reduced residues modulo \( q \), the
existence of certain hypothetical sets of zeros of Dirichlet L-functions modulo $q$ implies that at most $r(r - 1)$ of the sets $P_{Q; a(1), \ldots, a(r)}$ are unbounded, $\sigma$ running over all permutations of $\{a_1, \ldots, a_r\}$.

In this paper, we investigate the effect of zeros of $L$-functions lying off the critical line for two-way prime races. This case is harder, since it is unconditionally proved that, for certain races $(q; a, b)$, the set $P_{q; a, b}$ is unbounded. For example, Littlewood [11] proved that $P_{4:3,1}, P_{4:1,3}, P_{3:1,2}$ and $P_{3:2,1}$ are unbounded. Later Knapowski and Turán [9, 10] proved, for many $q, a, b$ that $\pi(x; q, b) - \pi(x; q, a)$ changes sign infinitely often and more recently Sneed [13] showed that $P_{q; a, b}$ is unbounded for every $q \leq 100$ and all possible pairs $(a, b)$.

Nevertheless, we prove that the existence of certain zeros off the critical line would imply that the set $P_{q; a, b}$ has asymptotic density zero, in contrast to a conditional result of Kaczorowski [7] on GRH, which asserts that $P_{q; 1, b}$ and $P_{q; b, 1}$ have positive lower densities for all $(b, q) = 1$.

Let $q \geq 3$ be a positive integer and $a, b$ be distinct reduced residues modulo $q$. Moreover, for any set $S$ of real numbers we define $S(X) = S \cap [2, X]$.

**Theorem 1.1** Let $q \geq 3$ and suppose that $a$ and $b$ are distinct reduced residues modulo $q$. Let $\chi$ be a non-principal Dirichlet character with $\chi(a) \neq \chi(b)$, and put $\xi = \arg(\chi(a) - \chi(b)) \in [0, 2\pi)$. Suppose $\frac{1}{2} < \sigma < 1, 0 < \delta < \sigma - \frac{1}{2}, A > 0$ and $B = B(\xi, \sigma, \delta, A)$ is a multiset of complex numbers satisfying the conditions listed in Section 2. If $L(\rho, \chi) = 0$, for all $\rho \in B$, $L(s, \chi)$, has no other zeros in the region $\{s : \Re(s) \geq \sigma - \delta, \Im(s) \geq 0\}$, and for all other non-principal characters $\chi'$ modulo $q$, $L(s, \chi') \neq 0$ in the region $\{s : \Re(s) \geq \sigma - \delta, \Im(s) \geq 0\}$, then

$$\lim_{X \to \infty} \frac{\meas(P_{q, a, b}(X))}{X} = 0.$$ 

**Remarks** A character $\chi$ with $\chi(a) \neq \chi(b)$ exists whenever $a$ and $b$ are distinct modulo $q$. The sets $B$ have the property that any $\rho \in B$ has real part in $[\sigma - \delta, \sigma]$, imaginary part greater than $A$ and multiplicity $O((\log \Im(\rho))^{3/4})$ (that is, the multiplicities are much smaller than known bounds on the multiplicity of zeros of Dirichlet L-functions). The number of elements of $B$ (counted with multiplicity) with imaginary part less than $T$ is $O((\log T)^{5/4})$, and thus $B$ is quite a ‘thin’ set. Also, we note that if $L(\beta + iy, \chi') = 0$, then $L(\beta - iy, \overline{\chi}) = 0$, which is a consequence of the functional equation for Dirichlet L-functions (see, for example, [1, Chapter 9]). The point of Theorem 1.1 is that proving

$$\limsup_{X \to \infty} \frac{\meas(P_{q, a, b}(X))}{X} > 0$$

requires showing that the multiset of zeros of $L(s, \chi)$ cannot contain any of the multisets $B$. This is beyond what is possible with existing technology (see, for example, [6] for the best known estimates for multiplicities of zeros). In other words, Theorem 1.1 claims that under certain suppositions the set $P_{q, a, b}(X)$ has the zero asymptotic density. This implies that its logarithmic density is also zero, in contrast to conditional results from [12].

Our method works as well for the difference $\pi(x) - \text{li}(x)$, the error term in the prime number theorem. Littlewood [11] established that this quantity changes sign infinitely often. Let $P_1$ be the set of real numbers $x \geq 2$ such that $\pi(x) > \text{li}(x)$. In [8], Kaczorowski proved, assuming the Riemann Hypothesis, that both $P_1$ and $\overline{P}_1$ have positive lower densities. Assuming the Riemann Hypothesis and that the non-negative imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$ are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown that $P_1$ has a positive logarithmic density $\delta_1 \approx 0.00000026$. In contrast to these results, we prove that the existence of certain zeros
of \( \zeta(s) \) off the critical line would imply that the set \( P_1 \) has asymptotic density zero (or asymptotic density 1).

**Theorem 1.2** Suppose \( \frac{1}{2} < \sigma < 1, \ 0 < \delta < \frac{1}{2} - \sigma \) and \( A > 0 \). (i) If \( \xi = 0 \), \( B = B(\xi, \sigma, \delta, A) \) satisfies the conditions of Section 2, \( \zeta(\rho) = 0 \) for all \( \rho \in B \), and \( \zeta(s) \) has no other zeros in the region \( \{ s : \text{Re}(s) \geq \sigma - \delta, \text{Im}(s) \geq 0 \} \), then

\[
\lim_{X \to \infty} \frac{\text{meas}(P_1(X))}{X} = 0.
\]

(ii) If \( \xi = \pi \), \( B \) satisfies the conditions of Section 2, \( \zeta(\rho) = 0 \) for all \( \rho \in B \), and \( \zeta(s) \) has no other zeros in the region \( \{ s : \text{Re}(s) \geq \sigma - \delta, \text{Im}(s) \geq 0 \} \), then

\[
\lim_{X \to \infty} \frac{\text{meas}(P_1(X))}{X} = 1.
\]

We omit the proof of Theorem 1.2, as it is nearly identical to the proof of Theorem 1.1 in the case \( q = 4 \).

**2. The construction of \( B \)**

For \( j \geq 1 \), we fix any real numbers \( \gamma_j, \delta_j \) and \( \theta_j \) satisfying

\[
\exp(j^8) \leq \gamma_j \leq 2 \exp(j^8), \quad \left| \delta_j - \frac{1}{j^8} \right| \leq \frac{1}{j^9} \quad \text{and} \quad \left| \theta_j - \frac{\pi}{16} \right| \leq \frac{1}{j^{17}}.
\]

We choose \( j_0 \) so large that, for all \( j \geq j_0, \gamma_j > A \) and \( \sigma - \delta \leq \sigma - \delta_j \). Then we take \( B \) to be the union, over \( j \geq j_0 \) and \( 1 \leq k \leq j^3 \), of \( m(k, j) = k(j^3 + 1 - k) \) copies of \( \rho_{j,k} \), where

\[
\rho_{j,k} = \sigma - \delta_j + i(k\gamma_j + \theta_j).
\]

**3. Preliminary results**

The following classical-type explicit formula was established in [2, Lemma 1.1] when \( x' = x \). The slightly more general result below, which is more convenient for us, is proved in exactly the same way.

**Lemma 3.1** Let \( \beta \geq \frac{1}{2} \) and for each non-principal character \( \chi \mod q \), let \( B(\chi) \) be the sequence of zeros (duplicates allowed) of \( L(s, \chi) \) with \( \text{Re}(s) > \beta \) and \( \text{Im}(s) > 0 \). Suppose further that all \( L(s, \chi) \) are zero-free on the real segment \( \beta < s < 1 \). If \( (a, q) = (b, q) = 1, \) \( x \) is sufficiently large and \( x' \geq x \), then

\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = \sum_{\substack{\chi \not\equiv 0 \pmod{q} \chi \operatorname{mod} q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\rho \in B(\chi)} f(\rho) + O(x^{\beta} \log^2 x),
\]

\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = \sum_{\substack{\chi \not\equiv 0 \pmod{q} \chi \operatorname{mod} q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\rho \in B(\chi)} f(\rho) + O(x^{\beta} \log^2 x),
\]

\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = \sum_{\substack{\chi \not\equiv 0 \pmod{q} \chi \operatorname{mod} q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\rho \in B(\chi)} f(\rho) + O(x^{\beta} \log^2 x),
\]

\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = \sum_{\substack{\chi \not\equiv 0 \pmod{q} \chi \operatorname{mod} q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\rho \in B(\chi)} f(\rho) + O(x^{\beta} \log^2 x),
\]

\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = \sum_{\substack{\chi \not\equiv 0 \pmod{q} \chi \operatorname{mod} q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\rho \in B(\chi)} f(\rho) + O(x^{\beta} \log^2 x),
\]

\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = \sum_{\substack{\chi \not\equiv 0 \pmod{q} \chi \operatorname{mod} q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\rho \in B(\chi)} f(\rho) + O(x^{\beta} \log^2 x),
\]

\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = \sum_{\substack{\chi \not\equiv 0 \pmod{q} \chi \operatorname{mod} q}} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\rho \in B(\chi)} f(\rho) + O(x^{\beta} \log^2 x),
\]
where
\[
f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho} \int_2^x \frac{t^\rho}{t \log^2 t} \, dt = \frac{x^\rho}{\rho \log x} + O\left(\frac{x^{\Re(\rho)}}{|\rho|^2 \log^2 x}\right).
\]

**Remark** For Theorem 1.2, we use a similar explicit formula for \(\pi(x)\) in terms of the zeros \(B(\xi)\) of the Riemann zeta function which satisfy \(\Re(\rho) > \beta\) and \(\Im(\rho) > 0\):
\[
\pi(x) = \text{li}(x) - 2\Re \sum_{\rho \in B(\xi)} f(\rho) + O(x^{\beta \log^2 x}).
\]

Using properties of the Fejér kernel, we prove the following key proposition.

**Proposition 3.2** Let \(\gamma \geq 1\), \(L \geq 4\) and \(X \geq 2\). Define
\[
F_{\gamma,L}(x) = \sum_{k=1}^{L-1} (L - k) \cos(k \gamma \log x).
\]

Then
\[
\text{meas}\left\{ x \in [1, X] : F_{\gamma,L}(x) \geq -\frac{L}{4} \right\} \ll \frac{X}{\sqrt{L}}.
\]

**Proof.** The Fejér kernel satisfies the following identity:
\[
\frac{1}{L} \left( \frac{\sin(L\theta/2)}{\sin(\theta/2)} \right)^2 = 1 + 2 \sum_{k=1}^{L-1} \left( 1 - \frac{k}{L} \right) \cos(k\theta).
\]
This yields
\[
F_{\gamma,L}(x) = \frac{\sin^2(L \gamma \log x/2)}{2 \sin^2(\gamma \log x/2)} - \frac{L}{2}.
\]
Therefore, if \(F_{\gamma,L}(x) \geq -L/4\), then
\[
\sin^2\left(\frac{\gamma \log x}{2}\right) \leq \frac{2}{L} \sin^2\left(\frac{L \gamma \log x}{2}\right) \leq \frac{2}{L}.
\]
Hence,
\[
\left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon := \frac{1}{\sqrt{2L}},
\]
where \(\|t\|\) denotes the distance to the nearest integer. We observe that the condition \(\|\gamma \log x/2\pi\| \leq \varepsilon\) means that, for some integer \(k\), we have
\[
k - \varepsilon \leq \frac{\gamma \log x}{2\pi} \leq k + \varepsilon,
\]
or equivalently $e^{2\pi(k+\varepsilon)/\gamma} \leq x \leq e^{2\pi(k-\varepsilon)/\gamma}$. Thus,

$$\text{meas}\left\{ x \in [1, X] : F_{\gamma, L}(x) \geq -\frac{L}{4} \right\} \leq \text{meas}\left\{ x \in [1, X] : \left\lfloor \frac{\gamma \log x}{2\pi} \right\rfloor \leq \varepsilon \right\} \leq \sum_{0 \leq k \leq \gamma \log X/2\pi + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} - e^{2\pi(k-\varepsilon)/\gamma} \ll \frac{\varepsilon}{\gamma} \sum_{0 \leq k \leq \gamma \log X/2\pi + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} \ll \varepsilon X.$$ 

\[\square\]

4. Proof of Theorem 1.1

Suppose that $X$ is large and $\sqrt{X} \leq x \leq X$. For brevity, let

$$\Delta = \phi(q)(\pi(x; q, a) - \pi(x; q, b)).$$

It follows from Lemma 3.1 with $x' = \max(x, \max\{j^3\gamma_j : \gamma_j \leq x\})$ that

$$\Delta = -\frac{2}{\log x} \text{Re} \left( (\tilde{\chi}(a) - \tilde{\chi}(b)) \sum_{\gamma_j \leq x} \sum_{k=1}^{j^3} x^{\sigma - \delta_j - i(k\gamma_j + \theta_j)} m(k, j) \right)$$

$$+ O \left( \frac{x^\sigma}{\log^2 x} \sum_{\gamma_j \leq x} \frac{x^{-\delta_j}}{\gamma_j^2} \sum_{k=1}^{j^3} \frac{m(k, j)}{k^2} + x^{\sigma - \delta} \log^2 x \right)$$

$$= 2 \frac{x^\sigma}{\log x} \text{Re} \left( i(\tilde{\chi}(a) - \tilde{\chi}(b)) \sum_{\gamma_j \leq x} \frac{x^{-\delta_j}}{\gamma_j} \sum_{k=1}^{j^3} x^{i(k\gamma_j + \theta_j)} (j^3 + 1 - k) \right)$$

$$+ O \left( \frac{x^\sigma}{\log x} \sum_{\gamma_j \leq x} \frac{j^4 x^{-\delta_j}}{\gamma_j^2} + x^{\sigma - \delta} \log^2 x \right). \tag{2}$$

Note that

$$\frac{x^{-\delta_j}}{\gamma_j} = \exp \left( -\frac{\log x}{j^8} \left( 1 + O \left( \frac{1}{j} \right) \right) - j^8 + O(1) \right).$$

The maximum of this function over $j$ occurs around $J = J(x) := [(\log x)^{1/16}]$. In this case, we have $\log x = J^{16}(1 + O(1/J))$ so that

$$\frac{x^{-\delta_j}}{\gamma_j} = \exp(-2J^8 + O(J^7)) = \exp(-2(\log x)^{1/2} + O((\log x)^{7/16})). \tag{3}$$

We will prove that most of the contribution to the main term on the right-hand side of (2) comes for the $j$s in the range $J - J^{3/4} \leq j \leq J + J^{3/4}$. First, if $j \geq 3J/2$ or $j \leq J/2$, then

$$\frac{x^{-\delta_j}}{\gamma_j} \ll \exp(-4J^8) \ll \exp(-3(\log x)^{1/2}/\gamma_j).$$
Now suppose that $J/2 < j < J - J^{3/4}$ or $J + J^{3/4} < j < 3J/2$. Write $j = J + r$ with $J^{3/4} < |r| < J/2$. For $x > 0$, $x + 1/x = 2 + (x - 1)^2/x$, hence

$$
\left(1 + \frac{r}{J}\right)^8 + \left(1 + \frac{r}{J}\right)^{-8} \geq \left(1 + \left|\frac{r}{J}\right|\right)^8 + \left(1 + \left|\frac{r}{J}\right|\right)^{-8} \geq 2 + \left(\frac{8r/J^2}{1 + 8r/J}\right) \geq 2 + 12(r/J)^2.
$$

We infer from (3) that

$$
\frac{x^{-\delta_j}}{\gamma_j} = \exp \left( -\frac{J^6}{J^8} \left(1 + O\left(\frac{1}{J}\right)\right) - j^8 \right) \\
= \exp \left( -J^8 \left(\left(1 + \frac{r}{J}\right)^8 + \left(1 + \frac{r}{J}\right)^{-8}\right) + O(J^7) \right) \\
\leq \exp \left( -2J^8 \left(1 + \frac{6}{\sqrt{J}}\right) + O(J^7) \right) \\
\ll \exp(-2(\log x)^{1/3}) \frac{x^{-\delta_j}}{\gamma_j}.
$$

Since $\gamma_j \leq x$ implies that $j \ll (\log x)^{1/8}$, the contribution of the terms $1 \leq j < J - J^{3/4}$ or $J + J^{3/4} < j$ to the main term of (2) is

$$
\ll \exp(-2(\log x)^{1/3}) \frac{x^{-\delta_j}}{\gamma_j} \sum_{j \leq (\log x)^{1/4}} \sum_{k=1}^{j^3} (j^3 + 1 - k) \ll \exp(-2(\log x)^{1/3}) \frac{x^{-\delta_j}}{\gamma_j}.
$$

(4)

Similarly, we have

$$
\frac{x^{-\delta_j}}{\gamma_j^2} = \exp \left( -\log x \frac{1}{j^8} \left(1 + O\left(\frac{1}{j}\right)\right) - 2j^8 + O(1) \right) \\
\ll \exp(-2\sqrt{2}(\log x)^{1/2}(1 + o(1))) \\
\ll \exp(-2(\log x)^{1/3}) \frac{x^{-\delta_j}}{\gamma_j},
$$

which follows from (3) along with the fact that the maximum of $f(t) = -\log x/t^8 - 2t^8$ occurs at $t = (\log x/2)^{1/16}$. Hence, using (3), the contribution of the error term of (2) is

$$
\ll \exp(-2(\log x)^{1/3}) \frac{x^{-\delta_j}}{\gamma_j} \sum_{j \leq (\log x)^{1/4}} j^4 + x^{\sigma-\delta} \log^2 x \ll \exp(-2(\log x)^{1/3}) \frac{x^{-\delta_j}}{\gamma_j}.
$$

(5)

Therefore, inserting the bounds (4) and (5) in (2), we deduce that

$$
\Delta = \frac{2\sigma}{\log x} \text{Re} \left( i(\bar{\chi}(a) - \bar{\chi}(b)) \sum_{|j - J| \leq J^{3/4}} \frac{x^{-\delta_j}}{\gamma_j} \sum_{k=1}^{j^3} \exp(i(k\gamma_j + \theta_j) \log x)(j^3 + 1 - k) \right) \\
+ O\left( \exp(-2(\log x)^{1/3}) \frac{x^{-\delta_j}}{\gamma_j} \right).
$$

(6)
Let \( J - J^{3/4} \leq j \leq J + J^{3/4} \). Then \( j^{16} = J^{16}(1 + O(J^{-1/4})) \). Hence, we get

\[
\theta_j \log x = \left( \arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) \frac{\log x}{j^{16}} + O\left( \frac{\log x}{j^{17}} \right)
\]

\[
= \left( \arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) + O\left( \frac{1}{j^{1/4}} \right).
\]

This implies

\[
i(\bar{\chi}(a) - \bar{\chi}(b)) \exp(i\theta_j \log x) = |\chi(a) - \chi(b)| \left( 1 + O\left( \frac{1}{j^{1/4}} \right) \right),
\]

since \( e^{i \arg z} = z/|z| \). Inserting this estimate in (6), we obtain

\[
\Delta = \left( 1 + O\left( \frac{1}{\log^{1/64} x} \right) \right) 2|\chi(a) - \chi(b)| \sum_{|j - J| \leq J^{3/4}} \frac{x^{\sigma - \delta_j}}{\gamma_j \log x} F_{\gamma_j, j^{3/4} + 1}(x)
\]

\[
+ O\left( \exp(-(\log x)^{1/3}) \frac{x^{\sigma - \delta_j}}{\gamma_j} \right).
\]

(7)

For \( x \in [\sqrt{X}, X] \), we have \( \frac{1}{4}(\log X)^{1/16} \leq J - J^{3/4} \) and \( J + J^{3/4} \leq 4(\log X)^{1/16} \) if \( X \) is sufficiently large, since \( J = (\log x)^{1/16} + O(1) \). We define

\[
\Omega := \left\{ x \in [\sqrt{X}, X] : F_{\gamma_j, j^{3/4}}(x) \leq -\frac{j^{3}}{4} \text{ for all } \frac{1}{4}(\log X)^{1/16} \leq j \leq 4(\log X)^{1/16} \right\}.
\]

Then it follows from Proposition 3.2 that

\[
\text{meas } \Omega = X + O\left( \frac{X}{j^{3/2}} + \sqrt{X} \right)
\]

\[
= X(1 + O((\log x)^{-1/32})).
\]

(8)

Furthermore, if \( x \in \Omega \), then we infer from (7) that

\[
\Delta \leq -\frac{1}{3}|\chi(a) - \chi(b)| \sum_{|j - J| \leq J^{3/4}} \frac{j^{3}x^{\sigma - \delta_j}}{\gamma_j \log x} + O\left( \exp(-(\log x)^{1/3}) \frac{x^{\sigma - \delta_j}}{\gamma_j} \right)
\]

\[
\leq -\frac{1}{3}|\chi(a) - \chi(b)| \frac{j^{3}x^{\sigma - \delta_j}}{\gamma_j \log x} (1 + o(1)) \leq -x^\sigma / \exp((2 + o(1))\sqrt{x}) < 0
\]

as \( X \to \infty \), which completes the proof.
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