THE SHANKS–RÉNYI PRIME NUMBER RACE WITH MANY CONTESTANTS

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Abstract. Under certain plausible assumptions, M. Rubinstein and P. Sarnak solved the Shanks–Rényi race problem by showing that the set of real numbers \( x \geq 2 \) such that \( \pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r) \) has a positive logarithmic density \( \delta_{q,a_1,\ldots,a_r} \). Furthermore, they established that if \( r \) is fixed, \( \delta_{q,a_1,\ldots,a_r} \to 1/r! \) as \( q \to \infty \). In this paper, we investigate the size of these densities when the number of contestants \( r \) tends to infinity with \( q \). In particular, we deduce a strong form of a recent conjecture of Feuerverger and Martin which states that \( \delta_{q,a_1,\ldots,a_r} = o(1) \) in this case. Among our results, we prove that \( \delta_{q,a_1,\ldots,a_r} \sim 1/r! \) in the region \( r = o(\sqrt{\log q}) \) as \( q \to \infty \). We also bound the order of magnitude of these densities beyond this range of \( r \). For example, we show that when \( \log q \leq r \leq \phi(q) \), \( \delta_{q,a_1,\ldots,a_r} \ll q^{-1+\epsilon} \).

1. Introduction

A classical problem in analytic number theory is the so-called “Shanks–Rényi prime number race” which concerns the distribution of prime numbers in arithmetic progressions. As colorfully described by Knapowski and Turán in [11], let \( q \geq 3 \) and \( 2 \leq r \leq \phi(q) \) be positive integers, and denote by \( A_r(q) \) the set of ordered \( r \)-tuples of distinct residue classes \((a_1, a_2, \ldots, a_r)\) modulo \( q \) which are coprime to \( q \). For \((a_1, a_2, \ldots, a_r) \in A_r(q)\) consider a game with \( r \) players called “1” through “\( r \)”, where at time \( x \), the player “\( j \)” has a score of \( \pi(x; q, a_j) \) (where \( \pi(x; q, a) \) denotes the number of primes \( p \leq x \) with \( p \equiv a \mod q \)). As \( x \to \infty \), will all \( r! \) orderings of the players occur for infinitely many integers \( x \)?

It is generally believed that the answer to this question is yes for all \( q \) and all \((a_1, a_2, \ldots, a_r) \in A_r(q)\). An old result of Littlewood [14] shows that this is indeed true in the special cases \((q, a_1, a_2) = (4, 1, 3)\) and \((q, a_1, a_2) = (3, 1, 2)\). Since then, this problem has been extensively studied by many authors, including Knapowski and Turán [11], Bays and Hudson [1, 2], Kaczorowski [8–10], Feuerverger and Martin [4], Martin [15], Ford and Konyagin [6, 7], Fiorilli and Martin [5], and Lamzouri [12] and [13].

A major breakthrough was made in 1994 by Rubinstein and Sarnak who completely solved this problem in [16], conditionally on the two following assumptions:

- The Generalized Riemann Hypothesis (GRH): all nontrivial zeros of Dirichlet \( L \)-functions have real part equal \( 1/2 \).
The Linear Independence Hypothesis (LI) (also known as the Grand Simplicity Hypothesis): the nonnegative imaginary parts of the nontrivial zeros of Dirichlet $L$-functions attached to primitive characters are linearly independent over $\mathbb{Q}$.

Rubinstein and Sarnak [16] proved, under these two hypotheses, the stronger result that for any $(a_1, \ldots, a_r) \in A_r(q)$ the set of real numbers $x \geq 2$ such that

$$\pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r),$$

has a positive logarithmic density, which shall be denoted throughout this paper by $\delta_{q;a_1,\ldots,a_r}$. (Recall that the logarithmic density of a subset $S$ of $\mathbb{R}$ is defined as

$$\delta_S := \lim_{x \to \infty} \frac{1}{\log x} \int_{t \in S \cap [2, x]} \frac{dt}{t},$$

provided that this limit exists.) To establish this result, they constructed an absolutely continuous measure $\mu_{q;a_1,\ldots,a_r}$ for which

$$\delta_{q;a_1,\ldots,a_r} = \int_{x_1 > x_2 > \cdots > x_r} d\mu_{q;a_1,\ldots,a_r}(x_1, \ldots, x_r).$$

Among the results they derived on these densities, Rubinstein and Sarnak [16] showed that in an $r$-way race with $r$ fixed, all biases disappear when $q \to \infty$. More specifically they proved

$$\lim_{q \to \infty} \max_{(a_1, \ldots, a_r) \in A_r(q)} |r! \delta_{q;a_1,\ldots,a_r} - 1| = 0. \tag{1.2}$$

Recently, Fiorilli and Martin [5] established an asymptotic formula for the density in a two-way race, which allows them to determine the exact rate at which $\delta_{q;a_1,\ldots,a_r}$ converges to $1/2$ as $q$ grows. Shortly after, the author [12] obtained an asymptotic formula for $\delta_{q;a_1,\ldots,a_r}$ for any fixed $r \geq 3$ as $q \to \infty$, in which the rate of convergence to $1/r!$ is surprisingly different from the case $r = 2$.

However, as far as the author of the present paper knows, no results have been obtained on the size of the densities $\delta_{q;a_1,\ldots,a_r}$ if $r \to \infty$ as $q \to \infty$. In [4], Feuerverger and Martin conjectured that in this case we should have $\delta_{q;a_1,\ldots,a_r} = o(1)$. They also asked whether one can prove a uniform version of the result of Rubinstein and Sarnak (1.2), namely that this statement holds in a certain range $r \leq r_0(q)$ for some $r_0(q) \to \infty$ as $q \to \infty$.

Conjecture 1.1 (Feuerverger–Martin). We have

$$\lim_{q \to \infty} \max_{(a_1, \ldots, a_r) \in A_r(q)} \delta_{q;a_1,\ldots,a_r} = 0,$$

for any arbitrary function $r = r(q)$ tending to infinity with $q$.

In the present paper, we investigate the order of magnitude of $\delta_{q;a_1,\ldots,a_r}$ when the number of contestants $r \to \infty$ as $q \to \infty$. In particular, answering the question of Feuerverger and Martin, we establish a uniform version of (1.2), and obtain a strong quantitative form of Conjecture 1.1.
Theorem 1.1. Assume GRH and LI. Let \( q \) be a large positive integer. Then, for any integer \( r \) such that \( 2 \leq r \leq \sqrt{\log q} \) we have

\[
\delta_{q,a_1,\ldots,a_r} = \frac{1}{r!} \left( 1 + O \left( \frac{r^2}{\log q} \right) \right),
\]

uniformly for all \( r \)-tuples \( (a_1, \ldots, a_r) \in A_r(q) \).

As a consequence, Theorem 1.1 implies that (1.2) holds true in the range \( r = o(\sqrt{\log q}) \) as \( q \to \infty \). Indeed in this region of \( r \), all biases disappear when \( q \to \infty \), namely

\[
\delta_{q,a_1,\ldots,a_r} \sim \frac{1}{r!},
\]

uniformly for all \( r \)-tuples \( (a_1, \ldots, a_r) \in A_r(q) \). Moreover, one can also deduce that if \( c_0 > 0 \) is a suitably small constant and \( r \leq c_0 \sqrt{\log q} \), then uniformly for all \( r \)-tuples \( (a_1, \ldots, a_r) \in A_r(q) \) we have

\[
\delta_{q,a_1,\ldots,a_r} \approx \frac{1}{r!}.
\]

Note that \( 1/r! = \exp(-r \log r + r + O(\log r)) \) by Stirling’s formula. Our next result shows that the densities \( \delta_{q,a_1,\ldots,a_r} \) have roughly the same asymptotic decay in the range \( \sqrt{\log q} \ll r \leq (1 - \epsilon) \log q / \log \log q \), for any \( \epsilon > 0 \).

Theorem 1.2. Assume GRH and LI. For any \( \epsilon > 0 \), if \( q \) is large and \( \sqrt{\log q} \ll r \leq (1 - \epsilon) \log q / \log \log q \) is an integer, then

\[
\delta_{q,a_1,\ldots,a_r} = \exp \left( -r \log r + r + O \left( \log r + \frac{r^2}{\log q} \right) \right),
\]

uniformly for all \( r \)-tuples \( (a_1, \ldots, a_r) \in A_r(q) \).

It would be interesting to determine the order of magnitude of the densities \( \delta_{q,a_1,\ldots,a_r} \) beyond the region \( r \leq (1 - \epsilon) \log q / \log \log q \). Unfortunately, this range seems to be the limit of what can be achieved using our method. Nevertheless, we can use Theorem 1.2 to obtain an upper bound for \( \delta_{q,a_1,\ldots,a_r} \) beyond this range of \( r \).

Theorem 1.3. Assume GRH and LI. For any \( \epsilon > 0 \), if \( q \) is large and \( r \) is a positive integer such that \( (1 - \epsilon/2) \log q / \log \log q \leq r \leq \phi(q) \), then

\[
\max_{(a_1, \ldots, a_r) \in A_r(q)} \delta_{q,a_1,\ldots,a_r} \ll \epsilon \frac{1}{q^{1-\epsilon}}.
\]

The paper is organized as follows. In Section 2, following the work of Rubinstein and Sarnak, we shall construct the measure \( \mu_{q,a_1,\ldots,a_r} \) as a probability distribution corresponding to a certain random vector and study its covariance matrix and large deviations. In Section 3, we investigate the Fourier transform of \( \mu_{q,a_1,\ldots,a_r} \) and show that in a certain range \( \hat{\mu}_{q,a_1,\ldots,a_r} \) can be approximated by the Fourier transform of a multivariate normal distribution having the same covariance matrix. In Section 4 we study properties of multivariate normal distributions and prove Theorems 1.1, 1.2 and 1.3.
2. The measure $\mu_{q; a_1, \ldots, a_r}$

We begin by developing the necessary notation to construct the measure $\mu_{q; a_1, \ldots, a_r}$, following the work of Rubinstein and Sarnak [16]. For $(a_1, a_2, \ldots, a_r) \in A_r(q)$ we introduce the vector-valued function

$$E_{q; a_1, \ldots, a_r}(x) := (E(x; q, a_1), \ldots, E(x; q, a_r)),$$

where

$$E(x; q, a) := \frac{\log x}{\sqrt{x}} (\phi(q)\pi(x; q, a) - \pi(x)).$$

The normalization is such that, if we assume GRH, $E_{q; a_1, \ldots, a_r}(x)$ varies roughly boundedly as $x$ varies. Moreover, for a nontrivial character $\chi$ modulo $q$, we denote by $\{\gamma_\chi\}$ the sequence of imaginary parts of the nontrivial zeros of $L(s, \chi)$. Let $\chi_0$ denote the principal character modulo $q$ and define $S = \cup_{\chi \neq \chi_0 \text{ mod } q} \{\gamma_\chi\}$. Furthermore, let $\{U(\gamma_\chi)\}_{\gamma_\chi \in S}$ be a sequence of independent random variables uniformly distributed on the unit circle.

Rubinstein and Sarnak established, under GRH and LI, that the vector-valued function $E_{q; a_1, \ldots, a_r}$ has a limiting distribution $\mu_{q; a_1, \ldots, a_r}$, where $\mu_{q; a_1, \ldots, a_r}$ is the probability measure corresponding to the random vector

$$X_{q; a_1, \ldots, a_r} = (X(q, a_1), \ldots, X(q, a_r)),$$

where

$$X(q, a) = -C_q(a) + \sum_{\chi \neq \chi_0 \text{ mod } q} \sum_{\gamma_\chi > 0} \frac{2\text{Re}(\chi(a)U(\gamma_\chi))}{\sqrt{\frac{1}{4} + \gamma_\chi^2}},$$

and

$$C_q(a) := -1 + \sum_{b^2 \equiv a \text{ mod } q} 1.$$ 

Note that for $(a, q) = 1$ the function $C_q(a)$ takes only two values: $C_q(a) = -1$ if $a$ is a non-square modulo $q$, and $C_q(a) = C_q(1)$ if $a$ is a square modulo $q$. Furthermore, an elementary argument shows that $C_q(a) < d(q) \ll q^\epsilon$ for any $\epsilon > 0$, where $d(q) = \sum_{m|q} 1$ is the usual divisor function.

To investigate the distribution of the random vector $X_{q; a_1, \ldots, a_r}$, we shall first compute its covariance matrix $\text{Cov}_{q; a_1, \ldots, a_r}$ (the covariance matrix generalizes the notion of variance to multiple dimensions). Recall that the $j, k$ entry of the covariance matrix corresponds to the covariance between the $j$-th and $k$-th entry of the random vector.

**Lemma 2.1.** The entries of $\text{Cov}_{q; a_1, \ldots, a_r}$ are

$$\text{Cov}_{q; a_1, \ldots, a_r}(j, k) = \begin{cases} \text{Var}(q) & \text{if } j = k \\ B_q(a_j, a_k) & \text{if } j \neq k, \end{cases}$$

where

$$\text{Var}(q) := 2 \sum_{\chi \neq \chi_0 \text{ mod } q} \sum_{\gamma_\chi > 0} \frac{1}{\frac{1}{4} + \gamma_\chi^2},$$

and

$$B_q(a, b) := \sum_{\chi \neq \chi_0 \text{ mod } q} \sum_{\gamma_\chi > 0} \frac{\chi(b) + \chi(b \frac{a}{b})}{\frac{1}{4} + \gamma_\chi^2}.$$
Proof. First, note that $E(X(q,a)) = -C_q(a)$ since $E(U(\gamma_\chi)) = 0$ for all $\gamma_\chi$. Therefore, $\text{Cov}_{q,a_1,...,a_r}(j,k)$ equals

$$E\left( (X(q,a_j) + C_q(a_j))(X(q,a_k) + C_q(a_k)) \right)$$

$$= \sum_{\chi \neq \chi_0} \sum_{\gamma_\chi > 0} \sum_{\psi \neq \chi_0} \sum_{\gamma_\psi > 0} \frac{(\chi(a_j)U(\gamma_\chi) + \chi(a_j)U(\gamma_\psi))}{\sqrt{\frac{1}{4} + \gamma_\chi^2}} \frac{(\psi(a_k)U(\gamma_\psi) + \psi(a_k)U(\gamma_\psi))}{\sqrt{\frac{1}{4} + \gamma_\psi^2}}.$$

Since $E(U(\gamma_\chi)U(\gamma_\psi)) = 0$ for all $\gamma_\chi, \gamma_\psi$ and

$$E\left( U(\gamma_\chi)U(\gamma_\psi) \right) = \begin{cases} 1 & \text{if } \chi = \psi \text{ and } \gamma_\chi = \gamma_\psi, \\ 0 & \text{otherwise}, \end{cases}$$

we deduce that

$$\text{Cov}_{q,a_1,...,a_r}(j,k) = \sum_{\chi \neq \chi_0} \sum_{\gamma_\chi > 0} \chi(a_j/a_k) + \chi(a_k/a_j),$$

which implies the result. \qed

Our next lemma gives the asymptotic behavior of $\text{Var}(q)$ along with the maximal order of $B_q(a_j,a_k)$. This was established in [12], and we should also note that it follows implicitly from the results of [5].

**Lemma 2.2.** Assume GRH. Then

(2.1) $\text{Var}(q) = \phi(q) \log q + O(\phi(q) \log \log q)$,

and

(2.2) $\max_{(a,b) \in A_2(q)} B_q(a,b) \asymp \phi(q)$.

Proof. First, the asymptotic formula (2.1) is proved in Lemma 3.1 of [12]. Now, the fact that $B_q(a_j,a_k) \ll \phi(q)$ is proved in Corollary 5.4 of [12], while Proposition 5.1 of [12] implies $B_q(a,-a) \gg \phi(q)$. \qed

Here and throughout we shall use the notations $\|t\| = \sqrt{\sum_{j=1}^r t_j^2}$ and $|t|_\infty = \max_{1 \leq j \leq r} |t_j|$ for the Euclidean norm and the maximum norm of $t \in \mathbb{R}^r$, respectively. Our next result is an upper bound for the tail of the distribution $\mu_{q,a_1,...,a_r}$. This was established in Proposition 4.1 of [12] in the case where $r$ is fixed.

**Lemma 2.3.** Let $q$ be large and $2 \leq r \leq \phi(q)$ be a positive integer. Then for $R \geq \sqrt{\phi(q) \log q}$ we have

$$\mu_{q,a_1,...,a_r}(|x|_\infty > R) \leq 2r \exp \left( -\frac{R^2}{4\phi(q) \log q} \right),$$

uniformly for all $(a_1, \ldots, a_r) \in A_r(q)$. 

Proof. First, we have
\[
\mu_{q;a_1,\ldots,a_r}(|x|_\infty > R) = P(|X_{q;a_1,\ldots,a_r}|_\infty > R) \\
\leq \sum_{j=1}^{r} P(X(q,a_j) > R) + \sum_{j=1}^{r} P(X(q,a_j) < -R).
\]
We shall bound only \(P(X(q,a_j) > R)\), since the corresponding bound for \(P(X(q,a_j) < -R)\) can be obtained similarly. Let \(s > 0\) and \((a,q) = 1\). Then we have
\[
\mathbb{E}(e^{sX(q,a)}) = e^{-sC_q(a)} \prod_{\chi \neq \chi_0} \prod_{\gamma_X > 0} \mathbb{E}\left(\frac{2s\text{Re}(\chi(a)U(\gamma_X))}{\sqrt{1 + \gamma_X^2}}\right) \\
= e^{-sC_q(a)} \prod_{\chi \neq \chi_0} \prod_{\gamma_X > 0} I_0\left(\frac{2s}{\sqrt{1 + \gamma_X^2}}\right),
\]
where \(I_0(t) := \sum_{n=0}^{\infty} (t/2)^{2n}/n!^2\) is the modified Bessel function of order 0. Hence, using the Chernoff bound along with the fact that \(I_0(s) \leq \exp(s^2/4)\) for all \(s \in \mathbb{R}\) we derive
\[
P(X(q,a) > R) \leq e^{-sR} \mathbb{E}(e^{sX(q,a)}) \leq \exp\left(-sR - sC_q(a) + \frac{s^2}{2} \text{Var}(q)\right).
\]
The lemma follows upon choosing \(s = R/(\phi(q) \log q)\), since \(C_q(a) = q^{\sigma(1)}\) and \(\text{Var}(q) \sim \phi(q) \log q\) by Lemma 2.2. \(\square\)

3. The Fourier transform \(\hat{\mu}_{q;a_1,\ldots,a_r}\)
Throughout the remaining part of the paper we shall assume both GRH and LI. Moreover, we will use the following normalization for the Fourier transform of an integrable function \(f : \mathbb{R}^n \to \mathbb{C}\)
\[
\hat{f}(t_1, \ldots, t_n) = \int_{\mathbb{R}^n} e^{-i(t_1 x_1 + \cdots + t_n x_n)} f(x_1, \ldots, x_n) dx_1 \cdots dx_n.
\]
Then if \(\hat{f}\) is integrable on \(\mathbb{R}^n\) we have the Fourier inversion formula
\[
f(x_1, \ldots, x_n) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(t_1 x_1 + \cdots + t_n x_n)} \hat{f}(t_1, \ldots, t_n) dt_1 \cdots dt_n.
\]
Similarly we write
\[
\hat{\nu}(t_1, \ldots, t_n) = \int_{\mathbb{R}^n} e^{-i(t_1 x_1 + \cdots + t_n x_n)} d\nu(x_1, \ldots, x_n)
\]
for the Fourier transform of a finite measure \(\nu\) on \(\mathbb{R}^n\).
Rubinstein and Sarnak [16] established the following explicit formula for the Fourier transform of $\mu_{q; a_1, \ldots, a_r}$:

\begin{equation}
\hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r) = \exp \left( i \sum_{j=1}^{r} C_q(a_j) t_j \right) \prod_{\chi \not= \chi_0, \chi \mod q} J_0 \left( \frac{2 \sum_{j=1}^{r} \chi(a_j) t_j}{\sqrt{\frac{1}{4} + \gamma_{\chi}}^2} \right),
\end{equation}

where $J_0(z) = \sum_{m=0}^{\infty} (-1)^m (z/2)^{2m}/m!$ is the Bessel function of order 0.

Our first result shows that in the range $\|t\| \leq \Var(q)^{-1/2 + o(1)}$, the Fourier transform $\hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r)$ is very close to the Fourier transform of a multivariate normal distribution whose covariance matrix equals $\Cov_{q; a_1, \ldots, a_r}$.

**Proposition 3.1.** Let $q$ be large, $2 \leq r \leq \log q$ be a positive integer, and $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$. Then in the range $\|t\| \leq \Var(q)^{-1/2} \log^2 q$ we have

\begin{equation}
\hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r) = \exp \left( -\frac{1}{2} t^T \Cov_{q; a_1, \ldots, a_r} t \right) \left( 1 + O \left( \frac{d(q) \log^3 q}{\sqrt{q}} \right) \right).
\end{equation}

**Proof.** First, the explicit formula (3.1) yields

\[ \log \hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r) = \sum_{\chi \not= \chi_0, \chi \mod q} \log J_0 \left( \frac{2 \sum_{j=1}^{r} \chi(a_j) t_j}{\sqrt{\frac{1}{4} + \gamma_{\chi}}^2} \right) + O \left( \|t\| \sum_{j=1}^{r} |C_q(a_j)| \right). \]

Using Lemma 2.2 along with the standard estimate $\phi(q) \gg q / \log \log q$, we deduce that the error term above is $\ll q^{-1/2} d(q) \log^3 q$. On the other hand note that

\[ \frac{2 \sum_{j=1}^{r} \chi(a_j) t_j}{\sqrt{\frac{1}{4} + \gamma_{\chi}}^2} \ll r \|t\| \leq 1 \]

if $q$ is large enough. Hence, using that $\log J_0(z) = -z^2/4 + O(z^4)$ for $|z| \leq 1$ we obtain

\begin{equation}
\log \hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r) = -\sum_{\chi \not= \chi_0, \chi \mod q} \sum_{\gamma_{\chi} > 0} \left| \sum_{j=1}^{r} \chi(a_j) t_j \right|^2 \left( \frac{1}{4} + \gamma_{\chi}^2 \right)^{-1/2} + O \left( r^4 \|t\|^4 \sum_{\chi \not= \chi_0, \chi \mod q} \sum_{\gamma_{\chi} > 0} \left( \frac{1}{4} + \gamma_{\chi}^2 \right)^{-1/2} + \frac{d(q) \log^3 q}{\sqrt{q}} \right).
\end{equation}

Since $\sum_{\chi \not= \chi_0, \chi \mod q} \sum_{\gamma_{\chi} > 0} 1/(\frac{1}{4} + \gamma_{\chi}^2)^2 \ll \Var(q)$, it follows that the error term in the above estimate is $\ll q^{-1/2} d(q) \log^3 q$. On the other hand, the main term on the RHS
of (3.2) equals
\[- \sum_{\chi \neq \chi_0} \sum_{\chi \mod q} \chi a_j \chi(a_k) t_j t_k = -\frac{1}{2} \sum_{1 \leq j, k \leq r} \text{Cov}_{q,a_1,\ldots,a_r}(j,k)t_j t_k \]
by Lemma 2.1.

Next, we show that $\hat{\mu}_{q,a_1,\ldots,a_r}(t_1,\ldots,t_r)$ is rapidly decreasing in the range $\|t\| \geq \text{Var}(q)^{-1/2}$. In particular, the following result is a refinement of Proposition 3.2 of [12], which takes into account the dependence of the upper bounds on $r$.

**Proposition 3.2.** There exists a constant $c_1 > 0$ such that, if $q$ is large and $2 \leq r \leq c_1 \log q$, then uniformly for all $(a_1,\ldots,a_r) \in \mathcal{A}_r(q)$ we have
\[
|\hat{\mu}_{q,a_1,\ldots,a_r}(t_1,\ldots,t_r)| \leq \begin{cases} 
\exp\left(-\frac{\phi(q)}{8r} \|t\|\right) & \text{if } \|t\| \geq 400, \\
\exp\left(-\frac{\phi(q)}{(\log q)^6} \|t\| \right) & \text{if } (\log q)^{-2} \leq \|t\| \leq 400, \\
\exp\left(-\frac{\phi(q) \log q}{4} \|t\|^2 \right) & \text{if } \|t\| \leq (\log q)^{-2}.
\end{cases}
\]

Before proving this result we first require the following lemma.

**Lemma 3.3.** Let $q$ be large and $2 \leq r \leq \phi(q)/4$ be an integer. For $a = (a_1,\ldots,a_r) \in \mathcal{A}_r(q)$ and $t \in \mathbb{R}^r$ we denote by $M_{q,a}(t)$ the set of nontrivial characters $\chi \mod q$ such that $\sum_{j=1}^r \chi(a_j)t_j \geq \|t\|/2$. Then
\[
|M_{q,a}(t)| \geq \frac{\phi(q)}{2r}.
\]

**Proof.** Let
\[
(3.3) \quad S(t) = \sum_{\chi \neq \chi_0} \sum_{\chi \mod q} \left| \sum_{j=1}^r \chi(a_j)t_j \right|^2 = \sum_{\chi \mod q} \left| \sum_{j=1}^r \chi(a_j)t_j \right|^2 - \left( \sum_{j=1}^r t_j \right)^2 
\]
\[
= \sum_{j=1}^r \sum_{k=1}^r t_j t_k \sum_{\chi \mod q} \chi(a_j)\overline{\chi(a_k)} - \left( \sum_{j=1}^r t_j \right)^2 = \phi(q) \sum_{j=1}^r t_j^2 - \left( \sum_{j=1}^r t_j \right)^2 
\]
\[
\geq (\phi(q) - r)\|t\|^2,
\]
by the Cauchy–Schwarz inequality. Therefore, using that $|\sum_{j=1}^r \chi(a_j)t_j|^2 \leq \left( \sum_{j=1}^r |t_j| \right)^2 \leq r\|t\|^2$, we deduce
\[
S(t) = \sum_{\chi \in M_{q,a}(t)} \left| \sum_{j=1}^r \chi(a_j)t_j \right|^2 + \sum_{\chi \notin M_{q,a}(t)} \left| \sum_{j=1}^r \chi(a_j)t_j \right|^2 \leq r|M_{q,a}(t)|\|t\|^2 + \frac{\phi(q)}{4} \|t\|^2.
\]
Combining this estimate with (3.3) completes the proof. \qed
Proof of Proposition 3.2. First, assume that \( \|t\| \geq 400 \). For any Dirichlet character \( \chi \) we define
\[
F(x, \chi) := \prod_{\gamma > 0} \frac{2x}{\sqrt{\frac{1}{4} + \gamma_x^2}}.
\]
Then it follows from Lemma 2.16 of [5] that for any nontrivial character \( \chi \mod q \) we have
\[
|F(x, \chi)F(x, \overline{\chi})| \leq e^{-x}
\]
for \( x \geq 200 \). Moreover, the explicit formula (3.1) implies
\[
|\hat{\mu}_{q,a_1, \ldots, a_r}(t_1, \ldots, t_r)| = \prod_{\chi \neq \chi_0 \mod q} F \left( \left| \sum_{j=1}^r \chi(a_j)t_j \right|, \chi \right).
\]
Note that \( \chi \in M_{q,a}(t) \) if and only if \( \overline{\chi} \in M_{q,a}(t) \). Hence
\[
\prod_{\chi \in M_{q,a}(t)} F \left( \left| \sum_{j=1}^r \chi(a_j)t_j \right|, \chi \right) = \prod_{\chi \in M_{q,a}(t)} F \left( \left| \sum_{j=1}^r \chi(a_j)t_j \right|, \overline{\chi} \right).
\]
Furthermore, if \( \chi \in M_{q,a}(t) \) then \( \left| \sum_{j=1}^r \chi(a_j)t_j \right| \geq \|t\|/2 \geq 200 \). Therefore combining (3.4) and (3.5) along with the trivial bound \( |F(x, \chi)| \leq 1 \) (since \( |J_0(x)| \leq 1 \)) we derive
\[
|\hat{\mu}_{q,a_1, \ldots, a_r}(t_1, \ldots, t_r)|^2 \leq \left( \prod_{\chi \in M_{q,a}(t)} F \left( \left| \sum_{j=1}^r \chi(a_j)t_j \right|, \chi \right) \right)^2 = \prod_{\chi \in M_{q,a}(t)} F \left( \left| \sum_{j=1}^r \chi(a_j)t_j \right|, \chi \right) F \left( \left| \sum_{j=1}^r \chi(a_j)t_j \right|, \overline{\chi} \right) \leq \exp \left( - \sum_{\chi \in M_{q,a}(t)} \left| \sum_{j=1}^r \chi(a_j)t_j \right| \right) \leq \exp \left( - \frac{1}{2} \|M_{q,a}(t)\| \|t\| \right).
\]
Thus, we infer from Lemma 3.3 that
\[
|\hat{\mu}_{q,a_1, \ldots, a_r}(t_1, \ldots, t_r)| \leq \exp \left( - \frac{1}{4} \|M_{q,a}(t)\| \|t\| \right) \leq \exp \left( - \phi(q) \|t\| \right),
\]
as desired.

Let \( \epsilon = (\log q)^{-2} \) and suppose that \( \epsilon \leq \|t\| \leq 400 \). If \( \chi \in M_{q,a}(t) \) then
\[
\frac{2}{\sqrt{\frac{1}{4} + \gamma_x^2}} \geq \frac{\epsilon}{\sqrt{\frac{1}{4} + \gamma_x^2}}.
\]
We also note that if \( q \) is sufficiently large then \( \epsilon \left( \frac{1}{4} + \gamma_x^2 \right)^{-1/2} \leq 2\epsilon \leq 1 \). Therefore, since \( J_0 \) is a positive decreasing function on \([0, 1]\) and \( |J_0(z)| \leq J_0(1) \) for all \( z \geq 1 \),
we get
\[ |\hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r)| \leq \prod_{\chi \in \mathcal{M}} \prod_{a(t) \gamma_x > 0} \left| J_0 \left( \frac{\epsilon}{\sqrt{\frac{1}{4} + \gamma_x^2}} \right) \right|. \]

Furthermore, using the standard bound \(|J_0(x)| \leq \exp(-x^2/4)\) for \(|x| \leq 1\), we deduce that
\[ (3.6) \quad |\hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r)| \leq \exp \left( -\frac{\epsilon^2}{4} \sum_{\chi \in \mathcal{M}} \sum_{a(t) \gamma_x > 0} \frac{1}{\frac{1}{4} + \gamma_x^2} \right). \]

Let \(N(T, \chi)\) denote the number of \(\gamma_x\) in the interval \([0, T]\). Then, we have the classical estimate (see Chapters 15 and 16 of [3])
\[ N(T, \chi) = \frac{T}{2\pi} \log \frac{q^* T}{2\pi e} + O(\log q T), \]
where \(q^*\) is the conductor of \(\chi\). Hence, if \(T = \log^2 q\) then \(N(T, \chi) \gg \log^2 q\). This yields
\[ \sum_{\gamma_x > 0} \frac{1}{\frac{1}{4} + \gamma_x^2} \geq \sum_{0 < \gamma_x \leq \log^2 q} \frac{1}{\frac{1}{4} + \gamma_x^2} \gg \frac{1}{\log^2 q}. \]

The upper bound on \(|\hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r)|\) then follows upon inserting this estimate in (3.6) and using Lemma 3.3.

Finally assume that \(|t| \leq (\log q)^{-2}\). If \(q\) is large enough then
\[ \frac{2 \left| \sum_{j=1}^r \chi(a_j) t_j \right|}{\sqrt{\frac{1}{4} + \gamma_x^2}} \ll r\|t\| \leq 1. \]

Hence, using that \(|J_0(x)| \leq \exp(-x^2/4)\) for \(|x| \leq 1\) we obtain from the explicit formula (3.1)
\[ (3.7) \quad |\hat{\mu}_{q; a_1, \ldots, a_r}(t_1, \ldots, t_r)| \leq \exp \left( -\sum_{\chi \neq \chi_0} \sum_{\gamma_x > 0} \frac{1}{\frac{1}{4} + \gamma_x^2} \right). \]

Furthermore, Lemma 2.2 yields
\[
\sum_{\chi \neq \chi_0} \sum_{\gamma_x > 0} \frac{1}{\frac{1}{4} + \gamma_x^2} = \sum_{\chi \neq \chi_0} \sum_{\gamma_x > 0} \frac{1}{\frac{1}{4} + \gamma_x^2} \sum_{1 \leq j, k \leq r} \chi(a_j) \chi(a_k) t_j t_k \\
= \frac{\text{Var}(q)}{2} (t_1^2 + \cdots + t_r^2) + \sum_{1 \leq j < k \leq r} B_q(a_j, a_k) t_j t_k \\
= \frac{\phi(q) \log q}{2} \|t\|^2 \left( 1 + O \left( \frac{r + \log \log q}{\log q} \right) \right),
\]
since
\[ \sum_{1 \leq j < k \leq r} |t_j t_k| \leq \left( \sum_{j=1}^{r} |t_j| \right)^2 \leq r \| t \|^2, \]
by the Cauchy–Schwarz inequality. Thus, if \( r \leq c_1 \log q \) where \( c_1 > 0 \) is suitably small, then
\[ \sum_{\chi \neq \chi_0 \mod q} \sum_{\gamma \geq 0} \left| \sum_{j=1}^{r} \chi(a_j t_j) \right|^2 \geq \frac{\phi(q) \log q}{4} \| t \|^2. \]
Inserting this estimate in (3.7) completes the proof. \( \square \)

4. The asymptotic behavior of the densities \( \delta_q; a_1, \ldots, a_r \): Proof of Theorems 1.1, 1.2 and 1.3

We showed in the previous section that in a small region around 0, the Fourier transform of \( \mu_q; a_1, \ldots, a_r \) can be approximated by the Fourier transform of a multivariate normal distribution whose covariance matrix equals \( \text{Cov}_q; a_1, \ldots, a_r \). If we normalize by \( \sqrt{\text{Var}(q)} \) then Proposition 3.1 above implies that in the range \( \| t \| \leq \log^2 q \) we have
\[ \hat{\mu}_q; a_1, \ldots, a_r \left( \frac{t_1}{\sqrt{\text{Var}(q)}}, \ldots, \frac{t_r}{\sqrt{\text{Var}(q)}} \right) = \exp \left( -\frac{1}{2} t^T C t \right) \left( 1 + O \left( \frac{d(q) \log^3 q}{\sqrt{q}} \right) \right), \]
where \( C \) is an \( r \times r \) symmetric matrix whose entries are
\[ C_{jk} = \begin{cases} 1 & \text{if } j = k, \\ \frac{B_q(a_j, a_k)}{\text{Var}(q)} & \text{if } j \neq k. \end{cases} \]

Let \( \mathcal{M}_r(\epsilon) \) denote the set of \( r \times r \) symmetric matrices \( A = (a_{jk}) \) such that \( a_{jj} = 1 \) for all \( 1 \leq j \leq r \) and \( |a_{jk}| \leq \epsilon \) for all \( 1 \leq j \neq k \leq r \). In order to prove Theorems 1.1-1.3, we need to investigate multivariate normal distributions whose covariance matrices belong to \( \mathcal{M}_r(\epsilon) \) where \( \epsilon \ll 1/\log q \) is small. To this end we shall study the density function of a multivariate normal distribution, which is given by
\[ f(x) = \frac{1}{(2\pi)^{r/2} \sqrt{\det(A)}} \exp \left( -\frac{1}{2} x^T A^{-1} x \right), \]
if \( A \) is the covariance matrix of the distribution.

Our first lemma shows that the determinant of any matrix \( A \in \mathcal{M}_r(\epsilon) \) is close to 1 if \( \epsilon \) is small enough.

**Lemma 4.1.** If \( \epsilon \leq 1/(2r) \) then for any \( A \in \mathcal{M}_r(\epsilon) \) we have \( \det(A) = 1 + O(\epsilon^2 r^2) \).

**Proof.** Let \( S_r \) be the set of all permutations \( \sigma \) of \( \{1, \ldots, r\} \). Then we have
\[ \det(A) = \sum_{\sigma \in S_r} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)} = 1 + \sum_{\sigma \in S_r, \sigma \neq 1} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)}, \]
where \( 1 \) denotes the identity permutation. For \( 0 \leq k \leq r \) let \( S_r(k) \) be the set of permutations \( \sigma \in S_r \) such that the equation \( \sigma(j) = j \) has exactly \( r - k \) solutions in
\{1, \ldots, r\}. Then \(S_r(0) = \{1\}, \ S_r(1) = \emptyset\) and more generally one has

\[
|S_r(k)| \leq \binom{r}{r-k}(k-1)! \leq r^k, \text{ for } 2 \leq k \leq r.
\]

Moreover, note that \(|a_{1\sigma(1)} \cdots a_{r\sigma(r)}| \leq \epsilon^k\), for all \(\sigma \in S_r(k)\).

Hence, we deduce

\[
\sum_{\sigma \in S_r \atop \sigma \neq 1} \sigma a_{1\sigma(1)} \cdots a_{r\sigma(r)} = \sum_{k=2}^{r} \sum_{\sigma \in S_r(k)} \sigma a_{1\sigma(1)} \cdots a_{r\sigma(r)} \ll \sum_{k=2}^{r} (\epsilon r)^k \ll \epsilon^2 r^2.
\]

Inserting this estimate in (4.3) implies the result.

In order to understand the behavior of the density function \(f(x)\) we need to determine the size of the entries \(\{\tilde{a}_{jk}\}\) of \(A^{-1}\), if \(A \in \mathcal{M}_r(\epsilon)\). The next lemma shows that if \(\epsilon\) is small then the diagonal entries are close to 1 and the off-diagonal ones are small.

**Lemma 4.2.** If \(\epsilon \leq 1/(2r)\) then for any \(A \in \mathcal{M}_r(\epsilon)\) we have

\[
\tilde{a}_{jj} = \begin{cases} 1 + O(\epsilon^2 r^2) & \text{if } j = k, \\ O(\epsilon) & \text{if } j \neq k. \end{cases}
\]

**Proof.** Recall that

\[
\tilde{a}_{jj} = \frac{1}{\det(A)}(-1)^{j+k}M_{kj},
\]

where \(M_{kj}\) is the minor of the entry \(a_{kj}\), which is given by \(M_{kj} = \det(A_{kj})\) and \(A_{kj}\) is the matrix obtained from \(A\) by deleting the \(k\)th row and the \(j\)th column.

First, we determine the size of the diagonal entries \(\tilde{a}_{jj}\). In this case, remark that \(A_{jj} \in \mathcal{M}_{r-1}(\epsilon)\). Hence, it follows from Lemma 4.1 that

\[
\tilde{a}_{jj} = \frac{\det(A_{jj})}{\det(A)} = 1 + O(\epsilon^2 r^2).
\]

Now, we handle the off-diagonal entries. For \(1 \leq j \neq k \leq r\), let \(B_{j,k}\) denote the set of all bijections \(\sigma\) from \(\{1, \ldots, r\} \setminus \{j\}\) to \(\{1, \ldots, r\} \setminus \{k\}\). Then, we have

\[
|M_{jk}| = |\det(A_{jk})| \leq \sum_{\sigma \in B_{j,k}} \prod_{1 \leq n \neq j \leq r} |a_{n\sigma(n)}|.
\]

For \(0 \leq l \leq r-1\) we define \(B_{j,k}(l)\) to be the set of bijections \(\sigma \in B_{j,k}\) such that the equation \(\sigma(m) = m\) has exactly \(r-1-l\) solutions. Since \(\sigma(k) \neq k\) then it follows that \(B_{j,k}(0) = \emptyset\), and more generally one has

\[
|B_{j,k}(l)| \leq \binom{r-2}{r-1-l}(l-1)! \leq r^{l-1}, \text{ for } 1 \leq l \leq r-1.
\]

Hence we obtain

\[
|M_{jk}| \leq \sum_{l=1}^{r-1} \sum_{\sigma \in B_{j,k}(l)} \prod_{1 \leq n \neq j \leq r} |a_{n\sigma(n)}| \ll \sum_{l=1}^{r-1} r^{l-1} \epsilon^l \ll \epsilon.
\]

Combining this bound with Lemma 4.1 yield the desired bound \(\tilde{a}_{jk} \ll \epsilon\). \(\square\)
We know that the Fourier transform of a multivariate Gaussian of covariance matrix $A$ is (up to normalization) a multivariate Gaussian of covariance $A^{-1}$. The last ingredient we need to prove Theorems 1.1–1.3 is an approximate version of this statement when $A \in \mathcal{M}_r(\epsilon)$.

**Lemma 4.3.** Let $r \geq 2$ be a positive integer, $R \geq 10\sqrt{r}$ be a real number and $x \in \mathbb{R}^r$. If $\epsilon \leq 1/(2r)$ then for any $A \in \mathcal{M}_r(\epsilon)$ we have

$$
(2\pi)^{-r} \int_{\|t\| \leq R} e^{i(t_1 x_1 + \cdots + t_r x_r)} \exp\left(-\frac{1}{2} t^T A t\right) dt
$$

$$
= \frac{1}{(2\pi)^{r/2} \sqrt{\det(A)}} \exp\left(-\frac{1}{2} x^T A^{-1} x\right)
$$

$$
+ O\left(\exp\left(-\frac{R^2}{5}\right)\right).
$$

**Proof.** Since $\exp\left(-\frac{1}{2} t^T A t\right)$ is the Fourier transform of the multivariate normal distribution whose density equals

$$
f(x) = \frac{1}{(2\pi)^{r/2} \sqrt{\det(A)}} \exp\left(-\frac{1}{2} x^T A^{-1} x\right),
$$

then the Fourier inversion formula yields

$$
(2\pi)^{-r} \int_{t \in \mathbb{R}^r} e^{i(t_1 x_1 + \cdots + t_r x_r)} \exp\left(-\frac{1}{2} t^T A t\right) dt
$$

$$
= \frac{1}{(2\pi)^{r/2} \sqrt{\det(A)}} \exp\left(-\frac{1}{2} x^T A^{-1} x\right).
$$

Moreover, since $|a_{jk}| \leq 1/(2r)$ for $j \neq k$ then

$$
\left| \sum_{1 \leq j \neq k \leq r} a_{jk} t_j t_k \right| \leq \frac{1}{2r} \left( \sum_{j=1}^r |t_j| \right)^2 \leq \frac{1}{2} \sum_{j=1}^r t_j^2,
$$

by the Cauchy–Schwarz inequality. This implies

$$
t^T A t = \sum_{j=1}^r \sum_{k=1}^r a_{jk} t_j t_k \geq \frac{1}{2} \sum_{j=1}^r t_j^2.
$$

Hence, we get

$$
(2\pi)^{-r} \int_{\|t\| > R} \exp\left(-\frac{1}{2} t^T A t\right) dt
$$

$$
\leq (2\pi)^{-r} \int_{\|t\| > R} \exp\left(-\frac{1}{4} \|t\|^2\right) dt \ll \exp\left(-\frac{R^2}{5}\right),
$$

which in view of (4.4) completes the proof. \qed
Proof of Theorem 1.1. To lighten the notation we shall write $\delta_q$ for $\delta_{q,a_1,\ldots,a_r}$ and $\mu_q$ for $\mu_{q,a_1,\ldots,a_r}$. Let $R = 3 \sqrt{\text{Var}(q) \log q}$. First, using Lemma 2.3 we derive

$$
(4.6) \quad \delta_q = \int_{y_1 > y_2 > \cdots > y_r} d\mu_q(y_1, \ldots, y_r) = \int_{y_1 > y_2 > \cdots > y_r} d\mu_q(y_1, \ldots, y_r) \\
+ O \left( \exp \left( -2 \log^2 q \right) \right).
$$

Next, we apply the Fourier inversion formula to the measure $\mu_q$ to obtain

$$
\int_{y_1 > y_2 > \cdots > y_r} d\mu_q(y_1, \ldots, y_r) \\
= (2\pi)^{-r} \int_{y_1 > y_2 > \cdots > y_r} \int_{s \in \mathbb{R}^r} e^{i(s_1 y_1 + \cdots + s_r y_r)} \hat{\mu}_q(s_1, \ldots, s_r) \, ds \, dy.
$$

Since the Fourier transform $\hat{\mu}_q(s_1, \ldots, s_r)$ is rapidly decreasing, we shall deduce that the main contribution to the integral over $\mathbb{R}^r$ of $e^{i(s_1 y_1 + \cdots + s_r y_r)} \hat{\mu}_q(s_1, \ldots, s_r)$ comes from a small ball centered at 0. Indeed, we infer from Proposition 3.2 that

$$
\int_{s \in \mathbb{R}^r} e^{i(s_1 y_1 + \cdots + s_r y_r)} \hat{\mu}_q(s_1, \ldots, s_r) \, ds = \int_{||s|| \leq \epsilon} e^{i(s_1 y_1 + \cdots + s_r y_r)} \hat{\mu}_q(s_1, \ldots, s_r) \, ds \\
+ O \left( \exp \left( -2 \log^2 q \right) \right),
$$

where $\epsilon = 3(\text{Var}(q))^{-1/2} \log q$. Hence we obtain

$$
(4.7) \quad \delta_q = (2\pi)^{-r} \int_{y_1 > y_2 > \cdots > y_r} \int_{||s|| \leq \epsilon} e^{i(s_1 y_1 + \cdots + s_r y_r)} \hat{\mu}_q(s_1, \ldots, s_r) \, ds \, dy \\
+ O \left( \exp \left( -2 \log^2 q \right) \right),
$$

since $R^r \ll \exp(r \log q)$. Now, we make the change of variables

$$
t_j := \sqrt{\text{Var}(q)} s_j \quad \text{and} \quad x_j := \frac{y_j}{\sqrt{\text{Var}(q)}}, \quad \text{for all } 1 \leq j \leq r
$$

to obtain

$$
(4.8) \quad \delta_q = (2\pi)^{-r} \int_{x_1 > x_2 > \cdots > x_r} \int_{||t|| \leq 3 \log q} e^{i(t_1 x_1 + \cdots + t_r x_r)} \hat{\mu}_q \\
\times \left( \frac{t_1}{\sqrt{\text{Var}(q)}}, \ldots, \frac{t_r}{\sqrt{\text{Var}(q)}} \right) \, dt \, dx + O \left( \exp \left( -2 \log^2 q \right) \right).
$$

Replacing $\hat{\mu}_q \left( \frac{t_1}{\sqrt{\text{Var}(q)}}, \ldots, \frac{t_r}{\sqrt{\text{Var}(q)}} \right)$ by the approximation (4.1) that we derived in Proposition 3.1 yields

$$
\delta_q = (2\pi)^{-r} \int_{x_1 > x_2 > \cdots > x_r} \int_{||t|| \leq 3 \log q} e^{i(t_1 x_1 + \cdots + t_r x_r)} \exp \left( -\frac{1}{2} t^T \mathbf{C} t \right) \, dt \, dx + E_1,
$$

where

$$
E_1 \ll q^{-1/3} (\log q)^{3r} \ll q^{-1/4}.
$$
since \( d(q) = q^{o(1)} \) and \( t^T C t \geq 0 \) by (4.5). Furthermore, applying Lemma 4.3 we derive

\[
\delta_q = \frac{1}{(2\pi)^{r/2} \sqrt{\det(C)}} \int_{|x|_\infty \leq 3 \log q} \exp \left(-\frac{1}{2} x^T C^{-1} x \right) dx + O \left(q^{-1/4} \right).
\]

Since \( C_{jk} = B_q(a_j, a_k)/\Var(q) \ll (\log q)^{-1} \) for \( j \neq k \) by Lemma 2.2, there exists an absolute constant \( \alpha_0 > 0 \) such that \( C \in \mathcal{M}_r(\beta) \) with \( \beta = \alpha_0/\log q \). Therefore, appealing to Lemma 4.2 we obtain

\[
x^T C^{-1} x = \left(1 + O \left(\frac{r^2}{\log^2 q} \right) \right) \sum_{j=1}^r x_j^2 + O \left( \frac{1}{\log q} \left( \sum_{j=1}^r |x_j|^2 \right)^2 \right)
= \left(1 + O \left(\frac{r}{\log q} \right) \right) \|x\|^2,
\]

which follows from the Cauchy–Schwarz inequality. Hence we deduce

\[
-\frac{1}{2} \left(1 + \frac{\alpha_1 r}{\log q} \right) \|x\|^2 \leq -\frac{1}{2} x^T C^{-1} x \leq -\frac{1}{2} \left(1 - \frac{\alpha_1 r}{\log q} \right) \|x\|^2,
\]

for some absolute constant \( \alpha_1 > 0 \). This implies

\[
\int_{|x|_\infty \geq 3 \log q} \exp \left(-\frac{1}{2} x^T C^{-1} x \right) dx 
\leq \int_{|x|_\infty \geq 3 \log q} \exp \left(-\frac{1}{4} \|x\|^2 \right) dx \ll \exp \left(-\log^2 q \right).
\]

Inserting this estimate in (4.9) and using Lemma 4.1 we get

\[
\delta_q = \left(1 + O \left(\frac{r^2}{\log^2 q} \right) \right) \frac{1}{(2\pi)^{r/2}} \int_{|x|_\infty \geq 3 \log q} \exp \left(-\frac{1}{2} x^T C^{-1} x \right) dx + O \left(q^{-1/4} \right).
\]

Let \( \kappa \) be a real number such that \( |\kappa| \leq \alpha_1 r/\log q \). Since the function \( \|x\|^2 \) is symmetric in the variables \( \{x_j\}_{1 \leq j \leq r} \) we obtain

\[
\frac{1}{(2\pi)^{r/2}} \int_{|x|_\infty \geq 3 \log q} \exp \left(-\frac{1}{2} (1 + \kappa) \|x\|^2 \right) dx 
= \frac{1}{r!} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp \left(-\frac{1}{2} (1 + \kappa) y^2 \right) dy \right)^r 
= \frac{1}{r! (1 + \kappa)^{r/2}} = \frac{1}{r!} \exp \left(O \left(\frac{r^2}{\log q} \right) \right).
\]

The theorem follows upon combining this estimate with (4.10) and (4.11). \( \square \)

**Proof of Theorem 1.2.** The result can be obtained by proceeding along the same lines as the proof of Theorem 1.1, except that we make a different choice of parameters in
this case. Indeed, choosing $R = 5\sqrt{\text{Var}(q) \log r}$ and using Lemma 2.3 and Proposition 3.2, we obtain analogously to (4.8)

$$
\delta_q = (2\pi)^{-r} \int_{|x| \leq 5\sqrt{\text{Var}(q)}} e^{i(t_1 x_1 + \cdots + t_r x_r)} \mu_q
\times \left( \frac{t_1}{\sqrt{\text{Var}(q)}}, \ldots, \frac{t_r}{\sqrt{\text{Var}(q)}} \right) dtdx
+ O(\exp(-4r\log r)).
$$

Moreover, we infer from (4.1) that

$$
\delta_q = (2\pi)^{-r} \int_{|x| \leq 5\sqrt{\text{Var}(q)}} e^{i(t_1 x_1 + \cdots + t_r x_r)} \exp(-\frac{1}{2} t^T C t) dtdx + E_2,
$$

where

$$
E_2 \ll \frac{d(q) \log^3 q}{\sqrt{q}} (2\pi)^{-r} \int_{|x| \leq 5\sqrt{\text{Var}(q)}} dtdx \int_{|t| \leq 3\log q} \exp\left(-\frac{1}{2} t^T C t\right) dt + O(\exp(-4r\log r)).
$$

Note that

$$
\int_{|x| \leq 5\sqrt{\text{Var}(q)}} dx = \frac{1}{r!} \int_{|x| \leq 5\sqrt{\text{Var}(q)}} dx = \frac{(10\sqrt{r\log r})^r}{r!} = \exp\left(-\frac{r \log \log r}{2} + O(r \log \log r)\right),
$$

by Stirling’s formula. On the other hand, it follows from (4.5) that

$$
\frac{1}{(2\pi)^r} \int_{|t| \leq 3\log q} \exp\left(-\frac{1}{2} t^T C t\right) dt \leq \frac{1}{(2\pi)^r} \int_{t \in \mathbb{R}^r} \exp\left(-\frac{\|t\|^2}{4}\right) dt = \frac{1}{\pi^{r/2}}.
$$

Therefore, inserting these estimates in (4.15) and using the classical bound $d(q) = \exp(O(\log q / \log \log q))$ we deduce

$$
E_2 \ll \exp\left(-\frac{1}{2} (\log q + r \log r)\right) + O\left(\frac{\log q}{\log \log q} + r \log \log r\right) + \exp(-4r \log r).
$$

Continuing along the same line as in the proof of Theorem 1.1, we obtain analogously to (4.11)

$$
\delta_q = \left(1 + O\left(\frac{r^2}{\log^2 q}\right)\right) \frac{1}{(2\pi)^{r/2}} \int_{x_1 > x_2 > \cdots > x_r} \exp\left(-\frac{1}{2} x^T C^{-1} x\right) dx + E_3,
$$

where

$$
E_3 \ll \exp\left(-\frac{1}{2} (\log q + r \log r)\right) + O\left(\frac{\log q}{\log \log q} + r \log \log r\right) + \exp(-4r \log r).
$$
Furthermore, it follows from (4.10) and (4.12) that
\[
\frac{1}{(2\pi)^{r/2}} \int_{x_1 > x_2 > \cdots > x_r} \exp \left(-\frac{1}{2}x^T C^{-1} x\right) \, dx
= \frac{1}{r!} \exp \left(O \left( \frac{r^2}{\log q} \right) \right)
= \exp \left(-r \log r + r + O \left( \log r + \frac{r^2}{\log q} \right) \right),
\]
by Stirling’s formula. Inserting this estimate in (4.16) completes the proof. □

**Proof of Theorem 1.3.** Since \( \mu_{q,a_1,\ldots,a_r} \) is absolutely continuous with respect to the Lebesgue measure, it follows from (1.1) that
\[
\delta_{q,a_1,\ldots,a_{r-1}} = \delta_{q,a_r,a_1,\ldots,a_{r-1}} + \delta_{q,a_1,a_r,\ldots,a_{r-1}} + \cdots + \delta_{q,a_1,\ldots,a_{r-1},a_r}.
\]
Hence, if \( 2 \leq s < r \leq \phi(q) \) are positive integers then
\[
(4.17) \quad \max_{(a_1,\ldots,a_r) \in A_r(q)} \delta_{q,a_1,\ldots,a_r} < \max_{(b_1,\ldots,b_s) \in A_s(q)} \delta_{q,b_1,\ldots,b_s}.
\]
On the other hand, using Theorem 1.2 with \( s = \lfloor (1 - \epsilon/2) \log q / \log \log q \rfloor \), we get
\[
\max_{(b_1,\ldots,b_s) \in A_s(q)} \delta_{q,b_1,\ldots,b_s} = \exp \left(-s \log s + s + O \left( \log s + \frac{r^2}{\log q} \right) \right) \ll \epsilon \frac{1}{q^{1-\epsilon}}.
\]
The theorem follows upon combining this inequality with (4.17). □

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**References**


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