Large deviations of the limiting distribution in the Shanks–Rényi prime number race

BY YOUNESS LAMZOURI†

Department of Mathematics, University of Illinois at Urbana–Champaign,
1409 W. Green Street, Urbana, IL, 61801, *U.S.A.*
e-mail: `lamzouri@math.uiuc.edu`

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Abstract

Let $q \geqslant 3$, $2 \leqslant r \leqslant \phi(q)$ and $a_1, \ldots, a_r$ be distinct residue classes modulo q that are relatively prime to q. Assuming the Generalized Riemann Hypothesis (GRH) and the Linear Independence Hypothesis (LI), M. Rubinstein and P. Sarnak [11] showed that the vector-valued function $E_{q;a_1,\ldots,a_r}(x) = (E(x; q, a_1), \ldots, E(x; q, a_r))$, where $E(x; q, a) = (\log x/\sqrt{x})(\phi(q)\pi(x; q, a) - \pi(x))$, has a limiting distribution $\mu_{q;a_1,\ldots,a_r}$ which is absolutely continuous on $\mathbb{R}^r$. Furthermore, they proved that for r fixed, $\mu_{q;a_1,\ldots,a_r}$ tends to a multidimensional Gaussian as $q \to \infty$. In the present paper, we determine the exact rate of this convergence, and investigate the asymptotic behavior of the large deviations of $\mu_{q;a_1,\ldots,a_r}$.

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1. *Introduction*

A classical problem in analytic number theory is the so-called "Shanks and Rényi prime number race" (see [3]) which is described in the following way. Let $q \geqslant 3$ and $2 \leqslant r \leqslant \phi(q)$ be positive integers. For an ordered r-tuple of distinct reduced residues $(a_1, a_2, \ldots, a_r)$ modulo q we denote by $P_{q;a_1,\ldots,a_r}$ the set of real numbers $x \geqslant 2$ such that

$$\pi(x; q, a_1) > \pi(x; q, a_2) > \ldots > \pi(x; q, a_r).$$

Will the sets $P_{q;a_{\sigma(1)},\ldots,a_{\sigma(r)}}$ contain arbitrarily large values, for any permutation σ of $\{1, 2, \ldots, r\}$?

A result of J. E. Littlewood [7] from 1914 shows that the answer is yes in the cases $(q, a_1, a_2) = (4, 1, 3)$ and $(q, a_1, a_2) = (3, 1, 2)$. Similar results to other moduli in the case $r = 2$ were subsequently derived by S. Knapowski and P. Turán [3] (under some hypotheses on the zeros of Dirichlet L-functions), and special cases of the prime number race with $r \geqslant 3$ were considered by J. Kaczorowski [4, 5].

In 1994, M. Rubinstein and P. Sarnak [11] completely solved this problem, conditionally on the assumptions of the Generalized Riemann Hypothesis (GRH) and the Linear Independence Hypothesis (LI) (which is the assumption that the nonnegative imaginary parts of the zeros of all Dirichlet L-functions attached to primitive characters modulo q are linearly independent over $\mathbb{Q}$). To describe their results, we first define some notation. For any real

number $x \geqslant 2$ we introduce the vector-valued function

$$E_{q;a_1,...,a_r}(x) := (E(x; q, a_1), \ldots, E(x; q, a_r)),$$

where

$$E(x; q, a) := \frac{\log x}{\sqrt{x}} (\phi(q)\pi(x; q, a) - \pi(x)).$$

The normalization is such that, if we assume GRH, $E_{q;a_1,...,a_r}(x)$ varies roughly boundedly as x varies. Rubinstein and Sarnak showed, assuming GRH, that the function $E_{q;a_1,...,a_r}(x)$ has a limiting distribution $\mu_{q;a_1,...,a_r}$. More precisely, they proved

$$\lim_{X \to \infty} \frac{1}{\log X} \int_2^X f(E_{q;a_1,...,a_r}(x)) \frac{dx}{x} = \int_{\mathbb{R}^r} f(x_1, \ldots, x_r) d\mu_{q;a_1,...,a_r}, \qquad (1\cdot1)$$

for all bounded, continuous functions f on $\mathbb{R}^r$. Furthermore, assuming both GRH and LI, they showed that $\mu_{q;a_1,...,a_r}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^r$ if $r < \phi(q)$. (When $r = \phi(q)$, $\mu_{q;a_1,...,a_r}$ is shown to be absolutely continuous with respect to the Lebesgue measure on the hyperplane $\sum_{j=1}^r x_j = 0$.) As a consequence, under GRH and LI, the logarithmic density of the set $P_{q;a_1,...,a_r}$ defined by

$$\lim_{x \to \infty} \frac{1}{\log x} \int_{t \in P_{q;a_1,...,a_r} \cap [2,x]} \frac{dt}{t}$$

exists and is positive.

Here and throughout we shall use the notations $||\mathbf{x}|| = \sqrt{\sum_{j=1}^r x_j^2}$ and $|\mathbf{x}|_\infty = \max_{1 \leqslant i \leqslant r} |x_i|$ for the Euclidean norm and the maximum norm on $\mathbb{R}^r$ respectively. In [**11**], Rubinstein and Sarnak also studied the behavior of the tail $\mu_{q;a_1,...,a_r}(||\mathbf{x}|| > V)$ when q is fixed. They showed, under GRH, that for all distinct reduced residues $a_1, \ldots, a_r$ modulo q we have

$$\exp(-\exp(c_2(q)V)) \ll_q \mu_{q;a_1,...,a_r}(||\mathbf{x}|| > V) \ll_q \exp(-c_1(q)\sqrt{V}), \qquad (1\cdot2)$$

for some $c_1(q), c_2(q) > 0$ which depend on q.

In this paper we investigate large deviations of the distribution $\mu_{q;a_1,...,a_r}$ uniformly as $q \to \infty$, under the additional assumption of LI. For a non-trivial character χ modulo q, we denote by $\{\gamma_\chi\}$ the sequence of imaginary parts of the non-trivial zeros of $L(s, \chi)$. Let χ_0 denote the principal character modulo q and define $S = \bigcup_{\chi \neq \chi_0 \bmod q} \{\gamma_\chi\}$. Moreover, let $\{U(\gamma_\chi)\}_{\gamma_\chi \in S}$ be a sequence of independent random variables uniformly distributed on the unit circle. Rubinstein and Sarnak established, under GRH and LI, that the distribution $\mu_{q;a_1,...,a_r}$ is the same as the probability measure corresponding to the random vector

$$X_{q;a_1,...,a_r} = (X(q, a_1), \ldots, X(q, a_r)), \qquad (1\cdot3)$$

where

$$X(q, a) = -C_q(a) + \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{2\text{Re}(\chi(a)U(\gamma_\chi))}{\sqrt{\frac{1}{4} + \gamma_\chi^2}},$$

χ_0 is the principal character modulo q and

$$C_q(a) := -1 + \sum_{\substack{b^2 \equiv a \bmod q \\ 1 \leqslant b \leqslant q}} 1.$$

Note that for $(a, q) = 1$ the function $C_q(a)$ takes only two values: $C_q(a) = -1$ if a is a non-square modulo q, and $C_q(a) = C_q(1)$ if a is a square modulo q. Furthermore, an elementary argument shows that $C_q(a) < d(q) \ll_\epsilon q^\epsilon$ for any $\epsilon > 0$, where $d(q) = \sum_{m|q} 1$ is the usual divisor function.

Let $\text{Cov}_{q;a_1,\dots,a_r}$ be the covariance matrix of $X_{q;a_1,\dots,a_r}$. A straightforward computation shows that the entries of $\text{Cov}_{q;a_1,\dots,a_r}$ are

$$\text{Cov}_{q;a_1,\dots,a_r}(j, k) = \begin{cases} \text{Var}(q) & \text{if } j = k \\ B_q(a_j, a_k) & \text{if } j \neq k, \end{cases}$$

where

$$\text{Var}(q) := \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi} \frac{1}{\frac{1}{4} + \gamma_\chi^2} = 2 \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{1}{\frac{1}{4} + \gamma_\chi^2}$$

and

$$B_q(a, b) := \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{\chi\left(\frac{b}{a}\right) + \chi\left(\frac{a}{b}\right)}{\frac{1}{4} + \gamma_\chi^2}$$

for $(a, b) \in \mathcal{A}(q)$, where $\mathcal{A}(q)$ is the set of ordered pairs of distinct reduced residues modulo q. Assuming GRH, Rubinstein and Sarnak showed that

$$\text{Var}(q) \sim \phi(q) \log q \text{ and } B_q(a, b) = o(\phi(q) \log q) \text{ as } q \to \infty, \tag{1·4}$$

uniformly for all $(a, b) \in \mathcal{A}(q)$. Combining these estimates with an explicit formula for the Fourier transform of $\mu_{q;a_1,\dots,a_r}$ (see (3·1) below) they established, under GRH and LI, that

$$\mu_{q;a_1,\dots,a_r}\left(||\mathbf{x}|| > \lambda\sqrt{\text{Var}(q)}\right) = (2\pi)^{-r/2} \int_{||\mathbf{x}||>\lambda} \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} + o_q(1), \tag{1·5}$$

for any fixed $\lambda > 0$.

We refine their result using the approach developed in [6]. More precisely, we prove that the asymptotic formula (1·5) holds uniformly in the range $0 < \lambda \leqslant \sqrt{\log \log q}$ with an optimal error term $O_r(1/\log^2 q)$.

THEOREM 1. *Assume GRH and LI. Fix an integer $r \geqslant 2$. Let q be large and $a_1, \dots, a_r$ be distinct reduced residues modulo q. Then, in the range $0 < \lambda \leqslant \sqrt{\log \log q}$ we have*

$$\mu_{q;a_1,\dots,a_r}\left(||\mathbf{x}|| > \lambda\sqrt{\text{Var}(q)}\right) = (2\pi)^{-r/2} \int_{||\mathbf{x}||>\lambda} \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} + O_r\left(\frac{1}{\log^2 q}\right).$$

Moreover, there exists an r-tuple of distinct reduced residue classes $(a_1, \dots, a_r)$ modulo q, such that in the range $1/4 < \lambda < 3/4$ we have

$$\left| \mu_{q;a_1,\dots,a_r}\left(||\mathbf{x}|| > \lambda\sqrt{\text{Var}(q)}\right) - (2\pi)^{-r/2} \int_{||\mathbf{x}||>\lambda} \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} \right| \gg_r \frac{1}{\log^2 q}.$$

Since $\text{Var}(q) \sim \phi(q) \log q$, it follows from Theorem 1 that

$$\mu_{q;a_1,\dots,a_r}(||\mathbf{x}|| > V) = \exp\left(-\frac{V^2}{2\phi(q) \log q}(1 + o(1))\right) \tag{1·6}$$

in the range $(\phi(q) \log q)^{1/2} \ll V \leqslant (1 + o(1))(\phi(q) \log q \log \log q)^{1/2}$. Our next result shows that a similar behavior holds in the much larger range $(\phi(q) \log q)^{1/2} \ll V \ll \phi(q) \log q$.

THEOREM 2. *Assume GRH and LI. Fix an integer $r \geqslant 2$ and a real number $A \geqslant 1$. Let q be large. Then for all distinct reduced residues $a_1, \ldots, a_r$ modulo q, we have*

$$\exp\left(-c_2(r, A)\frac{V^2}{\phi(q) \log q}\right) \ll \mu_{q;a_1,\ldots,a_r}(||\mathbf{x}|| > V) \ll \exp\left(-c_1(r, A)\frac{V^2}{\phi(q) \log q}\right),$$

uniformly in the range $(\phi(q) \log q)^{1/2} \ll V \leqslant A\phi(q) \log q$, where $c_2(r, A) > c_1(r, A)$ are positive numbers that depend only on r and A.

Using an analogous approach, we prove that (1·6) does not hold when $V/(\phi(q) \log q) \to \infty$ as $q \to \infty$, which shows that a transition occurs at $V \asymp \phi(q) \log q$.

THEOREM 3. *Assume GRH and LI. Fix an integer $r \geqslant 2$ and let q be large. If $V/(\phi(q) \log q) \to \infty$ and $V/(\phi(q) \log^2 q) \to 0$ as $q \to \infty$, then for all distinct reduced residues $a_1, \ldots, a_r$ modulo q, we have*

$$\exp\left(-c_4(r)\frac{V^2}{\phi(q) \log q}\exp\left(c_6(r)\frac{V}{\phi(q) \log q}\right)\right) \ll \mu_{q;a_1,\ldots,a_r}(||\mathbf{x}|| > V)$$

and

$$\mu_{q;a_1,\ldots,a_r}(||\mathbf{x}|| > V) \ll \exp\left(-c_3(r)\frac{V^2}{\phi(q) \log q}\exp\left(c_5(r)\frac{V}{\phi(q) \log q}\right)\right),$$

where $c_4(r) > c_3(r)$, and $c_6(r) > c_5(r)$ are positive numbers which depend only on r.

In the range $V/(\phi(q) \log^2 q) \to \infty$ one can prove, using the same ideas, that there are positive constants $c_8(r) > c_7(r)$ such that

$$\exp\left(-\exp\left(c_8(r)\sqrt{\frac{V}{\phi(q)}}\right)\right) \leqslant \mu_{q;a_1,\ldots,a_r}(||\mathbf{x}|| > V) \leqslant \exp\left(-\exp\left(c_7(r)\sqrt{\frac{V}{\phi(q)}}\right)\right). \tag{1·7}$$

In particular, these bounds show that the asymptotic behavior of the tail $\mu_{q;a_1,\ldots,a_r}(||\mathbf{x}|| > V)$ changes again at $V \asymp \phi(q) \log^2 q$. Assuming the RH and using the LI for the Riemann zeta function, H. L. Montgomery [9] had previously obtained a similar result for μ_1, the limiting distribution of the error term in the prime number theorem $\pi(x) - \mathrm{Li}(x)$. His result states that

$$\exp\left(-c_{10}\sqrt{V}\exp\left(\sqrt{2\pi V}\right)\right) \leqslant \mu_1(|x| > V) \leqslant \exp\left(-c_9\sqrt{V}\exp\left(\sqrt{2\pi V}\right)\right),$$

for some absolute constants $c_{10} > c_9 > 0$. A more precise estimate was subsequently derived by W. Monach [8], namely

$$\mu_1(|x| > V) = \exp\left(-(e^{-A_0} + o(1))\sqrt{2\pi V}\exp\left(\sqrt{2\pi V}\right)\right), \tag{1·8}$$

where A_0 is an absolute constant defined in Theorem 4 below.

In our case, it appears that $\mu_{q;a_1,\ldots,a_r}(|\mathbf{x}|_\infty > V)$ is more natural to study, if one wants to gain a better understanding of the decay rate of large deviations of $\mu_{q;a_1,\ldots,a_r}$ in the range $V/(\phi(q) \log^2 q) \to \infty$. We achieved this using the saddle-point method. We also note that in contrast to our previous results, r can vary uniformly in $[2, \phi(q) - 1]$ as $q \to \infty$.

THEOREM 4. *Assume GRH and LI. Let q be large, and $2 \leqslant r \leqslant \phi(q) - 1$ be an integer. If $V/(\phi(q) \log^2 q) \to \infty$ as $q \to \infty$, then for all distinct reduced residue classes $a_1, \ldots, a_r$ modulo q, the tail $\mu_{q;a_1,\ldots,a_r}(|\mathbf{x}|_\infty > V)$ equals*

$$\exp\left(-e^{-L(q)}\sqrt{\frac{2(\phi(q)-1)V}{\pi}} \, \exp\left(\sqrt{L(q)^2 + \frac{2\pi V}{\phi(q)-1}}\right)\left(1 + O\left(\left(\frac{\phi(q)\log^2 q}{V}\right)^{1/4}\right)\right)\right),$$

where

$$L(q) = \frac{\phi(q)}{\phi(q)-1}\left(\log q - \sum_{p|q} \frac{\log p}{p-1}\right) + A_0 - \log \pi$$

and

$$A_0 := \int_0^1 \frac{\log I_0(t)}{t^2} dt + \int_1^\infty \frac{\log I_0(t) - t}{t^2} dt + 1 = 1.2977474,$$

where $I_0(t) = \sum_{n=0}^\infty (t/2)^{2n}/n!^2$ is the modified Bessel function of order 0.

Lastly, we should also mention that in the range $V \geqslant \sqrt{\phi(q)} \log q$ one may allow r to vary uniformly, as in Theorem 4, if one is willing to replace $\mu_{q;a_1,\ldots,a_r}(||x|| > V)$ by $\mu_{q;a_1,\ldots,a_r}(|\mathbf{x}|_\infty > V)$ in the statements of Theorems 2 and 3.

2. *The intermediate range: proof of Theorems 2 and 3*

In this section we investigate the behavior of the tail $\mu_{q;a_1,\ldots,a_r}(||x|| > V)$ when $V \gg \sqrt{\phi(q)} \log q$ as long as $V/(\phi(q) \log^2 q) \to 0$ as $q \to \infty$. Recall that $\mu_{q;a_1,\ldots,a_r}$ is also the probability measure corresponding to the random vector $X_{q;a_1,\ldots,a_r}$ (defined in (1·3)). Our idea starts with the observation that the random variables

$$Y(q, a) := X(q, a) - \mathbb{E}(X(q, a)) = \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{2\mathrm{Re}(\chi(a)U(\gamma_\chi))}{\sqrt{\frac{1}{4} + \gamma_\chi^2}}, \tag{2·1}$$

are identically distributed for all reduced residues a modulo q. Indeed, for all $(a, q) = 1$ the random variables $\{\tilde{U}(\gamma_\chi)\}_{\gamma_\chi \in S}$, where $\tilde{U}(\gamma_\chi) = \chi(a)U(\gamma_\chi)$, are independent and uniformly distributed on the unit circle. Hence, if $(a, q) = 1$ then $Y(q, a)$ has the same distribution as

$$Y(q) := \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{2\cos(2\pi\theta(\gamma_\chi))}{\sqrt{\frac{1}{4} + \gamma_\chi^2}}, \tag{2·2}$$

where $\{\theta(\gamma_\chi)\}_{\gamma_\chi \in S}$ are independent random variables uniformly distributed on $[0, 1]$. Our first lemma shows, in our range of V, that large deviations of $\mu_{q;a_1,\ldots,a_r}$ are closely related to those of $Y(q)$.

LEMMA 2·1. *The random variable $Y(q)$ is symmetric. Moreover, if q is sufficiently large, $r \geqslant 2$ is fixed and $V \geqslant \sqrt{\phi(q)}$, then for all distinct reduced residues $a_1, \ldots, a_r$ modulo q we have*

$$\mathrm{P}(Y(q) > 2V) \leqslant \mu_{q;a_1,\ldots,a_r}(||x|| > V) \leqslant 2r\mathrm{P}\left(Y(q) > \frac{V}{2\sqrt{r}}\right).$$

Proof. Note that $\mathbb{E}(\exp(it\cos(2\pi\theta(\gamma_\chi)))) = J_0(t)$, where $J_0(t) := \sum_{m=0}^{\infty}(-1)^m$ $(t/2)^{2m}/m!^2$ is the Bessel function of order 0. Therefore, since the Fourier transform (characteristic function) of $Y(q)$ is

$$\mathbb{E}(e^{itY(q)}) = \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi > 0} J_0\left(\frac{2t}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right),$$

and J_0 is an even function then $Y(q)$ is symmetric. Now,

$$\mu_{q;a_1,\dots,a_r}(||x|| > V) = \mathrm{P}(||X_{q;a_1,\dots,a_r}|| > V) \geqslant \mathrm{P}(X(q,a_1) > V) = \mathrm{P}(Y(q) > V + C_q(a_1)).$$

The lower bound follows from the fact that $C_q(a_1) < d(q) = q^{o(1)}$. On the other hand, if $|X(q,a_j)| \leqslant V/\sqrt{r}$ for all $1 \leqslant j \leqslant r$, then $||X_{q;a_1,\dots,a_r}|| \leqslant V$. Hence

$$\mu_{q;a_1,\dots,a_r}(||x|| > V) \leqslant \sum_{j=1}^{r} \mathrm{P}\left(|X(q,a_j)| > \frac{V}{\sqrt{r}}\right) \leqslant \sum_{j=1}^{r} \mathrm{P}\left(|Y(q)| > \frac{V}{\sqrt{r}} - |C_q(a_j)|\right).$$

Using that $V \geqslant \sqrt{\phi(q)}$, $|C_q(a)| = q^{o(1)}$ for all $(a,q) = 1$, and $Y(q)$ is symmetric we obtain from the last inequality

$$\mu_{q;a_1,\dots,a_r}(||x|| > V) \leqslant r\mathrm{P}\left(|Y(q)| > \frac{V}{2\sqrt{r}}\right) = 2r\mathrm{P}\left(Y(q) > \frac{V}{2\sqrt{r}}\right),$$

if q is sufficiently large. This establishes the lemma.

Let

$$S_q(T) := \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{0 < \gamma_\chi \leqslant T} \frac{1}{\sqrt{\frac{1}{4}+\gamma_\chi^2}},$$

and

$$K_q(T) := \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > T} \frac{1}{\frac{1}{4}+\gamma_\chi^2}.$$

To investigate large deviations of $Y(q)$, we shall establish the following result, which is very similar to Montgomery and Odlyzko [**10**, theorem 1].

THEOREM 2·2. *Let V be a positive real number. If $S_q(T) \leqslant V/4$ then*

$$\mathrm{P}(Y(q) \geqslant V) \leqslant \exp\left(-\frac{V^2}{16K_q(T)}\right). \tag{2·3}$$

Moreover, if $S_q(T) \geqslant 8V$ then

$$\mathrm{P}(Y(q) \geqslant V) \geqslant 2^{-50}\exp\left(-\frac{120V^2}{K_q(T)}\right). \tag{2·4}$$

In order to prove this result, we shall use the properties of the Laplace transform (or the moments generating function) of $Y(q)$, which is defined by $\mathcal{L}(\lambda) := \mathbb{E}(e^{\lambda Y(q)})$. Since

$\mathbb{E}(\exp(t \cos(2\pi\theta(\gamma_\chi)))) = I_0(t)$, we have the following product formula for $\mathcal{L}(\lambda)$

$$\mathcal{L}(\lambda) = \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi > 0} I_0\left(\frac{2\lambda}{\sqrt{\frac{1}{4} + \gamma_\chi^2}}\right) \quad \text{for } \lambda \geqslant 0. \tag{2.5}$$

Note that the infinite product in (2·5) is absolutely convergent for $\lambda \geqslant 0$, since $I_0(t) \leqslant \exp(t^2/4)$ by Lemma 2·3 below and $\sum_{\chi \neq \chi_0} \sum_{\gamma_\chi > 0} (\frac{1}{4} + \gamma_\chi^2)^{-1} < \infty$.

Recall that $I_0(\lambda) = \sum_{n=0}^\infty (\lambda/2)^{2n}/n!^2$, and so $I_0(\lambda) \geqslant 1$ for $\lambda \geqslant 0$. First, we establish some classical estimates for $I_0(\lambda)$.

LEMMA 2·3. *For all $\lambda \geqslant 0$ we have*

$$I_0(\lambda) \leqslant \min\left(e^\lambda, e^{\lambda^2/4}\right). \tag{2.6}$$

Moreover, we have

$$I_0(\lambda) \geqslant \begin{cases} e^{\lambda^2/16} & \text{if } 0 \leqslant \lambda \leqslant 4 \\ e^{\lambda/8} & \text{if } \lambda \geqslant 4. \end{cases} \tag{2.7}$$

Proof. First, note that $I_0(\lambda) = \sum_{n=0}^\infty (\lambda/2)^{2n}/n!^2 \leqslant \sum_{n=0}^\infty (\lambda/2)^{2n}/n! = \exp(\lambda^2/4)$. Now, using that $e^x \leqslant 1 + 4x$ for $0 \leqslant x \leqslant 1$, we deduce that $I_0(\lambda) \geqslant 1 + \lambda^2/4 \geqslant \exp(\lambda^2/16)$ for $0 \leqslant \lambda \leqslant 4$. On the other hand, we have the following integral representation of $I_0(\lambda)$

$$I_0(\lambda) = \int_0^1 e^{\lambda \cos(2\pi\theta)} d\theta, \tag{2.8}$$

which shows that $I_0(\lambda) \leqslant e^\lambda$ for all $\lambda \geqslant 0$. Moreover, since $\cos(2\pi\theta) \geqslant 1/2$ if $\theta \in [0, 1/6] \cup [5/6, 1]$ then $I_0(\lambda) \geqslant e^{\lambda/2}/3 \geqslant e^{\lambda/8}$ if $\lambda \geqslant 4$, which completes the proof.

To prove Theorem 2·2 we shall use the following probability inequalities.

LEMMA 2·4. *Let X be a random variable on a probability space (Ω, μ) such that $\mathbb{E}(e^{tX}) < \infty$, for all $t \geqslant 0$. Then, for all $\lambda > 0$ we have*

$$\mathrm{P}(X \geqslant V) \leqslant e^{-\lambda V} \mathbb{E}(e^{\lambda X}). \tag{2.9}$$

Moreover, if λ and V are positive real numbers such that $\mathbb{E}(e^{\lambda X}) = 2e^{\lambda V}$, then

$$\mathrm{P}(X \geqslant V) \geqslant \frac{\mathbb{E}(e^{\lambda X})^2}{4\mathbb{E}(e^{2\lambda X})}. \tag{2.10}$$

Proof. The first inequality is standard and known as Chernoff's bound. It is a simple consequence of Markov's inequality. Indeed, we have

$$\mathrm{P}(X \geqslant V) = \mathrm{P}(e^{\lambda X} \geqslant e^{\lambda V}) \leqslant e^{-\lambda V} \mathbb{E}(e^{\lambda X}).$$

Now we establish (2·10). Let $Z = e^{\lambda X}$. Then $\mathrm{P}(X \geqslant V) = \mathrm{P}(Z \geqslant e^{\lambda V}) = \mathrm{P}(Z \geqslant \mathbb{E}(Z)/2)$, by our assumption on λ and V. Moreover, since $\int_{Z \leqslant \mathbb{E}(Z)/2} Z d\mu \leqslant \mathbb{E}(Z)/2$, then applying the Cauchy-Schwarz inequality, we obtain

$$\left(\frac{\mathbb{E}(Z)}{2}\right)^2 \leqslant \left(\int_{Z \geqslant \mathbb{E}(Z)/2} Z d\mu\right)^2 \leqslant \mathbb{E}(Z^2) \mathrm{P}\left(Z \geqslant \frac{\mathbb{E}(Z)}{2}\right),$$

which completes the proof.

Proof of Theorem 2·2. We follow the proof of [**10**, theorem 1]. We shall first establish (2·3). Using (2·5) and (2·9) we obtain

$$P(Y(q) \geqslant V) \leqslant e^{-\lambda V} \mathcal{L}(\lambda) \leqslant e^{-\lambda V} \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi > 0} I_0 \left(\frac{2\lambda}{\sqrt{\frac{1}{4} + \gamma_\chi^2}} \right),$$

for any $\lambda > 0$. Now, we use the first bound of (2·6) for $I_0(2\lambda(\frac{1}{4} + \gamma_\chi^2)^{-1/2})$ when $\gamma_\chi \leqslant T$ and the second bound of (2·6) when $\gamma_\chi > T$, which yields

$$P(Y(q) \geqslant V) \leqslant \exp(-\lambda V + 2\lambda S_q(T) + \lambda^2 K_q(T)) \leqslant \exp\left(-\frac{\lambda V}{2} + \lambda^2 K_q(T) \right)$$

since $S_q(T) \leqslant V/4$. Choosing $\lambda = V/(4K_q(T))$ gives the desired bound.

Now suppose that $S_q(T) \geqslant 8V$, and let λ_0 be a solution of the equation $\mathcal{L}(\lambda) = 2e^{\lambda V}$. Such a λ_0 must exist, since both sides of this equation are continuous functions of λ, the left-hand side is smaller than the right when $\lambda = 0$, and the reverse is true when $\lambda \geqslant \max(V^{-1}, 2\sqrt{1/4 + T^2})$, since by Lemma 2·3 we have

$$\mathcal{L}(\lambda) \geqslant \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{0 < \gamma_\chi \leqslant T} I_0 \left(\frac{2\lambda}{\sqrt{\frac{1}{4} + \gamma_\chi^2}} \right) \geqslant \exp\left(\frac{\lambda}{4} S_q(T) \right) \geqslant e^{2\lambda V} > 2e^{\lambda V}. \qquad (2\cdot11)$$

Moreover, it follows from Lemma 2·3 that $I_0(\lambda)^{16} \geqslant I_0(2\lambda)$ for all $\lambda \geqslant 0$, and hence $\mathcal{L}(\lambda)^{16} \geqslant \mathcal{L}(2\lambda)$. Therefore, using (2·10) we find that

$$P(Y(q) \geqslant V) \geqslant \frac{\mathcal{L}(\lambda_0)^2}{4\mathcal{L}(2\lambda_0)} \geqslant \frac{1}{4\mathcal{L}(\lambda_0)^{14}} = 2^{-16} e^{-14\lambda_0 V}. \qquad (2\cdot12)$$

Now, if $\lambda_0 V \leqslant 1$, then $P(Y(q) \geqslant V) \geqslant 2^{-50}$ and we are done. Thus, we may suppose that $\lambda_0 \geqslant 1/V$. In this case, we must have $\lambda_0 \leqslant 2\sqrt{1/4 + T^2}$ by (2·11). Therefore, we infer from (2·7) that

$$e^{2\lambda_0 V} \geqslant \mathcal{L}(\lambda_0) \geqslant \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi > T} I_0 \left(\frac{2\lambda_0}{\sqrt{\frac{1}{4} + \gamma_\chi^2}} \right) \geqslant \exp\left(\frac{\lambda_0^2}{4} K_q(T) \right),$$

which yields that $\lambda_0 \leqslant 8V/K_q(T)$. Inserting this bound in (2·12) completes the proof.

In order to apply Theorem 2·2 to our setting, we have to understand the asymptotic behavior of the sums $S_q(T)$ and $K_q(T)$. For a non-trivial character χ modulo q, we let q_χ^* be the conductor of χ, and χ^* be the unique primitive character modulo q_χ^* which induces χ. We begin by recording some standard estimates which will be useful in our subsequent work.

LEMMA 2·5. *Assume GRH. Let χ be a non-trivial character modulo q. Then*

$$\sum_{\gamma_\chi > 0} \frac{1}{\frac{1}{4} + \gamma_\chi^2} = \frac{1}{2} \log q_\chi^* + O(\log \log q).$$

Moreover, we have

$$\sum_{\chi \bmod q} \log q_\chi^* = \phi(q) \left(\log q - \sum_{p \mid q} \frac{\log p}{p - 1} \right),$$

and

$$\sum_{p|q} \frac{\log p}{p-1} \ll \log \log q.$$

Proof. The first estimate follows from [**2**, lemma 3·5], and the second is proved in [**2**, proposition 3·3]. Finally, we have

$$\sum_{p|q} \frac{\log p}{p-1} \leqslant \sum_{p \leqslant (\log q)^2} \frac{\log p}{p-1} + \frac{1}{\log q} \sum_{p|q} 1 \ll \log \log q,$$

which follows from the trivial bound $\sum_{p|q} 1 \leqslant \log q / \log 2$.

From this lemma, one can deduce the more precise asymptotic

$$\mathrm{Var}(q) = \phi(q) \log q + O(\phi(q) \log \log q).$$

Our next result gives the classical estimate for

$$N_q(T) := \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{0 < \gamma_\chi < T} 1.$$

LEMMA 2·6. *If $T \geqslant 2$, then*

$$N_q(T) = \frac{(\phi(q) - 1)}{2\pi} T \log T + \frac{R(q)}{2\pi} T + O(\phi(q) \log qT),$$

where

$$R(q) := \phi(q) \left(\log q - \sum_{p|q} \frac{\log p}{p-1} \right) - (\phi(q) - 1)(\log 2\pi + 1).$$

Proof. Using Lemma 2·5 and appealing to the classical estimates for the number of zeros of Dirichlet L-functions (see [**1**, chapters 15 and 16]), we get

$$N_q(T) = \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \left(\frac{T}{2\pi} \log \frac{q_\chi^* T}{2\pi} - \frac{T}{2\pi} + O(\log qT) \right)$$

$$= \frac{(\phi(q) - 1)}{2\pi} T \log T + \frac{R(q)}{2\pi} T + O(\phi(q) \log qT),$$

as desired.

Using Lemmas 2·5 and 2·6 we establish:

PROPOSITION 2·7. *There exists a constant $T_0 \geqslant 2$ such that if $T \geqslant T_0$, and $\log T / \log q \to 0$ as $q \to \infty$, then*

$$\frac{1}{20} \phi(q) \log q \log T \leqslant S_q(T) \leqslant 3\phi(q) \log q \log T$$

if q is sufficiently large, and

$$K_q(T) = \frac{\phi(q) \log q}{2\pi T} \left(1 + O\left(\frac{\log T}{\log q} + \frac{\log \log q}{\log q} + \frac{1}{T} \right) \right).$$

Proof. First, since $\log T = o(\log q)$ then Lemmas 2·5 and 2·6 yield

$$
\begin{aligned}
N_q(T) &= \frac{T}{2\pi}\phi(q)\log q + O\left(\phi(q)T\log T + T\phi(q)\log\log q + \phi(q)\log q\right) \\
&= \frac{T}{2\pi}\phi(q)\log q\left(1 + O\left(\frac{\log T}{\log q} + \frac{\log\log q}{\log q} + \frac{1}{T}\right)\right).
\end{aligned}
\tag{2·13}
$$

Hence, there exists a suitably large constant $T_1 \geqslant 2$ such that

$$
\frac{T}{10}\phi(q)\log q \leqslant N_q(T) \leqslant T\phi(q)\log q,
\tag{2·14}
$$

if $T \geqslant T_1$ and $\log T = o(\log q)$. Now assume that $T \geqslant T_1^2$. Then

$$
S_q(T) = \int_0^T \frac{dN_q(t)}{\sqrt{\frac{1}{4}+t^2}} = \frac{N_q(T)}{\sqrt{\frac{1}{4}+T^2}} + 2\int_0^T \frac{tN_q(t)}{\left(\frac{1}{4}+t^2\right)^{3/2}}dt.
$$

Then, using the lower bound of (2·14) we get

$$
S_q(T) \geqslant \frac{\phi(q)\log q}{10}\int_{T_1}^T \frac{2t^2}{\left(\frac{1}{4}+t^2\right)^{3/2}}dt \geqslant \frac{1}{20}\phi(q)\log q\log T,
$$

which follows from the fact that $2t^3 \geqslant (1/4+t^2)^{3/2}$ for $t \geqslant 1$. Similarly, we obtain from the upper bound of (2·14)

$$
\int_0^{T_1} \frac{tN_q(t)}{\left(\frac{1}{4}+t^2\right)^{3/2}}dt \ll_{T_1} \phi(q)\log q \quad \text{and} \quad \int_{T_1}^T \frac{tN_q(t)}{\left(\frac{1}{4}+t^2\right)^{3/2}}dt \leqslant \phi(q)\log q\log T.
$$

This implies

$$
S_q(T) \leqslant 2\phi(q)\log q\log T + O(\phi(q)\log q).
$$

Therefore, if $T \geqslant T_0$ for some suitably large constant T_0, then

$$
\frac{1}{20}\phi(q)\log q\log T \leqslant S_q(T) \leqslant 3\phi(q)\log q\log T.
$$

On the other hand, we have

$$
K_q(T) = \int_T^\infty \frac{dN_q(t)}{\frac{1}{4}+t^2} = -\frac{N_q(T)}{\frac{1}{4}+T^2} + 2\int_T^\infty \frac{tN_q(t)}{\left(\frac{1}{4}+t^2\right)^2}dt.
$$

Thus, inserting the estimate (2·13) into the previous identity gives

$$
K_q(T) = -\frac{T\phi(q)\log q}{2\pi(\frac{1}{4}+T^2)} + \frac{\phi(q)\log q}{\pi}\int_T^\infty \frac{t^2}{\left(\frac{1}{4}+t^2\right)^2}dt + E_1,
\tag{2·15}
$$

where

$$
E_1 \ll \frac{\phi(q)\log q}{T}\left(\frac{\log T}{\log q} + \frac{\log\log q}{\log q} + \frac{1}{T}\right).
$$

Moreover, the main term of (2·15) equals

$$
\frac{\phi(q)\log q}{2\pi T}\left(1 + O\left(\frac{1}{T}\right)\right),
$$

as desired.

Proof of Theorem 2. We first establish the upper bound. Taking $T = 0$ in (2·3) yields

$$P(Y(q) \geqslant V) \leqslant \exp\left(-\frac{V^2}{16\mathrm{Var}(q)}\right) \leqslant \exp\left(-\frac{V^2}{20\phi(q)\log q}\right), \qquad (2\cdot16)$$

if q is sufficiently large, since $\mathrm{Var}(q) \sim \phi(q)\log q$. Now, let $T = T(A)$ be a suitably large number such that $\log T \geqslant 200A$. Then Proposition 2·7 implies

$$S_q(T) \geqslant \frac{1}{20}\phi(q)\log q \log T \geqslant 8V,$$

since $V \leqslant A\phi(q)\log q$. On the other hand, using Proposition 2·7 we get

$$K_q(T) \geqslant \frac{\phi(q)\log q}{4\pi T},$$

if T is suitably large. Hence, applying (2·4) we derive

$$P(Y(q) \geqslant V) \geqslant \exp\left(-c_1(A)\frac{V^2}{\phi(q)\log q}\right), \qquad (2\cdot17)$$

for some positive number $c_1(A)$ which depends only on A. The theorem follows upon combining the bounds (2·16) and (2·17) with Lemma 2·1.

Proof of Theorem 3. The result can be deduced by proceeding along the same lines as in the proof of Theorem 2. Indeed, using Lemma 2·1, Theorem 2·2 and Proposition 2·7, the lower bound in Theorem 3 can be derived by taking $T = \exp(200V/(\phi(q)\log q))$. Similarly, to get the corresponding upper bound we choose $T = \exp(V/(20\phi(q)\log q))$.

3. *Approximating $\mu_{q;a_1,\dots,a_r}$ by a multivariate Gaussian distribution: Proof of Theorem* 1

Assuming GRH and LI, Rubinstein and Sarnak obtained an explicit formula for the Fourier transform of $\mu_{q;a_1,\dots,a_r}$ in terms of the non-trivial zeros of Dirichlet L-functions attached to non-principal characters modulo q. More specifically they showed that

$$\hat{\mu}_{q;a_1,\dots,a_r}(t_1,\dots,t_r) = \exp\left(i\sum_{j=1}^{r}C_q(a_j)t_j\right)\prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}}\prod_{\gamma_\chi > 0}J_0\left(\frac{2\left|\sum_{j=1}^{r}\chi(a_j)t_j\right|}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right). \qquad (3\cdot1)$$

First, we record an exponentially decreasing upper bound for $\hat{\mu}_{q;a_1,\dots,a_r}(\mathbf{t})$, which is established in [6].

LEMMA 3·1 ([6], proposition 3·2). *Assume GRH and LI. Fix an integer $r \geqslant 2$. Let q be large and $0 < \epsilon < 1/2$ be a real number. Then, uniformly for all r-tuples of distinct reduced residues $(a_1,\dots,a_r)$ modulo q we have*

$$|\hat{\mu}_{q;a_1,\dots,a_r}(t_1,\dots,t_r)| \leqslant \exp(-c_{11}(r)\phi(q)\|\mathbf{t}\|),$$

for $\|\mathbf{t}\| \geqslant 400$ and

$$|\hat{\mu}_{q;a_1,\dots,a_r}(t_1,\dots,t_r)| \leqslant \exp(-c_{12}(r)\epsilon^2\phi(q)\log q)$$

for $\epsilon \leqslant \|\mathbf{t}\| \leqslant 400$, where $c_{11}(r)$ and $c_{12}(r)$ are positive numbers that depend only on r.

Following the method developed by the author in [6], we shall derive an asymptotic formula for $\mu_{q;a_1,\dots,a_r}(\|\mathbf{x}\| > \lambda\sqrt{\mathrm{Var}(q)})$ in the range $0 < \lambda \leqslant \sqrt{\log\log q}$, from which Theorem 1 will be deduced.

THEOREM 3·2. *Assume GRH and LI. Fix an integer* $r \geqslant 2$. *Let* q *be large and* $a_1, \ldots, a_r$ *be distinct reduced residue classes modulo* q. *Then in the range* $0 < \lambda \leqslant \sqrt{\log \log q}$ *we have*

$$\mu_{q;a_1,\ldots,a_r}\left(||\mathbf{x}|| > \lambda\sqrt{\mathrm{Var}(q)}\right) = (2\pi)^{-r/2} \int_{||\mathbf{x}||>\lambda} \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x}$$
$$+ \frac{1}{2\mathrm{Var}(q)^2} \sum_{1 \leqslant j < k \leqslant r} B_q(a_j, a_k)^2 F_{j,k}(\lambda) + O_r\left(\frac{(\log\log q)^r}{\log^3 q}\right),$$

where

$$F_{j,k}(\lambda) = (2\pi)^{-r/2} \int_{||\mathbf{x}||>\lambda} \left(x_j^2 - 1\right)\left(x_k^2 - 1\right) \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x}.$$

Proof of Theorem 1. First, it follows from [6, corollary 5·4] that

$$\max_{(a,b)\in\mathcal{A}(q)} |B_q(a, b)| \ll \phi(q).$$

On the other hand, [6, proposition 5·1] yields

$$|B_q(a, -a)| \gg \phi(q),$$

for all $(a, q) = 1$. Hence, we deduce

$$\max_{(a,b)\in\mathcal{A}(q)} \frac{|B_q(a, b)|}{\mathrm{Var}(q)} \asymp \frac{1}{\log q}. \tag{3·2}$$

We also remark that this last estimate follows implicitly from the work of D. Fiorilli and G. Martin [2].

The first part of Theorem 1 follows from Theorem 3·2 upon using (3·2) and noting that $F_{j,k}(\lambda) \ll_r 1$. Let $1 \leqslant j < k \leqslant r$. If $1/4 < \lambda < 3/4$, then

$$(2\pi)^{-r/2} \int_{||\mathbf{x}||\leqslant\lambda} \left(x_j^2 - 1\right)\left(x_k^2 - 1\right) \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} \geqslant \delta_r,$$

for some positive number δ_r which depends only on r. Moreover, we have

$$(2\pi)^{-r/2} \int_{\mathbf{x}\in\mathbb{R}^r} \left(x_j^2 - 1\right)\left(x_k^2 - 1\right) \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} = 0,$$

since the antiderivative of $(x^2 - 1)e^{-x^2/2}$ is $-xe^{-x^2/2}$. Hence we deduce, in the range $1/4 < \lambda < 3/4$, that

$$F_{j,k}(\lambda) \leqslant -\delta_r, \text{ for all } 1 \leqslant j < k \leqslant r. \tag{3·3}$$

On the other hand, it follows from (3·2) that there exist distinct reduced residues a_1, a_2 modulo q, such that

$$\frac{B_q(a_1, a_2)^2}{\mathrm{Var}(q)^2} \gg \frac{1}{\log^2 q}. \tag{3·4}$$

Let $a_3, \ldots, a_r$ be distinct reduced residues modulo q that are different from a_1 and a_2.

Appealing to Theorem 3·2 along with (3·3) we get

$$\left| \mu_{q;a_1,\ldots,a_r}\left(||\mathbf{x}|| > \lambda\sqrt{\mathrm{Var}(q)}\right) - (2\pi)^{-r/2}\int_{||\mathbf{x}||>\lambda} \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right)d\mathbf{x}\right|$$

$$\geqslant |F_{1,2}(\lambda)|\frac{B_q(a_1,a_2)^2}{2\mathrm{Var}(q)^2} - \kappa_r\frac{(\log\log q)^r}{(\log q)^3}$$

for some $\kappa_r > 0$ (which depends only on r), if q is sufficiently large. Combining this inequality with (3·3) and (3·4) completes the proof.

The first step in the proof of Theorem 3·2 consists in using the explicit formula (3·1) to approximate the Fourier transform $\hat{\mu}_{q;a_1,\ldots,a_r}(t_1,\ldots,t_r)$ by a multivariate Gaussian in the range $||\mathbf{t}|| \ll \sqrt{\log q/\mathrm{Var}(q)}$. To lighten the notation, we set $\mu = \mu_{q;a_1,\ldots,a_r}$ throughout the remaining part of this section.

LEMMA 3·3. *Assume GRH and LI. Fix an integer $r \geqslant 2$ and a real number $A = A(r) > 0$. If $||\mathbf{t}|| \leqslant A\sqrt{\log q}$, then*

$$\hat{\mu}\left(\frac{t_1}{\sqrt{\mathrm{Var}(q)}},\ldots,\frac{t_r}{\sqrt{\mathrm{Var}(q)}}\right) = \exp\left(-\frac{t_1^2 + \cdots + t_r^2}{2}\right)\left(1 - \frac{1}{\mathrm{Var}(q)}\sum_{1\leqslant j<k\leqslant r} B_q(a_j,a_k)t_jt_k\right.$$

$$\left. + \frac{1}{2\mathrm{Var}(q)^2}\sum_{\substack{1\leqslant j_1<k_1\leqslant r \\ 1\leqslant j_2<k_2\leqslant r}} B_q(a_{j_1},a_{k_1})B_q(a_{j_2},a_{k_2})t_{j_1}t_{j_2}t_{k_1}t_{k_2} + O_r\left(\frac{||\mathbf{t}||^6}{(\log q)^3}\right)\right).$$

Proof. Using that $\mathrm{Var}(q) \sim \phi(q)\log q$ and $|C_q(a)| = q^{o(1)}$, we infer from (3·1) that

$$\log\hat{\mu}\left(\frac{t_1}{\sqrt{\mathrm{Var}(q)}},\ldots,\frac{t_r}{\sqrt{\mathrm{Var}(q)}}\right) = \sum_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}}\sum_{\gamma_\chi>0}\log J_0\left(\frac{2\left|\sum_{j=1}^r\chi(a_j)t_j\right|}{\sqrt{\frac{1}{4}+\gamma_\chi^2}\sqrt{\mathrm{Var}(q)}}\right) + O_r\left(\frac{||\mathbf{t}||}{\mathrm{Var}(q)^{1/4}}\right).$$

Moreover, since $\log J_0(s) = -s^2/4 + O(s^4)$ for $|s| \leqslant 1$, and $\sum_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}}\sum_{\gamma_\chi>0} 1/(\frac{1}{4}+\gamma_\chi^2)^2 \leqslant 2\mathrm{Var}(q)$, then

$$\log\hat{\mu}\left(\frac{t_1}{\sqrt{\mathrm{Var}(q)}},\ldots,\frac{t_r}{\sqrt{\mathrm{Var}(q)}}\right) = -\frac{1}{\mathrm{Var}(q)}\sum_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{\left|\sum_{j=1}^r\chi(a_j)t_j\right|^2}{\frac{1}{4}+\gamma_\chi^2} + O_r\left(\frac{||\mathbf{t}||}{\mathrm{Var}(q)^{1/4}}\right),$$

$$(3·5)$$

in the range $||\mathbf{t}|| \leqslant A\sqrt{\log q}$. Furthermore, the main term on the right-hand side of (3·5) equals

$$-\frac{1}{\mathrm{Var}(q)}\sum_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}}\sum_{\gamma_\chi>0}\frac{1}{\frac{1}{4}+\gamma_\chi^2}\sum_{1\leqslant j,k\leqslant r}\chi(a_j)\overline{\chi(a_k)}t_jt_k$$

$$(3·6)$$

$$= -\frac{t_1^2 + \cdots + t_r^2}{2} - \frac{1}{\mathrm{Var}(q)}\sum_{1\leqslant j<k\leqslant r}B_q(a_j,a_k)t_jt_k.$$

On the other hand, in our range of $\mathbf{t}$, we have

$$\exp\left(-\sum_{1\leqslant j<k\leqslant r}\frac{B_q(a_j,a_k)}{\mathrm{Var}(q)}t_jt_k\right)=1-\frac{1}{\mathrm{Var}(q)}\sum_{1\leqslant j<k\leqslant r}B_q(a_j,a_k)t_jt_k$$

$$+\frac{1}{2\mathrm{Var}(q)^2}\left(\sum_{1\leqslant j<k\leqslant r}B_q(a_j,a_k)t_jt_k\right)^2+O_r\left(\frac{||\mathbf{t}||^6}{\log^3 q}\right),$$

which follows from (3·2). Combining this estimate with (3·5) and (3·6) completes the proof.

Let $\Phi(x)=e^{-x^2/2}$ and denote by $\Phi^{(n)}$ the nth derivative of Φ. Then $\Phi^{(1)}(x)=-xe^{-x^2/2}$, $\Phi^{(2)}(x)=(x^2-1)e^{-x^2/2}$, and more generally we know that $\Phi^{(n)}(x)=(-1)^nH_n(x)e^{-x^2/2}$ where H_n is the nth Hermite polynomial. We record the following result, which corresponds to [6, lemma 4·2].

LEMMA 3·4. *Let $n_1,\ldots,n_r$ be fixed non-negative integers, and M be a large positive number. Then for any $(x_1,\ldots,x_r)\in\mathbb{R}^r$, we have*

$$\int_{||\mathbf{t}||<M}e^{i(t_1x_1+\cdots+t_rx_r)}\prod_{j=1}^r t_j^{n_j}e^{-t_j^2/2}d\mathbf{t}=(2\pi)^{r/2}\prod_{j=1}^r i^{n_j}H_{n_j}(x_j)e^{-x_j^2/2}+O(e^{-M^2/4}).$$

Our last ingredient to the proof of Theorem 3·2 is the following basic lemma.

LEMMA 3·5. *If $f_1,\ldots,f_r$ are real valued functions, such that f_i is odd for some $1\leqslant i\leqslant r$ then*

$$\int_{V_1<||x||<V_2}f_1(x_1)\cdots f_r(x_r)\exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right)d\mathbf{x}=0,$$

for all $V_2>V_1\geqslant 0$.

Proof. Using the change of variables $y_j=x_j$ if $j\neq i$ and $y_i=-x_i$, we deduce that the integral we seek to evaluate equals

$$-\int_{V_1<||y||<V_2}f_1(y_1)\cdots f_r(y_r)\exp\left(-\frac{y_1^2+\cdots+y_r^2}{2}\right)d\mathbf{y},$$

since f_i is odd, which establishes the lemma.

Proof of Theorem 3·2. Let $R:=\sqrt{\mathrm{Var}(q)}\log\log q$. First, appealing to Theorem 2 we get

$$\mu\big(||\mathbf{y}||>\lambda\sqrt{\mathrm{Var}(q)}\big)=\mu\big(\lambda\sqrt{\mathrm{Var}(q)}<||\mathbf{y}||<R\big)+O\big(\exp\big(-(\log\log q)^{3/2}\big)\big),\quad(3\cdot7)$$

if q is sufficiently large. Next, we apply the Fourier inversion formula to the measure μ, which yields

$$\mu\big(\lambda\sqrt{\mathrm{Var}(q)}<||\mathbf{y}||<R\big)=(2\pi)^{-r}\int_{\lambda\sqrt{\mathrm{Var}(q)}<||\mathbf{y}||<R}\int_{\mathbf{s}\in\mathbb{R}^r}e^{i(s_1y_1+\cdots+s_ry_r)}\hat{\mu}(s_1,\ldots,s_r)d\mathbf{s}\,d\mathbf{y}.$$

Let $A=A(r)\geqslant r$ be a suitably large constant. Then, using Lemma 3·1 with $\epsilon:=A(\mathrm{Var}(q))^{-1/2}\sqrt{\log q}$ we derive

$$\int_{\mathbf{s}\in\mathbb{R}^r}e^{i(s_1y_1+\cdots+s_ry_r)}\hat{\mu}(s_1,\ldots,s_r)d\mathbf{s}=\int_{||\mathbf{s}||\leqslant\epsilon}e^{i(s_1y_1+\cdots+s_ry_r)}\hat{\mu}(s_1,\ldots,s_r)d\mathbf{s}+O\left(\frac{1}{q^{2A}}\right).$$

Collecting the above estimates gives

$$\mu\big(||\mathbf{y}|| > \lambda\sqrt{\mathrm{Var}(q)}\big) = (2\pi)^{-r} \int_{\lambda\sqrt{\mathrm{Var}(q)} < ||\mathbf{y}|| < R} \int_{||\mathbf{s}|| \leqslant \epsilon} e^{i(s_1 y_1 + \cdots + s_r y_r)} \hat{\mu}(s_1, \ldots, s_r) d\mathbf{s}\, d\mathbf{y}$$
$$+ O\big(\exp\big(-(\log\log q)^{3/2}\big)\big),$$

using that $R^r q^{-2A} \ll q^{-r}$. Upon making the change of variables

$$t_j := \sqrt{\mathrm{Var}(q)}\, s_j, \text{ and } x_j := \frac{y_j}{\sqrt{\mathrm{Var}(q)}},$$

we infer that $\mu(||\mathbf{x}|| > \lambda\sqrt{\mathrm{Var}(q)})$ equals

$$(2\pi)^{-r} \int_{\lambda < ||\mathbf{x}|| < \log\log q} \int_{||\mathbf{t}|| \leqslant A\sqrt{\log q}} e^{i(t_1 x_1 + \cdots + t_r x_r)} \hat{\mu}\left(\frac{t_1}{\sqrt{\mathrm{Var}(q)}}, \ldots, \frac{t_r}{\sqrt{\mathrm{Var}(q)}}\right) d\mathbf{t}\, d\mathbf{x} \quad (3\cdot 8)$$
$$+ O\big(\exp\big(-(\log\log q)^{3/2}\big)\big),$$

Now we use the asymptotic expansion of $\hat{\mu}(t_1 \mathrm{Var}(q)^{-1/2}, \ldots, t_r \mathrm{Var}(q)^{-1/2})$ proved in Lemma 3·3. First, the contribution of the error term in this asymptotic to the integral in (3·8) is

$$\ll_r \frac{(\log\log q)^r}{(\log q)^3}.$$

Next we shall compute the contribution of the main terms. Appealing to Lemma 3·4, we obtain

$$(2\pi)^{-r} \int_{\lambda < ||\mathbf{x}|| < \log\log q} \int_{||\mathbf{t}|| \leqslant A\sqrt{\log q}} e^{i(t_1 x_1 + \cdots + t_r x_r)} \exp\left(-\frac{t_1^2 + \cdots + t_r^2}{2}\right) d\mathbf{t} d\mathbf{x}$$
$$= (2\pi)^{-r/2} \int_{\lambda < ||\mathbf{x}|| < \log\log q} \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} + O\left(\frac{1}{q^A}\right)$$
$$= (2\pi)^{-r/2} \int_{||\mathbf{x}|| > \lambda} \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} + O(\exp(-(\log\log q)^{3/2})),$$

if q is large enough. Similarly, we deduce from Lemma 3·4 that for $1 \leqslant j < k \leqslant r$ we have

$$(2\pi)^{-r} \int_{\lambda < ||\mathbf{x}|| < \log\log q} \int_{||\mathbf{t}|| \leqslant A\sqrt{\log q}} t_j t_k e^{i(t_1 x_1 + \cdots + t_r x_r)} \exp\left(-\frac{t_1^2 + \cdots + t_r^2}{2}\right) d\mathbf{t} d\mathbf{x}$$
$$= -(2\pi)^{-r/2} \int_{\lambda < ||\mathbf{x}|| < \log\log q} x_j x_k \exp\left(-\frac{x_1^2 + \cdots + x_r^2}{2}\right) d\mathbf{x} + O\left(\frac{1}{q^A}\right) = O\left(\frac{1}{q^A}\right),$$

which follows from Lemma 3·5. Furthermore, if $1 \leqslant j_1 < k_1 \leqslant r$ and $1 \leqslant j_2 < k_2 \leqslant r$, then a similar argument along with Lemmas 3·4 and 3·5 shows that

$$(2\pi)^{-r} \int_{\lambda < ||\mathbf{x}|| < \log\log q} \int_{||\mathbf{t}|| \leqslant A\sqrt{\log q}} t_{j_1} t_{k_1} t_{j_2} t_{k_2} e^{i(t_1 x_1 + \cdots + t_r x_r)} \exp\left(-\frac{t_1^2 + \cdots + t_r^2}{2}\right) d\mathbf{t} d\mathbf{x}$$

is $\ll q^{-A}$ if $j_1 \neq j_2$ or $k_1 \neq k_2$, and equals $F_{j,k}(\lambda) + O(\exp(-(\log\log q)^{3/2}))$, if $j_1 = j_2 = j$ and $k_1 = k_2 = k$. The theorem now follows upon collecting the above estimates and using Lemma 3·3.

4. *Very large deviations of $\mu_{q;a_1,\ldots,a_r}$: proof of Theorem* 4

In this section we prove a precise estimate for $\mu_{q;a_1,\ldots,a_r}(|x|_\infty > V)$ in the range $V/(\phi(q)\log^2 q) \to \infty$ as $q \to \infty$. The advantage of using the maximum norm is that the bounds in Lemma 2·1 can be made sharp, so that a precise estimate for large deviations of $\mu_{q;a_1,\ldots,a_r}$ would follow from a close investigation of the tails of $Y(q)$. Indeed, we have:

LEMMA 4·1. *Let $\epsilon > 0$ be small. If q is sufficiently large, $2 \leqslant r \leqslant \phi(q) - 1$ and $V \geqslant \sqrt{\phi(q)}$, then for all distinct reduced residues $a_1, \ldots, a_r$ modulo q, we have*

$$\mathrm{P}\,(Y(q) > V + q^\epsilon) \leqslant \mu_{q;a_1,\ldots,a_r}(|x|_\infty > V) \leqslant 2r\mathrm{P}\,(Y(q) > V - q^\epsilon).$$

Proof. The result can be derived along the same lines as in the proof of Lemma 2·1 upon noting that

$$\mu_{q;a_1,\ldots,a_r}(|x|_\infty > V) \leqslant \sum_{j=1}^r \mathrm{P}(|X(q, a_j)| > V) \leqslant \sum_{j=1}^r \mathrm{P}(|Y(q)| > V - |C_q(a_j)|),$$

and $|C_q(a)| = q^{o(1)}$ for all $(a, q) = 1$.

In view of this lemma, it suffices to obtain the analogous estimate for

$$\rho_q(V) := \mathrm{P}(Y(q) > V).$$

To this end we shall use the identity

$$\int_{-\infty}^\infty se^{sV}\rho_q(V)dV = \mathcal{L}(s) = \prod_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \prod_{\gamma_\chi > 0} I_0\left(\frac{2s}{\sqrt{\frac{1}{4} + \gamma_\chi^2}}\right), \quad \text{for all } s > 0. \qquad (4\cdot1)$$

To prove Theorem 4, we shall establish an asymptotic formula for $\log \mathcal{L}(s)$ in a wide range of s, and then use the saddle-point method to extract the desired estimate for $\rho_q(V)$ from that of $\mathcal{L}(s)$. We first collect some useful estimates and properties of the function $\log I_0(t)$.

LEMMA 4·2. $\log I_0(t)$ *is a smooth function with bounded derivative on $[0, +\infty)$ and satisfies*

$$\log I_0(t) = \begin{cases} O(t^2) & \text{if } 0 \leqslant t < 1 \\ t + O\,(\log 3t) & \text{if } t \geqslant 1. \end{cases}$$

Proof. The first estimate follows from the Taylor series $I_0(t) = \sum_{n=0}^\infty (t/2)^{2n}/n!^2$. On the other hand, taking $\epsilon = 1/2\pi t$ we obtain that

$$I_0(t) = \int_0^1 e^{t\cos(2\pi\theta)}d\theta \geqslant \int_0^\epsilon e^{t\cos(2\pi\theta)}d\theta \geqslant \epsilon e^{t\cos(2\pi\epsilon)} \geqslant \frac{e^t}{10\pi t}.$$

This together with the bound $I_0(t) \leqslant e^t$ yields the second estimate. Finally, since $I_0(t)$ is a positive smooth function on $[0, +\infty)$ then $\log I_0(t)$ is smooth and

$$|(\log I_0(t))'| = \left|\frac{\int_0^1 \cos(2\pi\theta)e^{t\cos(2\pi\theta)}d\theta}{\int_0^1 e^{t\cos(2\pi\theta)}d\theta}\right| \leqslant 1.$$

Let $f(t)$ be the real valued function defined by

$$f(t) = \begin{cases} \log I_0(t) & \text{if } 0 \leqslant t < 1 \\ \log I_0(t) - t & \text{if } t \geqslant 1. \end{cases} \tag{4.2}$$

We prove:

PROPOSITION 4·3. *If* $\log s / \log q \to \infty$ *as* $q \to \infty$, *then*

$$\mathcal{L}(s) = \exp\left(\frac{\phi(q)-1}{2\pi} s \log^2 s + \frac{D(q)}{\pi} s \log s + O(s\phi(q)\log q)\right),$$

where

$$D(q) = \phi(q)\left(\log q - \sum_{p|q} \frac{\log p}{p-1}\right) + (\phi(q)-1)\left(\int_0^\infty \frac{f(t)}{t^2} dt - \log \pi\right).$$

Proof. First, it follows from Lemma 4·2 that $\log I_0$ is a Lipschitz function. Therefore, using (4·1) and Lemma 2·6 we get

$$\log \mathcal{L}(s) = \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \log I_0\left(\frac{2s}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right) = \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 2} \log I_0\left(\frac{2s}{\gamma_\chi}\right) + E_2, \tag{4.3}$$

where

$$E_2 \ll s \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{0 < \gamma_\chi < 2} \frac{1}{\sqrt{\frac{1}{4}+\gamma_\chi^2}} + s \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 2} \left(\frac{1}{\gamma_\chi} - \frac{1}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right)$$

$$\ll s N_q(2) + s \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 2} \frac{1}{\frac{1}{4}+\gamma_\chi^2} \ll s\phi(q)\log q.$$

Furthermore, note that

$$\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 2} \log I_0\left(\frac{2s}{\gamma_\chi}\right) = 2s \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{2 < \gamma_\chi \leqslant 2s} \frac{1}{\gamma_\chi} + \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 2} f\left(\frac{2s}{\gamma_\chi}\right). \tag{4.4}$$

Now, we infer from Lemma 2·6 that

$$\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{2 < \gamma_\chi \leqslant 2s} \frac{1}{\gamma_\chi} = \int_2^{2s} \frac{dN_q(t)}{t} = \frac{N_q(2s)}{2s} + \int_2^{2s} \frac{N_q(t)}{t^2} dt + O(\phi(q)\log q)$$

$$= \frac{(\phi(q)-1)}{2\pi} \log s + \frac{1}{2\pi} \int_2^{2s} \left((\phi(q)-1)\frac{\log t}{t} + \frac{R(q)}{t}\right) dt + O(\phi(q)\log q)$$

$$= \frac{(\phi(q)-1)}{4\pi} \log^2 s + \frac{1}{2\pi}(R(q) + (\phi(q)-1)(1+\log 2))\log s + O(\phi(q)\log q).$$

On the other hand, the second sum on the right-hand side of (4·4) equals

$$\int_2^\infty f\left(\frac{2s}{t}\right) dN_q(t) = -\int_2^\infty f\left(\frac{2s}{t}\right)'\left(\frac{\phi(q)-1}{2\pi} t \log t + \frac{R(q)}{2\pi} t\right) dt + E_3, \tag{4.5}$$

where

$$E_3 \ll |f(s)N_q(2)| + \phi(q)\int_2^\infty \frac{s}{t^2}\left|f'\left(\frac{2s}{t}\right)\right|\log(qt)dt \ll s\phi(q)\log q,$$

which follows from Lemmas 2·6 and 4·2 along with the fact that $\lim_{t\to\infty} f(2s/t)N_q(t) = 0$. This yields

$$\int_2^\infty f\left(\frac{2s}{t}\right)dN_q(t) = \int_2^\infty f\left(\frac{2s}{t}\right)\left(\frac{\phi(q)-1}{2\pi}\log t + \frac{R(q)+\phi(q)-1}{2\pi}\right)dt$$
$$+ O(s\phi(q)\log q).$$

Making the change of variable $u = 2s/t$, we deduce that the integral on the right-hand side of the previous estimate equals

$$2s\int_0^s \frac{f(u)}{u^2}\left(\frac{\phi(q)-1}{2\pi}\log\left(\frac{2s}{u}\right) + \frac{R(q)+\phi(q)-1}{2\pi}\right)du$$
$$= \frac{(\phi(q)-1)}{\pi}s\log s\int_0^s\frac{f(u)}{u^2}du + O(s\phi(q)\log q)$$
$$= \frac{(\phi(q)-1)}{\pi}s\log s\int_0^\infty\frac{f(u)}{u^2}du + O(s\phi(q)\log q),$$

using that $R(q) \sim \phi(q)\log q$ by Lemma 2·5 and

$$\int_0^\infty \frac{f(u)\log u}{u^2}du < \infty \quad\text{and}\quad \int_s^\infty \frac{f(u)}{u^2}du \ll \int_s^\infty \frac{\log u}{u^2}du \ll \frac{\log s}{s},$$

which follow from Lemma 4·2. The proposition follows upon collecting the above estimates.

Proof of Theorem 4. Assume that V is such that $V/(\phi(q)\log^2 q) \to \infty$ as $q \to \infty$, and let $s = s(V) > 1$ be the unique solution to the equation

$$2\pi V = (\phi(q)-1)\log^2 s + 2(D(q)+\phi(q)-1)\log s. \tag{4·6}$$

Then

$$s = \exp\left(-\frac{D(q)}{\phi(q)-1} - 1 + \sqrt{\left(\frac{D(q)}{\phi(q)-1}+1\right)^2 + \frac{2\pi V}{\phi(q)-1}}\right). \tag{4·7}$$

Let $\epsilon, \delta > 0$ be small real numbers to be chosen later and define

$$s_1 := (1+\epsilon)s, \; s_2 = (1-\epsilon)s, \;\text{and}\; V_1 := (1+\delta)V, \; V_2 := (1-\delta)V.$$

Moreover, let

$$I_1 := \int_{V_1}^\infty se^{st}\rho_q(t)dt, \;\text{and}\; I_2 := \int_{-\infty}^{V_2} se^{st}\rho_q(t)dt.$$

Then

$$\frac{I_1}{\mathcal{L}(s)} = \frac{1}{(1+\epsilon)\mathcal{L}(s)}\int_{V_1}^\infty e^{-\epsilon st}s_1 e^{s_1 t}\rho_q(t)dt \leqslant e^{-\epsilon sV_1}\frac{\mathcal{L}(s_1)}{(1+\epsilon)\mathcal{L}(s)}. \tag{4·8}$$

Therefore, appealing to Proposition 4·3 we obtain that the right-hand side of the previous

inequality equals

$$\exp\left(-\epsilon s V_1 + \frac{\phi(q)-1}{2\pi}(s_1\log^2 s_1 - s\log^2 s) + \frac{D(q)}{\pi}(s_1\log s_1 - s\log s)\right.$$
$$\left. + O(s\phi(q)\log q)\right),$$

which in view of (4·6) becomes

$$\exp\left(-\epsilon\delta s V + \frac{\phi(q)-1}{\pi}s\log s((1+\epsilon)\log(1+\epsilon) - \epsilon) + O(s\phi(q)\log q)\right).$$

Combining this estimate with (4·8) and using that $(1+\epsilon)\log(1+\epsilon) - \epsilon \leqslant \epsilon^2$, we deduce

$$\frac{I_1}{\mathcal{L}(s)} \leqslant \exp\left(-\epsilon\delta s V + \epsilon^2\frac{\phi(q)-1}{\pi}s\log s + O(s\phi(q)\log q)\right).$$

Now, (4·6) yields $2\pi V \geqslant (\phi(q)-1)\log^2 s$. Therefore, choosing $\delta = 4\epsilon/\log s$ we get

$$\frac{I_1}{\mathcal{L}(s)} \leqslant \exp\left(-\frac{\epsilon^2}{2\pi}(\phi(q)-1)s\log s + O(s\phi(q)\log q)\right).$$

Furthermore, taking $\epsilon = C\sqrt{\log q/\log s}$ for a suitably large constant $C > 0$, we obtain

$$\frac{I_1}{\mathcal{L}(s)} \leqslant \exp\left(-s\phi(q)\log q\right).$$

A similar argument shows that

$$\frac{I_2}{\mathcal{L}(s)} \leqslant \exp\left(-s\phi(q)\log q\right).$$

Combining these two inequalities with Proposition 4·3 we deduce

$$\int_{V(1-\delta)}^{V(1+\delta)} se^{st}\rho_q(t)dt = \exp\left(\frac{\phi(q)-1}{2\pi}s\log^2 s + \frac{D(q)}{\pi}s\log s + O(s\phi(q)\log q)\right).$$

On the other hand, since $V \ll \phi(q)\log^2 s$ then

$$\int_{V(1-\delta)}^{V(1+\delta)} se^{st}dt = \exp\left(sV + O\left(s\phi(q)\sqrt{\log s\log q}\right)\right).$$

Hence, using that ρ_q is a non-increasing function we infer from the two previous estimates that

$$\rho_q(V(1+\delta)) \leqslant \exp\left(-\frac{\phi(q)-1}{\pi}s\log s + O\left(s\phi(q)\sqrt{\log s\log q}\right)\right) \leqslant \rho_q(V(1-\delta)). \tag{4·9}$$

Now, using that $D(q) \sim \phi(q)\log q$ by Lemma 2·5 we get

$$\log s = \sqrt{\frac{2\pi V}{\phi(q)-1}}\left(1 + O\left(\left(\frac{\phi(q)\log^2 q}{V}\right)^{1/2}\right)\right), \tag{4·10}$$

and

$$\sqrt{\frac{\log q}{\log s}} \ll \left(\frac{\phi(q)\log^2 q}{V}\right)^{1/4}. \tag{4·11}$$

Moreover, we have

$$
\exp\left(-\frac{D(q)}{\phi(q)-1}-1+\sqrt{\left(\frac{D(q)}{\phi(q)-1}+1\right)^2+\frac{2\pi V(1+O(\delta))}{\phi(q)-1}}\right)
$$

$$
=\exp\left(-\frac{D(q)}{\phi(q)-1}-1+\sqrt{\left(\frac{D(q)}{\phi(q)-1}+1\right)^2+\frac{2\pi V}{\phi(q)-1}}\right) \qquad (4\cdot12)
$$

$$
\times\left(1+O\left(\left(\frac{\phi(q)\log^2 q}{V}\right)^{1/4}\right)\right).
$$

Finally, the theorem follows upon using (4·6) and inserting the estimates (4·10)-(4·12) into (4·9).

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