ACTA ARITHMETICA
LXXIV.1 (1996)

On the Shanks–Rényi race problem

by

JERZY KACZOROWSKI (Poznań)

Dedicated to Professor Włodzimierz Staś
on the occasion of his 70-th birthday

1. Introduction and statement of results. The Shanks–Rényi race problem can be formulated as follows (cf. [9]). Let $q > 2$ be a natural number. Given a permutation $(a_1, \ldots, a_{\varphi(q)})$ of the reduced set of residue classes $(\bmod\, q)$, there exist infinitely many natural numbers m such that

$$\pi(m, q, a_1) > \pi(m, q, a_2) > \ldots > \pi(m, q, a_{\varphi(q)}),$$

where, as usual, we denote by $\pi(x, q, a)$ the number of primes $p \le x$ congruent to $a \pmod q$. This problem is one of the most interesting open problems concerning the distribution of primes in arithmetic progressions. The first result towards its solution has been proved in [6] and reads as follows (see also [7], [8]).

Assume the Generalized Riemann Hypothesis (*G.R.H.*) *for Dirichlet's L-functions* $\bmod\, q$, $q \ge 3$. *Then there exist infinitely many positive integers* m *with*

$$\pi(m, q, 1) > \max_{a \not\equiv 1 \,(\bmod\, q)} \pi(m, q, a).$$

Moreover, *the set of* m*'s satisfying this inequality has a positive lower density. The same statement holds for* m *satisfying*

$$\pi(m, q, 1) < \min_{a \not\equiv 1 \,(\bmod\, q)} \pi(m, q, a).$$

Hence there exist at least two permutations $(a_2, a_3, \ldots, a_{\varphi(q)})$ and $(b_2, b_3, \ldots, b_{\varphi(q)})$ of the residue classes $a \pmod q$, $(a, q) = 1$, $a \not\equiv 1 \pmod q$ such that

$$\pi(m, q, 1) > \pi(m, q, a_2) > \ldots > \pi(m, q, a_{\varphi(q)}) \tag{1}$$

Supported by KBN Grant nr. 2 1086 91 01.

and

$$\pi(m,q,b_2) > \ldots > \pi(m,q,b_{\varphi(q)}) > \pi(m,q,1). \tag{2}$$

This leaves $(\varphi(q)-1)!$ possibilities for each permutation. A natural question is whether or not, for a given q, one can construct explicit examples of permutations for which (1) and (2) hold. The aim of this paper is to solve this problem assuming of course the G.R.H.

Before stating our result let us introduce the following notation. Let $q > 1$ be a natural number. We write

$$m(q) = \begin{cases} 1/2 & \text{if } 2 \,\|\, q, \\ 2 & \text{if } 8 \mid q, \\ 1 & \text{otherwise,} \end{cases}$$

$$N_q = \frac{1}{\varphi(q)} m(q) 2^{\omega(q)},$$

where $\omega(q)$ denotes the number of distinct prime divisors of q and φ denotes Euler's totient function. For a prime number p let

$$p^{\nu_p(q)} \,\|\, q, \quad q_p = qp^{-\nu_p(q)}, \quad g_{p,q} = \operatorname{ord} p \pmod{q_p}.$$

Let now $(a,q) = 1$. Then we denote by $\overline{a}$ the inverse of $a \pmod q$: $a\overline{a} \equiv 1 \pmod q$. Moreover, we put

$$\varrho(q,a) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue} \pmod q, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda(q,a) = \sum_{\substack{p^\alpha \| q \\ a \equiv 1 \ (\mathrm{mod}\ q_p)}} \frac{\log p}{p^{\alpha-1}(p-1)} + \sum_{\substack{p^\alpha \mid q,\, \alpha < \nu_p(q) \\ a \equiv 1 \ (\mathrm{mod}\ qp^{-\alpha})}} \frac{\log p}{p^\alpha},$$

$$\delta(q,a) = \begin{cases} 1 & \text{if } a \equiv -1 \pmod q, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now that p is a prime number and that $a \pmod{q_p}$ belongs to the cyclic multiplicative group generated by $p \pmod{q_p}$. Then we denote by $l_p(a)$ the natural number uniquely determined by the following conditions:

$$1 \le l_p(a) \le g_{q,p}, \quad p^{l_p(a)} \equiv a \pmod{q_p}.$$

Then we set

$$\alpha(q,a) = \sum_{p \mid q} \frac{\log p}{\varphi(p^{\nu_p(q)}) p^{l_p(a)}} \left(1 - \frac{1}{p^{g_{q,p}}}\right)^{-1},$$

the summation being restricted to primes p for which $l_p(a)$ is defined; if there are no such primes p we put $\alpha(q,a) = 0$.

Let as usual

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m,\ p \text{ prime}, \\ 0 & \text{otherwise}, \end{cases}$$

and for real x let

$$\psi(x, q, a) = \sum_{n \le x,\, n \equiv a \ (\mathrm{mod}\ q)} \Lambda(n).$$

An easy consequence of Dirichlet's prime number theorem is that for every a prime to q there exists a constant $b(q, a)$ such that

$$\sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(q)} \log x + b(q, a) + o(1)$$

as x tends to infinity. We call $b(q, a)$ the *Dirichlet–Euler constants*.

Finally, we define the following quantities which play the crucial role in what follows:

$$\begin{aligned} r^+(q, a) &= \alpha(q, a) + b(q, a) + \tfrac{1}{2}\delta(q, a) \log 2 + \lambda(q, a), \\ r^-(q, a) &= r^+(q, a) - \lambda(q, a), \\ R^+(q, a) &= r^+(q, a) - \varrho(q, a) N_q, \\ R^-(q, a) &= r^-(q, a) - \varrho(q, a) N_q. \end{aligned}$$

Theorem. *Let $q \ge 5$, $q \ne 6$ be a natural number and assume the G.R.H. for Dirichlet's L-functions* $(\mathrm{mod}\ q)$. *Define permutations*

$$\begin{gathered} (a_2, a_3, \ldots, a_{\varphi(q)}), \quad (b_2, b_3, \ldots, b_{\varphi(q)}), \\ (c_2, c_3, \ldots, c_{\varphi(q)}), \quad (d_2, d_3, \ldots, d_{\varphi(q)}), \end{gathered}$$

of the set of residue classes

$$a \ (\mathrm{mod}\ q), \quad (a, q) = 1, \quad a \not\equiv 1 \ (\mathrm{mod}\ q)$$

so that the following inequalities hold:

$$\begin{gathered} R^+(q, \overline{a}_2) > R^+(q, \overline{a}_3) > \ldots > R^+(q, \overline{a}_{\varphi(q)}), \\ R^-(q, \overline{b}_2) > R^-(q, \overline{b}_3) > \ldots > R^-(q, \overline{b}_{\varphi(q)}), \\ r^+(q, \overline{c}_2) > r^+(q, \overline{c}_3) > \ldots > r^+(q, \overline{c}_{\varphi(q)}), \\ r^-(q, \overline{d}_2) > r^-(q, \overline{d}_3) > \ldots > r^-(q, \overline{d}_{\varphi(q)}). \end{gathered}$$

Then there exists a positive constant b_0 such that each of the sets of natural numbers

$$\begin{aligned} (3) \quad & \{m \in \mathbb{N} : \pi(m, q, a_2) > \ldots > \pi(m, q, a_{\varphi(q)}) > \pi(m, q, 1), \\ & \qquad \min_{a \not\equiv b \ (\mathrm{mod}\ q),\, (ab, q) = 1} |\pi(m, q, a) - \pi(m, q, b)| > b_0 \sqrt{m} / \log m\}, \\ (4) \quad & \{m \in \mathbb{N} : \pi(m, q, 1) > \pi(m, q, b_2) > \ldots > \pi(m, q, b_{\varphi(q)}), \\ & \qquad \min_{a \not\equiv b \ (\mathrm{mod}\ q),\, (ab, q) = 1} |\pi(m, q, a) - \pi(m, q, b)| > b_0 \sqrt{m} / \log m\}, \end{aligned}$$

$$\text{(5)}\quad \{m \in \mathbb{N} : \psi(m,q,c_2) > \ldots > \psi(m,q,c_{\varphi(q)}) > \psi(m,q,1),\\ \min_{a\not\equiv b\,(\mathrm{mod}\,q),\,(ab,q)=1} |\psi(m,q,a) - \psi(m,q,b)| > b_0\sqrt{m}\},$$

$$\text{(6)}\quad \{m \in \mathbb{N} : \psi(m,q,1) > \psi(m,q,d_2) > \ldots > \psi(m,q,d_{\varphi(q)}),\\ \min_{a\not\equiv b\,(\mathrm{mod}\,q),\,(ab,q)=1} |\psi(m,q,a) - \psi(m,q,b)| > b_0\sqrt{m}\},$$

has a positive natural density.

In the case when $r^{\pm}(q,b_i) = r^{\pm}(q,b_j)$ or $R^{\pm}(q,b_i) = R^{\pm}(q,b_j)$ for some $i \neq j$ the corresponding permutations are undefined. Then a weaker version of the theorem is still valid, in which all but one terms corresponding to the equal $r^{\pm}$'s or $R^{\pm}$'s are removed from (3)–(6). We conjecture, however, that all quantities $r^{\pm}$ and $R^{\pm}$ are distinct and this situation cannot occur.

Finally, let us remark that a well known and classical technique used in the proofs of the class number formula or the Kronecker limit formula for abelian extensions of rationals gives finite expressions for Dirichlet–Euler constants. For simplicity we consider the case of prime modulus only. Observe first of all that in this case the permutations (a_j), (b_j) and (c_j), (d_j) are equal. Let p be a prime number and let $\zeta = \exp(2\pi i/p)$ denote the primitive pth root of unity. We set

$$\text{(7)}\qquad D = \det[\log(1-\zeta^{i\bar{j}})], \qquad 1 \le i,j \le p-1.$$

Moreover, for $1 \le a \le p-1$, let

$$\text{(8)}\quad D_a = (-1)^a \det[\log(1-\zeta^{i\bar{j}})], \qquad 2 \le i \le p-1,\ 1 \le j \le p-1,\ j \neq a.$$

As we shall see in Lemma 6, $D \neq 0$ for all primes p.

Next let R denote the R-function introduced by Deninger [3]. By definition $R : (0,\infty) \to \mathbb{R}$ is the unique solution to the difference equation

$$R(x+1) - R(x) = \log^2 x$$

which is convex in some interval (A,∞), $A > 0$, and such that $\int_0^1 R(x)\,dx = 0$. As explained in [3], R is the right analogue of $\log\Gamma(x)$, and the understanding of its properties is comparable with that of Euler's gamma function. In particular, R has the following Weierstrass-type representation as a relatively fast convergent infinite series:

$$\text{(9)}\quad R(x) = A - C_1 x - \log^2 x - \sum_{n=1}^{\infty}\biggl(\log^2(x+n) - \log^2 n - 2x\frac{\log n}{n}\biggr),$$

where

$$C_1 = \lim_{n\to\infty}\biggl(2\sum_{k=1}^{n-1}\frac{\log k}{k} - \log^2 n\biggr) = -0.145631690967\ldots$$

denotes the Stieltjes constant,

$$A = \frac{1}{2}\left(\log^2(2\pi) + \frac{\pi^2}{12} - C^2 - C_1\right) = 2.006356455908\ldots$$

and $C = 0.577215664901\ldots$ is the Euler constant.

Let χ be a non-principal Dirichlet character $\pmod p$, and $\tau(\chi)$ the corresponding Gauss sum. Then

$$L(1,\chi) = -\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p-1}\overline{\chi(a)}\log(1-\zeta^a) = -\frac{\tau(\chi)}{p}\sum_{a=1}^{p-1}\overline{\chi(a)}\log(1-\zeta^a), \tag{10}$$

$$\frac{1}{L(1,\chi)} = \tau(\overline{\chi})\sum_{a=1}^{p-1}\overline{\chi(a)}\,\frac{D_a}{D}. \tag{11}$$

Writing, for real $0<\xi<1$ and $d\in\{0,1\}$,

$$W^*(\xi,d) = \begin{cases} \frac{1}{2}(C+\log(2\pi))\log(2-2\cos(2\pi\xi)) - R(\xi) & \text{if } d=0,\\ \pi(C+\log(2\pi))\xi + \pi\log\Gamma(\xi) & \text{if } d=1,\end{cases}$$

we have

$$L'(1,\chi) = -(-i)^{d(\chi)}\frac{\tau(\chi)}{p}\sum_{a=1}^{p-1}\overline{\chi(a)}W^*\left(\frac{a}{p}, d(\chi)\right), \tag{12}$$

where $d(\chi)\in\{0,1\}$ is such that $\chi(-1) = (-1)^{d(\chi)}$.

Formula (10) is well known and classical (see e.g. [10], Theorem 2.2).

In the proof of (11) we use (10) and the following relation:

$$\sum_{b=1}^{p-1}\log(1-\zeta^{a\bar b})\frac{D_b}{D} = \begin{cases} -1 & \text{if } a=1,\\ 0 & \text{if } 2\le a\le p-1.\end{cases} \tag{13}$$

We then have

$$\begin{aligned}\tau(\overline{\chi})L(1,\chi)\sum_{a=1}^{p-1}\overline{\chi(a)}\frac{D_a}{D} &= -\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi(ab)}\log(1-\zeta^b)\frac{D_a}{D}\\ &= -\sum_{c=1}^{p-1}\overline{\chi(c)}\sum_{a=1}^{p-1}\frac{D_a}{D}\log(1-\zeta^{\bar a c}) = \sum_{c=1}^{p-1}\overline{\chi(c)}\,\delta_{c,1} = 1.\end{aligned}$$

Formula (12) is equivalent to (3.4) and (3.6) of [3]. Let us remark here that in the case of odd χ the result is classical and is due to A. Berger [1] and M. Lerch [11] (cf. also [2]).

Let $\{\xi\} = \xi - [\xi]$ denote the fractional part of a real number ξ. For $\xi\in\mathbb{R}\backslash\mathbb{Z}$ we set

$$\begin{aligned}W(\xi) = {}&(C+\log(2\pi))\log(2(1-\cos(2\pi\{\xi\}))) - R(\{\xi\}) - R(1-\{\xi\})\\ &+ i\pi((C+\log(2\pi))(2\{\xi\}-1) + \log\Gamma(\{\xi\}) - \log\Gamma(1-\{\xi\})).\end{aligned}$$

We have

$$W(\xi) = W^*(\{\xi\},0) + W^*(\{-\xi\},0) + i(W^*(\{\xi\},1) - W^*(\{-\xi\},1)).$$

It is easy to see that (compare Lemma 5 below)

$$b(q,a) = \frac{1}{p-1}\sum_{\chi\neq\chi_0}\overline{\chi(a)}\Big\{-\frac{L'}{L}(1,\chi)\Big\} - \frac{\log p}{(p-1)^2} - \frac{C}{p-1},$$

and hence using the formulae for L' and $1/L$, after some easy but long computations we conclude that for $p \geq 3$, $1 \leq a \leq p-1$,

$$b(p,a) = \Re\sum_{b=1}^{(p-1)/2} W\Big(\frac{\overline{ab}}{p}\Big)\frac{D_b}{D} + c(p),$$

where

$$c(p) = \frac{\log^2 p - 2C\log p}{p-1}\Re\sum_{b=1}^{(p-1)/2}\frac{D_b}{D} - \frac{\log p}{(p-1)^2} - \frac{C}{p-1}.$$

The foregoing formulae give also finite expressions for $r^{\pm}(p,a)$ and $R^{\pm}(p,a)$ suitable for numerical computations. Hence there is no problem in finding permutations (a_j) and (c_j) for every given prime modulus p. The author performed calculations for $p \leq 29$. The output is displayed in the following tables:

p	Permutation (a_j)
5	$(3,2,4)$
7	$(5,6,3,4,2)$
11	$(6,10,8,7,4,9,2,3,5)$
13	$(7,8,9,2,6,12,10,11,5,3,4)$
17	$(6,9,5,7,16,14,4,10,3,15,12,11,13,2,8)$
19	$(13,18,10,4,3,5,11,7,2,12,8,17,9,15,14,16,6)$
23	$(12,22,14,10,8,21,19,6,16,17,15,5,20,18,3,11,7,4,9,13,2)$
29	$(15,10,28,6,8,25,11,26,9,12,17,24,22,27,21,13,2,7,18,14,3,19,16,23,20,4,5)$

p	Permutation (c_j)
5	$(3,4,2)$
7	$(4,5,6,3,2)$
11	$(6,4,9,10,8,3,7,5,2)$
13	$(9,7,8,12,10,2,6,11,5,3,4)$
17	$(9,6,16,5,7,4,14,15,13,2,10,3,12,11,8)$
19	$(13,18,4,10,5,11,7,3,17,9,2,12,8,16,6,15,14)$
23	$(12,8,22,14,10,6,16,21,18,19,3,17,15,4,5,9,20,11,13,7,2)$
29	$(15,10,28,6,25,9,8,11,24,22,26,13,12,7,17,27,16,21,23,2,20,18,4,14,3,19,5)$

The meaning of these numerical data is obvious. For instance, the fact that permutation (a_j), for $p = 7$ is equal to $(5,6,3,4,2)$ implies that the set of natural numbers m for which

$$\pi(m,7,1)>\pi(m,7,5)>\pi(m,7,6)>\pi(m,7,3)>\pi(m,7,4)>\pi(m,7,2)$$

has a positive lower density (assuming the G.R.H. for L-functions $\pmod 7$). Similar statement holds true for m's satisfying

$$\pi(m,7,5)>\pi(m,7,6)>\pi(m,7,3)>\pi(m,7,4)>\pi(m,7,2)>\pi(m,7,1).$$

The second table provides analogous information concerning $\psi(m,q,a)$.

2. Some lemmas. Let us denote by $\chi \pmod q$ Dirichlet's character $(\operatorname{mod} q)$. For such $\chi \pmod q$ we denote by $\chi' \pmod{q'}$, $q'\mid q$, the corresponding primitive character.

LEMMA 1. *Let $q>2$ and*

$$S_+(q,a)=\frac{1}{\varphi(q)}\sum_{\substack{\chi \,(\operatorname{mod} q)\\ \chi(-1)=1}}\overline{\chi(a)} \quad \text{and} \quad S_-(q,a)=\frac{1}{\varphi(q)}\sum_{\substack{\chi \,(\operatorname{mod} q)\\ \chi(-1)=-1}}\overline{\chi(a)}.$$

Then

$$S_+(q,a)=\begin{cases}1/2 & \text{if } a\equiv \pm 1 \pmod q,\\ 0 & \text{otherwise},\end{cases}$$

$$S_-(q,a)=\begin{cases}1/2 & \text{if } a\equiv 1 \pmod q,\\ -1/2 & \text{if } a\equiv -1 \pmod q,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The case of $a\equiv \pm1 \pmod q$ is obvious. Otherwise, by the orthogonality law for characters we have

$$S_+(q,a)+S_-(q,a)=0$$

and

$$S_+(q,a)-S_-(q,a)=S_+(q,-a)+S_-(q,-a)=0.$$

Hence $S_+(q,a)=S_-(q,a)=0$, and the lemma follows.

LEMMA 2. *For every $a \pmod q$, natural number m and a prime p we have*

$$\frac{1}{\varphi(q)}\sum_{\chi \,(\operatorname{mod} q)}\overline{\chi(a)}\chi'(p^m)=\begin{cases}\dfrac{1}{\varphi(p^{\nu_p(q)})} & \text{if } p^m\equiv a \pmod{q_p},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. See [5], p. 243.

LEMMA 3. *Let $q>1$ and $a\not\equiv 1 \pmod q$, $(a,q)=1$. Then*

$$\lambda(q,a)=-\frac{1}{\varphi(q)}\sum_{\chi \,(\operatorname{mod} q)}\overline{\chi(a)}\log q'.$$

Proof. For $(a,d)=1$ write

$$s_a(d)=\sum_{\chi \,(\mathrm{mod}\, d)}^{\quad *} \overline{\chi(a)},$$

where the star indicates that the summation is restricted to primitive characters. Then

$$\sum_{d\mid q} s_a(d)=\sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)}=\begin{cases}\varphi(q) & \text{if } a\equiv 1 \pmod q,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$s_a(d)=\sum_{\substack{k\mid d\\ a\equiv 1 \,(\mathrm{mod}\, k)}} \varphi(k)\mu\left(\frac{d}{k}\right)$$

and consequently

$$\begin{aligned}(14)\qquad \frac{1}{\varphi(q)}\sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)}\log q' &= \frac{1}{\varphi(q)}\sum_{d\mid q} s_a(d)\log d\\ &= \frac{1}{\varphi(q)}\sum_{\substack{k\mid q\\ a\equiv 1 \,(\mathrm{mod}\, k)}} \varphi(k)\sum_{\substack{d\\ k\mid d\mid q}} \mu\left(\frac{d}{k}\right)\log d\\ &= \frac{1}{\varphi(q)}\sum_{\substack{k\mid q\\ a\equiv 1 \,(\mathrm{mod}\, k)}} \varphi(k)\sum_{d\mid q/k} \mu(d)\log(kd).\end{aligned}$$

Since $a\not\equiv 1 \pmod q$, we have $k<q$. Therefore $q/k>1$ and

$$\sum_{d\mid q/k}\mu(d)\log(kd)=\sum_{d\mid q/k}\mu(d)\log d=-\Lambda\left(\frac{q}{k}\right).$$

Hence the expression in (14) equals

$$-\frac{1}{\varphi(q)}\sum_{\substack{k\mid q\\ a\equiv 1 \,(\mathrm{mod}\, k)}} \varphi(k)\Lambda\left(\frac{q}{k}\right)=-\sum_{\substack{p^\alpha\mid q\\ a\equiv 1 \,(\mathrm{mod}\, qp^{-\alpha})}}\frac{\varphi(qp^{-\alpha})}{\varphi(p^\alpha)}\log p=-\lambda(q,a),$$

and the result follows.

Lemma 4. *Let $q>1$, $(a,q)=1$ and*

$$(15)\qquad N(q,a)=\#\{b \pmod q : b^2\equiv a \pmod q\}.$$

Then $N(q,a)=\varphi(q)N_q\varrho(q,a)$.

Proof. Let $G(q)$ denote the group of reduced residue classes $(\mathrm{mod}\, q)$ and consider the group endomorphism

$$f: G(q)\to G(q),\qquad a\mapsto a^2 \pmod q.$$

Then $N(q,a) = \# f^{-1}(a) = \varrho(q,a)\#\ker f$ and $\ker f$ consists of course of elements of orders less than or equal to 2. Let $q = 2^{\alpha}p_1^{\alpha_1}\dots p_t^{\alpha_t}$ be the canonical factorization of q into prime powers and write $q_i = \varphi(p_i^{\alpha_i})$, $i = 1,\dots,t$. Then

$$G(q) = G(2^{\alpha}) \oplus G(p_1^{\alpha_1}) \oplus \dots \oplus G(p_t^{\alpha_t}) = G(2^{\alpha}) \oplus \mathbf{C}(q_1) \oplus \dots \oplus \mathbf{C}(q_t).$$

Each cyclic group $\mathbf{C}(q_i)$, $i = 1,\dots,t$, contains exactly two elements of order less than or equal to 2. Let $n_2(\alpha)$ denote the number of such elements in $G(2^{\alpha})$. Then

$$n_2(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \{0,1\}, \\ 2 & \text{if } \alpha = 2, \\ 4 & \text{if } \alpha \geq 3. \end{cases}$$

Hence $\#\ker f = n_2(\alpha)2^t = \varphi(q)N_q$, as required.

LEMMA 5. *Let $(a,q) = 1$. Then*

$$-\frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \overline{\chi(a)} \frac{L'}{L}(1, \overline{\chi}') = \alpha(q,\overline{a}) + b(q,\overline{a}) + \frac{C}{\varphi(q)}.$$

Proof. Inserting $-L'/L(1,\overline{\chi}') = \sum_n \Lambda(n)\overline{\chi'(n)}n^{-1}$ we see that the sum equals

$$\sum_{p^m} \frac{\log p}{p^m} \left\{ \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(\overline{a})}\chi'(p^m) - \frac{1}{\varphi(q)} \right\}. \tag{16}$$

Write

$$S = \frac{1}{\varphi(q)} \sum_{p \mid q} \frac{\log p}{p-1}.$$

By Lemma 2 the part of the sum in (16) corresponding to primes $p \mid q$ equals

$$\sum_{p \mid q} \log p \sum_{p^m \equiv \overline{a} \,(\mathrm{mod}\, q_p)} \frac{1}{\varphi(p^{\nu_p(q)})p^m} - S = \sum_{p \mid q} \frac{\log p}{\varphi(p^{\nu_p(q)})} \sum_{k=0}^{\infty} \frac{1}{p^{l_p(\overline{a}) + kg_{q,p}}} - S$$
$$= \alpha(q,\overline{a}) - S.$$

The sum over remaining primes equals

$$\lim_{N \to \infty} \Bigg\{ \sum_{\substack{n \leq N \\ n \equiv \overline{a} \,(\mathrm{mod}\, q)}} \frac{\Lambda(n)}{n} - \frac{1}{\varphi(q)} \sum_{\substack{n \leq N \\ (n,q)=1}} \frac{\Lambda(n)}{n} \Bigg\} = b(q,\overline{a}) - \frac{b(1,1)}{\varphi(q)} + S.$$

Since $b(1,1) = -C$, the assertion follows.

LEMMA 6. *Let D be as in* (7). *Then $D \neq 0$.*

Proof. We make use of the following well known Dedekind determinant relation (see e.g. [10], Theorem 6.1):

For every complex-valued function f defined on a finite abelian group G we have

$$\det_{a,b\in G} f(a^{-1}b) = \prod_{\chi\in\hat{G}} \sum_{a\in G} \chi(a) f(a^{-1}).$$

Applying this theorem to $G = \mathbb{Z}/p\mathbb{Z}$ and $f(a + p\mathbb{Z}) = \log(1 - \zeta^a)$, $a = 1, \ldots, p-1$, we obtain

$$D = \prod_{\chi \,(\mathrm{mod}\, p)} \sum_{a=1}^{p-1} \chi(a) \log(1 - \zeta^{\bar{a}}) = \log\Big(\prod_{a=1}^{p-1}(1-\zeta^a)\Big) \prod_{\chi\neq\chi_0} (-\tau(\overline{\chi})L(1,\chi)),$$

where we use the well known finite formula for $L(1,\chi)$. Hence $D \neq 0$ because $\tau(\overline{\chi})L(1,\chi) \neq 0$ for $\chi \neq \chi_0$ and $\prod_{a=1}^{p-1}(1-\zeta^a) = p$.

3. The behaviour of K-functions on the real axis. Let χ be a primitive Dirichlet character (mod q), $q \geq 1$. For $z \in H = \{x + iy, y > 0\}$, we define the *$K$-function* by

$$K(z,\chi) = \sum_{\Im\varrho>0} \frac{e^{\varrho z}}{\varrho}.$$

These functions have been studied in detail in [4]. We collect here some of their basic properties which are essential in this paper. For real x let

$$F(x,\chi) = \lim_{y\to 0^+} (K(x+iy,\chi) + \overline{K(x+iy,\overline{\chi})}).$$

The limit exists for every x and can be computed explicitly as follows (cf. [4], Theorem 4.1). Denote by $m(\varrho,\chi)$ the multiplicity of $L(s,\chi)$'s zero at $s = \varrho$; we put $m(\varrho,\chi) = 0$ when $L(\varrho,\chi) \neq 0$. Moreover, we denote by χ_0 the principal character and let

$$e(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0, \end{cases} \qquad e_1(\chi) = \begin{cases} 1 & \text{if } \chi(-1) = 1,\ \chi\neq\chi_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$d(\chi) = \begin{cases} 1 & \text{if } \chi(-1) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

For positive x we put

$$R(x,1) = \frac{1}{2}\log\frac{e^x - 1}{e^x + 1}, \qquad R(x,0) = \frac{1}{2}\log(1 - e^{-2x}),$$

$$\psi(x,\chi) = \sum_{n\leq x} \Lambda(n)\chi(n), \qquad \widetilde{\psi}(x,\chi) = \sum_{n\leq x} \frac{\Lambda(n)\chi(n)}{n},$$

$$\psi_0(x,\chi) = \frac{1}{2}(\psi(x+0,\chi) + \psi(x-0,\chi)),$$

$$\widetilde{\psi}_0(x,\chi) = \frac{1}{2}(\widetilde{\psi}(x+0,\chi) + \widetilde{\psi}(x-0,\chi)).$$

Under this notation and assuming the G.R.H., for $x > 0$ we have

$$F(x,\chi) + 2e^{x/2}m(1/2,\chi) = -\psi_0(e^x,\chi) + e(\chi)e^x - e_1(\chi)x - R(x,d(\chi)) + B(\chi), \tag{17}$$

$$B(\chi) = 2m(1/2,\chi) - e(\chi) - d(\chi)\log 2 - \frac{C}{2} - \frac{1}{2}\log\frac{\pi}{q} + F(0,\chi), \tag{18}$$

and for $x < 0$,

$$F(x,\chi) + 2e^{x/2}m(1/2,\chi) = \widetilde{\psi}_0(e^{|x|},\chi) + e(\chi)e^x + e(\chi)x + R(|x|, 1-d(\chi)) + C(\chi), \tag{19}$$

$$C(\chi) = B(\chi) + C + \log\frac{2\pi}{q}, \tag{20}$$

$C = 0.577\ldots$ being Euler's constant.

LEMMA 7. *Suppose $\chi \pmod q$ is a primitive Dirichlet character. Then for $\chi = \chi_0$ we have*

$$B(\chi) = -\log(2\pi) \quad \textit{and} \quad C(\chi) = C$$

whereas for $\chi \neq \chi_0$,

$$B(\chi) = \log\frac{q}{2\pi} - C + \frac{L'}{L}(1,\overline{\chi}), \qquad C(\chi) = \frac{L'}{L}(1,\overline{\chi}).$$

Proof. The numbers $F(0,\chi)$, $\chi = \chi' \pmod q$, have been computed in [7], Lemma 1:

$$F(0,\chi_0) = \frac{C}{2} + 1 - \frac{1}{2}\log(4\pi)$$

and for $\chi \neq \chi_0$,

$$F(0,\chi) = \frac{1}{2}\log\frac{q}{\pi} - \frac{C}{2} + (d(\chi)-1)\log 2 + \frac{L'}{L}(1,\overline{\chi}) - \sum_{\gamma=0}\frac{1}{\beta},$$

the summation being taken over non-trivial, real zeros β of $L(s,\chi)$ (if there are any). Substituting these results into (18) and (20) we obtain the assertion of Lemma 7.

Let $q \geq 1$ and a, $1 \leq a \leq q$, be two coprime integers. For $z \in H$ we write

$$F(z,q,a) = -2e^{-z/2}\frac{1}{\varphi(q)}\sum_{\chi \,(\mathrm{mod}\, q)}\overline{\chi(a)}K(z,\chi') - 2\frac{1}{\varphi(q)}\sum_{\chi \,(\mathrm{mod}\, q)}\overline{\chi(a)}m(1/2,\chi). \tag{21}$$

Further, for real x, let

$$P(x,q,a) = \lim_{y\to 0^+}\Re F(x+iy,q,a).$$

LEMMA 8. *For* $(a,q)=1$, $a\not\equiv 1 \pmod q$ *we have*

$$\lim_{x\to 0^+} P(x,q,a) = r^+(q,\overline{a}) - \frac{1}{\varphi(q)} \tag{22}$$

and

$$\lim_{x\to 0^-} P(x,q,a) = r^-(q,\overline{a}) - \frac{1}{\varphi(q)}.$$

Moreover, $\lim_{x\to 0^+} P(x,q,1) = -\infty$ *and* $\lim_{x\to 0^-} P(x,q,1) = \infty$.

Proof. We have

$$P(x,q,a) = -\frac{e^{-x/2}}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)}(F(x,\chi') + 2m(1/2,\chi)e^{x/2}). \tag{23}$$

Let $x>0$. Using (17) we split the sum into five parts:

$$P(x,q,a) = \sum_{i=1}^{5} S_i(x,q,a),$$

say. Trivially we have $S_i(0^+,q,a)=0$ for $i=1,3$ and $S_2(0^+,q,a) = -1/\varphi(q)$. Moreover, for positive x we have

$$S_4(x,q,a) = e^{-x/2}(S_+(q,a)R(x,0) + S_-(q,a)R(x,1)).$$

Hence, by Lemma 1,

$$S_4(x,q,a) = \begin{cases} 0 & \text{if } a\not\equiv \pm 1 \pmod q, \\ \frac12 e^{-x/2}(\log(e^x+1)-x) & \text{if } a\equiv -1 \pmod q, \\ \frac12 e^{-x/2}(\log(e^x-1)-x) & \text{if } a\equiv 1 \pmod q, \end{cases}$$

and therefore $S_4(0^+,q,a) = 0$, $\frac12\log 2$ or $-\infty$ respectively.

Since obviously $S_5(x,q,1) = O(1)$ as $x\to 0^+$, we have $\lim_{x\to 0^+} P(x,q,1) = -\infty$.

Let $a\not\equiv 1 \pmod q$. Then using Lemmas 7, 3 and 5 we obtain

$$\begin{aligned} S_5(0^+,q,a) &= -\frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)} B(\chi') \\ &= \frac{C}{\varphi(q)} - \frac{1}{\varphi(q)} \sum_{\chi\ne\chi_0} \overline{\chi(a)}\Big(\log\frac{q'}{2\pi} - C + \frac{L'}{L}(1,\overline{\chi}')\Big) \\ &= \frac{C}{\varphi(q)} + \lambda(q,a) - \frac{1}{\varphi(q)} \sum_{\chi\ne\chi_0} \overline{\chi(a)}\frac{L'}{L}(1,\overline{\chi}') \\ &= \alpha(q,\overline{a}) + b(q,\overline{a}) + \lambda(q,a). \end{aligned} \tag{24}$$

Gathering the foregoing formulae we obtain (22).

For $x < 0$ we proceed similarly. Using (19) in place of (17) we write

$$P(x, q, a) = \sum_{i=1}^{5} S_i'(x, q, a).$$

Then simple computations show that $S_i'(0^-, a, q) = S_i(0^+, q, a)$ for $i = 1, 2, 3$. Further,

$$S_4'(x, q, a) = -e^{-x/2}(S_+(q, a)R(x, 1) + S_-(q, a)R(x, 0))$$

and therefore

$$S_4'(0^-, q, a) = \begin{cases} 0 & \text{if } a \not\equiv \pm 1 \pmod q, \\ \frac{1}{2}\log 2 & \text{if } a \equiv -1 \pmod q, \\ \infty & \text{if } a \equiv 1 \pmod q. \end{cases}$$

Moreover, using (24) and (20) we have

$$S_5'(0^-, q, a) = -\frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)} C(\chi') = S_5(0^+, q, a) - \lambda(q, a).$$

Hence $P(0^-, q, 1) = \infty$ and for $a \not\equiv 1 \pmod q$

$$\begin{aligned} \lim_{x \to 0^-} P(x, q, a) &= \sum_{i=1}^{5} S_i'(0^-, q, a) = \sum_{i=1}^{5} S_i(0^+, q, a) - \lambda(q, a) \\ &= r^-(q, a) - \frac{1}{\varphi(q)}, \end{aligned}$$

which ends the proof.

Lemma 9. *For $x \ge 1$ we have*

$$P(x, q, a) = e^{-x/2}\left(\psi(e^x, q, a) - \frac{1}{\varphi(q)} e^x\right) + O(xe^{-x/2}).$$

Proof. This follows from (23) and (17) (cf. [5], proof of Corollary 7).

4. An auxiliary result on generalized Dirichlet series. We formulate now a subsidiary theorem which has been proved in [6]. We need some additional notation. Let $\mathcal{B}$ denote the class of all functions

$$F(z) = \sum_{n=1}^{\infty} \alpha_n e^{iw_n z}, \qquad z = x + iy, \ y > 0, \tag{25}$$

satisfying the following conditions:

1. $0 \le w_1 < w_2 < \ldots$ are real numbers.
2. $\alpha_n \in \mathbb{C}, n = 1, 2, \ldots$
3. The series in (25) converges absolutely for all $y > 0$.

4. The limit $P(x) = \lim_{y\to 0^+} P(x+iy)$, where $P(x+iy) = \Re F(x+iy)$, $y > 0$, exists for almost all real x; we put $P(x) = 0$ for the remaining x, so that P is well defined on the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \Im z \ge 0\}$.

5. We have

$$\lim_{y\to 0^+} \sup_{x\in\mathbb{R}} \int_{-1/2}^{1/2} |P(x+t) - P(x+t+iy)|^2\, dt = 0.$$

Lemma 10 (see [6], Lemma 3). *Let $F_j \in \mathcal{B}$ for $j = 1, \ldots, n$ and let $x_0 \in \mathbb{R}$ be a continuity point of the mapping*

$$P : \overline{H} \ni z \mapsto (P_1(z), \ldots, P_n(z)) \in \mathbb{R}^n, \qquad P_j = \Re F_j,\ j = 1, \ldots, n.$$

Then for every open neighbourhood $U \subset \mathbb{R}^n$ of $P(x_0)$ there exist constants $b_0 = b_0(U) > 0$ and $l_0 = l_0(U) > 0$ such that

$$\mu(P^{-1}(U) \cap \mathcal{J}) > b_0$$

for every interval $\mathcal{J}$ of length $\ge l_0$, μ being the Lebesgue measure on the real axis.

5. Proof of the Theorem. Let us start by proving that the set in (5) has a positive lower density. To this end we apply Lemma 10 with $n = \varphi(q)$ and

$$F_j(z) = \begin{cases} F(z, q, c_{j+1}) & \text{for } j = 1, \ldots, \varphi(q) - 1, \\ F(z, q, 1) & \text{for } j = \varphi(q), \end{cases}$$

the functions $F(z, q, a)$ being defined by (21). Then F_j's belong to the class $\mathcal{B}$ as proved in [6]. Moreover, they are holomorphic and therefore also continuous on the interval $(0, \log 2)$. From Lemma 8 and from the definition of the permutation $(c_2, c_3, \ldots, c_{\varphi(q)})$ we know that there exists $x_0 \in (0, \log 2)$ such that

$$P(x_0, q, c_2) > P(x_0, q, c_3) > \ldots > P(x_0, q, c_{\varphi(q)}) > P(x_0, q, 1).$$

Let

$$\delta = \min_{a \not\equiv b \,(\mathrm{mod}\, q)} |P(x_0, q, a) - P(x_0, q, b)| > 0$$

and let U be the open ball with centre $P(x_0)$ and radius $\delta/3$. Here and in what follows the minimum $\min_{a \not\equiv b \,(\mathrm{mod}\, q)}$ is restricted to a and b prime to q.

Then for $(y_1, \ldots, y_{\varphi(q)}) \in U$ we have $y_1 > \ldots > y_{\varphi(q)}$ and $\min_{i\ne j} |y_i - y_j| > \delta/3$. Hence for $x \in P^{-1}(U)$,

$$P(x, q, c_2) > P(x, q, c_3) > \ldots > P(x, q, c_{\varphi(q)}) > P(x, q, 1)$$

and

$$\min_{a \not\equiv b \,(\mathrm{mod}\, q)} |P(x_0, q, a) - P(x_0, q, b)| > \delta/3.$$

Write

$$E(x,q,a) = \frac{\psi(x,q,a) - x/\varphi(q)}{\sqrt{x}}.$$

Then by what we have already proved and Lemma 9 we have, for sufficiently large $x \in P^{-1}(U)$,

$$E(e^x,q,c_2) > \ldots > E(e^x,q,c_{\varphi(q)}) > E(e^x,q,1)$$

and

$$\min_{a\not\equiv b \,(\mathrm{mod}\, q)} |E(e^x,q,a) - E(e^x,q,b)| > \delta/4,$$

which is equivalent to

$$\psi(e^x,q,c_2) > \ldots > \psi(e^x,q,c_{\varphi(q)}) > \psi(e^x,q,1)$$

and

$$\min_{a\not\equiv b \,(\mathrm{mod}\, q)} |\psi(e^x,q,a) - \psi(e^x,q,b)| > (\delta/4)e^{x/2}.$$

Now we conclude that the set in (5) has indeed a positive lower density upon changing the variable $e^x = t$ and observing that for every real t we have $\psi(t,q,a) = \psi([t],q,a)$.

The proof for (6) is very similar. We choose $x_0 \in (-\log 2, 0)$ sufficiently close to 0 and we apply Lemma 10 to the functions

$$F_j(z) = \begin{cases} F(z,q,1) & \text{for } j = 1, \\ F(z,q,d_j) & \text{for } j = 2,\ldots,\varphi(q). \end{cases}$$

The proofs for (3) and (4) need some modifications but in principle they go along similar lines. For instance, consider (3); modifications needed for (5) will then become obvious. We consider the functions

$$F_j(z) = \begin{cases} F(z,q,a_{j+1}) - \varrho(q,a_{j+1})N_q & \text{for } j = 1,\ldots,\varphi(q)-1, \\ F(z,q,1) - N_q & \text{for } j = \varphi(q). \end{cases}$$

According to the definition of the sequence (a_j), for sufficiently small positive x_0 we have $\Re F_1(x_0) > \ldots > \Re F_{\varphi(q)}(x_0)$ and $\delta = \min_{i\neq j} |\Re F_i(x_0) - \Re F_j(x_0)| > 0$. Write

$$E^*(x,q,a) = E(x,q,a) - \varrho(q,a)N_q.$$

Then of course $\Re F_j(x) = E^*(x,q,a_{j+1}) + o(1)$ $(j = 1,\ldots,\varphi(q)-1)$, and $\Re F_{\varphi(q)}(x) = E^*(x,q,1) + o(1)$ as $x \to \infty$ by Lemma 9. Using Lemma 10 we conclude that the set of natural numbers m satisfying

$$E^*(m,q,a_2) > \ldots > E^*(m,q,a_{\varphi(q)}) > E^*(m,q,1),$$
$$\min_{a\not\equiv b \,(\mathrm{mod}\, q)} |E^*(m,q,a) - E^*(m,q,b)| > \delta/4$$

has a positive lower density.

Let

$$\theta(x,q,a)=\sum_{p\le x,\, p\equiv a\,(\mathrm{mod}\,q)}\log p.$$

Then by partial summation we have

$$\pi(m,q,a)-\frac{1}{\varphi(q)}\mathrm{li}\,m=\frac{\theta(m,q,a)-m/\varphi(q)}{\log m}+O\left(\frac{\sqrt{m}}{\log^2 m}\right)$$

and

$$\psi(m,q,a)=\theta(m,q,a)+\frac{N(q,a)}{\varphi(q)}\sqrt{m}+O(m^{1/3}),$$

$N(q,a)$ being defined by (15). Hence, using Lemma 4 we obtain

$$E^*(m,q,a)=\frac{\pi(m,q,a)-(\mathrm{li}\,m)/\varphi(q)}{\sqrt{m}/\log m}+o(1)$$

as $m\to\infty$, and thus the result follows.

References

[1] A. Berger, *Sur une sommation de quelques séries*, Nova Acta Regiae Soc. Sci. Upsaliensis (2) 12 (1883), 29–31.
[2] S. Chowla and A. Selberg, *On Epstein's zeta-functions*, *II*, J. Reine Angew. Math. 227 (1967), 86–110.
[3] C. Deninger, *On the analogue of the formula of Chowla and Selberg for real quadratic fields*, J. Reine Angew. Math. 351 (1984), 171–191.
[4] J. Kaczorowski, *The k-functions in multiplicative number theory*, *I. On complex explicit formulae*, Acta Arith. 56 (1990), 195–211.
[5] —, *The k-functions in multiplicative number theory*, *IV. On a method of A. E. Ingham*, ibid. 57 (1991), 231–244.
[6] —, *A contribution to the Shanks–Rényi race problem*, Quart. J. Math. 44 (1993), 451–458.
[7] —, *On the Shanks–Rényi race problem* mod 5, J. Number Theory 50 (1995), 106–118.
[8] —, *Results on the distribution of primes*, J. Reine Angew. Math. 446 (1994), 89–113.
[9] S. Knapowski and P. Turán, *Comparative prime-number theory*, *I*, Acta Math. Hungar. 13 (1962), 299–314.
[10] S. Lang, *Cyclotomic Fields*, Springer, 1990.
[11] M. Lerch, *Sur quelques formules relatives au nombre des classes*, Bull. Sci. Math. (2) 21 (1897), 302–303.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
A. MICKIEWICZ UNIVERSITY
MATEJKI 48/49
60-769 POZNAŃ, POLAND

Received on 18.6.1994
and in revised form on 17.3.1995 (2626)