JOURNAL OF NUMBER THEORY 50, 106–118 (1995)

On the Shanks–Rényi Race Problem mod 5

JERZY KACZOROWSKI*

Institute of Mathematics, A. Mickiewicz University, Poznań, Poland

Communicated by A. Hildebrand

Received August 1, 1992; revised July 23, 1993

It is proved that under the Generalized Riemann Hypothesis for Dirichlet L-functions (mod 5) for every permutation (a_1, a_2, a_3, a_4) of the sequence $(1, 2, 3, 4)$ the set of integers m satisfying $\psi(m, 5, a_1) > \psi(m, 5, a_2) > \psi(m, 5, a_3) > \psi(m, 5, a_4)$ has positive lower density. © 1995 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF RESULTS

The race problem belongs to the class of the most interesting open questions concerning the distribution of primes. As usual let $\pi(x, q, a)$, $x \geq 1$, $q \geq 1$, $1 \leq a \leq q$, $(a, q) = 1$, denote the number of primes $p \equiv a \pmod q$ with $p \leq x$. Then the problem can be formulated as follows (cf. [4]): *For every permutation*

$$a_1, a_2, ..., a_{\varphi(q)}$$

of reduced residue classes (mod q) *there are infinitely many values m for which*

$$\pi(m, q, a_1) > \pi(m, q, a_2) > \cdots > \pi(m, q, a_{\varphi(q)}). \tag{1}$$

We speak about "the race mod q" because the problem has an interpretation as a game. Let us suppose that each reduced residue class (mod q) is a "player." The player a scores a point when by the enumeration of all positive integers in increasing order a prime $\equiv a \pmod q$ occurs. Then after the first m steps the player a has exactly $\pi(m, q, a)$ points. Inequalities (1) can be described by saying that the player a_1 wins, a_2 is on the second position, and so on.

The general race problem seems very difficult. The only paper dealing with it is the author's forthcoming note [3], where it is proved that under

* Supported by KBN Grant 2 1086 91 01.

0022-314X/95 $6.00

the Generalized Riemann Hypothesis (GRH) for Dirichlet L-functions (mod q) the player 1 wins and loses infinitely many times. More precisely it is proved that the set of integers m satisfying

$$\pi(m, q, 1) > \max_{a \not\equiv 1 \pmod q} \pi(m, q, a)$$

has positive lower density and the same holds for m's satisfying

$$\pi(m, q, 1) < \min_{a \not\equiv 1 \pmod q} \pi(m, q, a).$$

In connection with this result *The strong Race Hypothesis* has been formulated. Namely we assert that for each permutation

$$a_1, a_2, ..., a_{\varphi(q)}$$

of the reduced set of residue classes (mod q) the set of m satisfying (1) has positive lower density. It can be observed that some modifications of the original race problem are of interest as well. For example, we can replace $\pi(m, q, a)$ in (1) by $\psi(m, q, a)$, where

$$\psi(x, q, a) = \sum_{\substack{n \equiv a \pmod q \\ n \leqslant x}} \Lambda(n),$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } \quad n = p^\alpha, p\text{-prime} \\ 0 & \text{otherwise} \end{cases}$$

and ask about the distribution of m's satisfying

$$\psi(m, q, a_1) > \psi(m, q, a_2) > \cdots > \psi(m, q, a_{\varphi(q)}).$$

In this paper we consider the first non-trivial case of $q = 5$.

THEOREM 1. *Let us assume the GRH for Dirichlet L-functions* (mod 5). *Then for every permutation* (a_1, a_2, a_3, a_4) *of the sequence* (1, 2, 3, 4) *the set of m's satisfying*

$$\psi(m, 5, a_1) > \psi(m, 5, a_2) > \psi(m, 5, a_3) > \psi(m, 5, a_4)$$

has positive lower density.

In fact we prove the following stronger theorem.

THEOREM 2. *Let us assume GRH for Dirichlet L-functions (mod 5). Then there exist three positive constants* c_0, b_0, b_1, *such that for every* *permutation* (a_1, a_2, a_3, a_4) *of the sequence* $(1, 2, 3, 4)$ *and for arbitrary* $T \geqslant 1$ *we have*

$$\#\{T \leqslant m \leqslant c_0 T : \psi(m, 5, a_1) > \psi(m, 5, a_2) > \psi(m, 5, a_3) > \psi(m, 5, a_4),$$
$$\min_{\substack{i \neq j \\ 1 \leqslant i, j \leqslant 4}} |\psi(m, 5, i) - \psi(m, 5, j)| \geqslant b_0 \sqrt{m}\} \geqslant b_1 T.$$

2. THE k-FUNCTIONS

In the proof of Theorem 2 we use k-functions introduced in [2]. For the reader's convenience we reproduce here some basic facts about them.

Let $\chi \pmod q$, $q \geqslant 1$, be a primitive Dirichlet character. For z from the upper half-plane $H = \{z \in C : \Im z > 0\}$ we write

$$k(z, \chi) = \sum_{\gamma > 0} e^{\rho z},$$

$$K(z, \chi) = \sum_{\gamma > 0} \frac{1}{\rho} e^{\rho z},$$

where the summation is taken over all non-trivial $L(s, \chi)$ zeros $\rho = \beta + i\gamma$ with positive imaginary parts. Of course both functions are holomorphic on H. The function k can be continued analytically to a meromorphic function on the Riemannian surface M of logarithmic type having infinitely many simple poles. Moreover it satisfies two functional equations on M (cf. Theorems 3.1, 3.2, and 3.3 in [2]). Hence K can be continued analytically along every path on M not passing through the singularities of k; it becomes therefore a multivalued function on this surface. Every pole of k becomes a logarithmic branch point of K. The only singularities on the real axis can appear at $x = 0$ or $x = \log n$, $n \in N$. For further details the reader is referred to [2]. In this paper we need k-functions on H together with their boundary values on the positive part of real axis.

Let $x \in R$ and

$$F(x, \chi) = \lim_{y \to 0^+} \{K(x + iy, \chi) + \overline{K(x + iy, \bar{\chi})}\}.$$

It can be proved that this limit exists for all real x (cf. [2, p. 200]). Moreover, for $x > 0$ we have (cf. [2, Theorem 4.1])

$$F(x, \chi) = -\psi_0(e^x, \chi) - \sum_{\gamma = 0} e^{\beta x}/\beta + e(\chi) e^x - e_1(\chi) x - R(x, d(\chi)) + B(\chi), \tag{2}$$

where the summation is over all non-trivial, real $L(s, \chi)$ zeros (if there are any) and

$$\psi_0(x, \chi) = \tfrac{1}{2}(\psi(x+0, \chi) + \psi(x-0, \chi)),$$

$$\psi(x, \chi) = \sum_{n \leqslant x} \Lambda(n)\, \chi(n),$$

$$e(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0 \text{ (the principal character)} \\ 0 & \text{otherwise,} \end{cases}$$

$$d(\chi) = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{otherwise,} \end{cases}$$

$$e_1(\chi) = \begin{cases} 1 & \text{if } d(\chi) = 0,\ \chi \neq \chi_0 \\ 0 & \text{otherwise,} \end{cases}$$

$$R(x, d(\chi)) = \begin{cases} \tfrac{1}{2}\log(1 - e^{-2x}) & \text{if } d(\chi) = 0 \\ \tfrac{1}{2}\log((e^x - 1)/(e^x + 1)) & \text{if } d(\chi) = 1. \end{cases}$$

The constants $B(\chi)$ are given by

$$B(\chi) = \sum_{\gamma = 0} \frac{1}{\beta} - e(\chi) - d(\chi)\log 2 - C/2 - \frac{1}{2}\log\frac{\pi}{q} + F(0, \chi), \tag{3}$$

where $C = 0.57721566\ldots$ denotes the Euler's constant. (Note that there is a misprint in formula (4.6) of [2].)

LEMMA 1. *Let $\chi \pmod q$ be a primitive Dirichlet character. Then for $q = 1$ we have*

$$F(0, \chi) = C/2 + 1 - \tfrac{1}{2}\log(4\pi)$$

and for $q > 1$

$$F(0, \chi) = \frac{1}{2}\log\frac{q}{\pi} - C/2 + (d(\chi) - 1)\log 2 + \frac{L'}{L}(1, \bar{\chi}) - \sum_{\gamma = 0} \frac{1}{\beta}.$$

Proof. We have $\overline{L(s, \chi)} = L(\bar{s}, \bar{\chi})$ so that $L(\rho, \chi) = 0$ implies $L(\bar{\rho}, \bar{\chi}) = 0$. Hence

$$F(0, \chi) = \lim_{y \to 0^+} \{K(iy, \chi) + \overline{K(iy, \bar{\chi})}\}$$

$$= \lim_{y \to 0^+} \left\{ \sum_{\substack{\Im\rho > 0 \\ L(\rho, \chi) = 0}} \frac{1}{\rho} e^{-\gamma y} + \sum_{\substack{\Im\rho > 0 \\ L(\rho, \bar{\chi}) = 0}} \frac{1}{\bar{\rho}} e^{-\gamma y} \right\}$$

$$= \lim_{y \to 0^+} \sum_{\substack{\rho \\ L(\rho, \chi) = 0}} \frac{1}{\rho} e^{-\gamma y} - \sum_{\gamma = 0} \frac{1}{\beta}.$$

By Abel's theorem the last limit equals

$$\sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{1}{\rho} = \lim_{T\to\infty} \sum_{|\gamma|\leq T} \frac{1}{\rho}.$$

In the case $q=1$ (the Riemann zeta function) the sum over real zeros is empty and thus it vanishes, and the limit equals

$$C/2 + 1 - \tfrac{1}{2}\log(4\pi)$$

(cf. [1, Sect. 12]). The first part of the lemma therefore follows.

Let us assume now that $q>1$. We consider the integral

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \left\{-\frac{L'}{L}(s,\chi)\right\}\frac{ds}{s}$$
$$= \lim_{T\to\infty}\int_{2-iT}^{2+iT} \left\{-\frac{L'}{L}(s,\chi)\right\}\frac{ds}{s}.$$

Substituting $-(L'/L)(s,\chi) = \sum_{n=1}^{\infty} (\Lambda(n)\,\chi(n)/n^s)$ and changing the order of summation and integration we easily obtain

$$I = 0. \tag{4}$$

On the other hand, shifting the line of integration to the left we get

$$I = -\sum_{\rho}\frac{1}{\rho} + \mathrm{Res}_{s=0}\left\{-\frac{L'}{L}(s,\chi)\frac{1}{s}\right\} + J, \tag{5}$$

where

$$J = \frac{1}{2\pi i}\int_{1/2-i\infty}^{-1/2+i\infty} \left\{-\frac{L'}{L}(s,\chi)\right\}\frac{ds}{s}.$$

From the functional equation for $L(s,\chi)$ we have

$$\frac{L'}{L}(s,\chi) = -\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s+d(\chi)}{2}\right)$$
$$-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right) - \frac{L'}{L}(1-s,\bar{\chi}). \tag{6}$$

We split J into three integrals

$$J=\log\frac{q}{\pi}\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{ds}{s}$$
$$+\frac{1}{2}\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{\Gamma'}{\Gamma}\left(\frac{1-s+d(\chi)}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\frac{ds}{s}$$
$$+\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{L'}{L}(1-s,\bar{\chi})\frac{ds}{s}$$
$$=A+B+C, \tag{4}$$

say. We have

$$A=\frac{1}{2\pi i}\log\frac{q}{\pi}\lim_{T\to\infty}\left(i\operatorname{Arg}\left(-\frac{1}{2}+iT\right)-i\operatorname{Arg}\left(-\frac{1}{2}-iT\right)\right)$$
$$=-\frac{1}{2}\log\frac{q}{\pi}. \tag{8}$$

Next we have

$$2B=\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{\Gamma'}{\Gamma}\left(\frac{1-s+d(\chi)}{2}\right)\frac{ds}{s}+\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\frac{ds}{s}.$$

The change of variable $1-s=w$ in the first integral and the shift of the line of integration to the right in the second integral yield

$$2B=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\frac{ds}{1-s}+\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\frac{ds}{s}$$
$$=-\operatorname{Res}_{s=0}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\frac{1}{s}\right\}+\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\frac{ds}{s(1-s)}.$$

The last integral equals zero, which can easily be proved by moving the path of integration to the vertical line $\Re s=N$, using

$$\left|\frac{\Gamma'}{\Gamma}(s)\right|\ll\log|s|\qquad(\Re s\geqslant 1),$$

and making $N\to\infty$. Hence

$$B=-\frac{1}{2}\operatorname{Res}_{s=0}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\frac{1}{s}\right\}. \tag{9}$$

Since $(1/2\pi i)\int_{-1/2-i\infty}^{-1/2+i\infty}(x^s/s)\,ds=0$ for $x>1$, we have

$$C=-\sum_{n=2}^{\infty}\frac{\Lambda(n)\,\overline{\chi(n)}}{n}\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{n^s}{s}\,ds=0. \tag{10}$$

Gathering (4), (5), (7), (8), (9), and (10) we obtain

$$\sum_\rho \frac{1}{\rho}=\operatorname{Res}_{s=0}\left\{\left(-\frac{L'}{L}(s,\chi)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+d(\chi)}{2}\right)\right)\frac{1}{s}\right\}-\frac{1}{2}\log\frac{q}{\pi},$$

which by the functional equation (6) equals

$$\operatorname{Res}_{s=0}\left\{\left(\log\frac{q}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s+d(\chi)}{2}\right)+\frac{L'}{L}(1-s,\bar\chi)\right)\frac{1}{s}\right\}-\frac{1}{2}\log\frac{q}{\pi}$$
$$=\frac{1}{2}\log\frac{q}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+d(\chi)}{2}\right)+\frac{L'}{L}(1,\bar\chi).$$

Since

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+d(\chi)}{2}\right)=(d(\chi)-1)\log 2-C/2$$

this ends the proof.

3. Calculations Related to Dirichlet Characters mod 5

There are four Dirichlet characters (mod 5):

	1	2	3	4
χ_0	1	1	1	1
χ_1	1	-1	-1	1
χ_2	1	i	$-i$	-1
χ_3	1	$-i$	i	-1

(11)

Let us observe that the corresponding L-functions do not vanish at $s=\frac{1}{2}$. This is well known for $\chi=\chi_0$ (cf. [6, Chap. III, Problem 6]), and for other functions follows at once from the inequalities

$$L\left(\frac{1}{2},\chi_1\right)=\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{(5k+1)(5k+2)}(\sqrt{5k+1}+\sqrt{5k+2})}-\frac{1}{\sqrt{(5k+3)(5k+4)}(\sqrt{5k+3}+\sqrt{5k+4})}\right)>0,$$

$$\Re L\left(\frac{1}{2}, \chi_2\right) = \Re L\left(\frac{1}{2}, \chi_3\right) = \sum_{k=3}^{\infty}\left(\frac{1}{\sqrt{5k+1}} - \frac{1}{\sqrt{5k+4}}\right) > 0.$$

We need also the values $L(1, \chi)$ and $L'(1, \chi)$. $L(1, \chi)$ can be expressed by elementary functions (see, e.g., [5, Proposition 8.6]). However, in our case we can compute them simply by taking sufficiently many initial terms of the series

$$L(1, \chi_1) = \sum_{k=0}^{\infty}\left(\frac{1}{5k+1} - \frac{1}{5k+2} - \frac{1}{5k+3} + \frac{1}{5k+4}\right),$$

$$L'(1, \chi_1) = -\sum_{k=0}^{\infty}\left(\frac{\log(5k+1)}{5k+1} - \frac{\log(5k+2)}{5k+2} - \frac{\log(5k+3)}{5k+3} + \frac{\log(5k+4)}{5k+4}\right).$$

We find

$$L(1, \chi_1) = 0.430408...$$

and

$$L'(1, \chi_1) = 0.356240....$$

Similarly, we have

$$L(1, \chi_2) = 0.864806... + i0.204153...$$

$$L'(1, \chi_2) = 0.154556... - i0.044165...$$

$$L(1, \chi_3) = \overline{L(1, \chi_2)}, \qquad L'(1, \chi_3) = \overline{L'(1, \chi_2)}.$$

Consequently

$$\frac{L'}{L}(1, \chi_1) = 0.827679...$$

$$\frac{L'}{L}(1, \chi_2) = 0.157864... - i0.088336... \tag{12}$$

$$\frac{L'}{L}(1, \chi_3) = \overline{\frac{L'}{L}(1, \chi_2)}. \tag{13}$$

LEMMA 2. *Let*

$$B_j = -\tfrac{1}{4} \sum_{\chi (\text{mod } 5)} \overline{\chi(j)}\, B(\chi'), \qquad j = 1, 2, 3, 4,$$

where $B(\chi)$ *are defined by* (3). *Then*

$$B_1 = 0.777858\ldots$$
$$B_2 = 0.420807\ldots$$
$$B_3 = 0.509143\ldots$$
$$B_4 = 0.130067\ldots.$$

Proof. From (3), Lemma 1, and (11) we obtain

$$B_1 = -\frac{1}{4}\left(-3C + 3\log 5 - 4\log(2\pi) + \frac{L'}{L}(1,\chi_1) + \frac{L'}{L}(1,\chi_2) + \frac{L'}{L}(1,\chi_3)\right),$$

$$B_2 = -\frac{1}{4}\left(C - \log 5 - \frac{L'}{L}(1,\chi_1) + i\left(\frac{L'}{L}(1,\chi_2) - \frac{L'}{L}(1,\chi_3)\right)\right),$$

$$B_3 = -\frac{1}{4}\left(C - \log 5 - \frac{L'}{L}(1,\chi_1) - i\left(\frac{L'}{L}(1,\chi_2) - \frac{L'}{L}(1,\chi_3)\right)\right),$$

$$B_4 = -\frac{1}{4}\left(C - \log 5 + \frac{L'}{L}(1,\chi_1) - \frac{L'}{L}(1,\chi_2) - \frac{L'}{L}(1,\chi_3)\right),$$

which together with (12) and some numerical computations give the assertion.

Let us write

$$F_j(z) = -\tfrac{1}{2} e^{-z/2} \sum_{\chi (\mathrm{mod}\ 5)} \overline{\chi(j)}\, K(z, \chi'), \qquad j = 1, 2, 3, 4, \quad z \in H, \tag{14}$$

where χ' denotes the primitive character induced by χ. Moreover for real x let

$$\begin{aligned} P_j(x) &= \lim_{y \to 0^+} \Re F_j(x + iy) \\ &= -\tfrac{1}{4} e^{-x/2} \sum_{\chi (\mathrm{mod}\ 5)} \overline{\chi(j)}\, F(x, \chi'), \end{aligned} \tag{15}$$

$$[x]_0 = \tfrac{1}{2}([x+0] + [x-0]),$$

$$\psi_0(x, q, a) = \tfrac{1}{2}(\psi(x+0, q, a) + \psi(x-0, q, a)).$$

LEMMA 3. *Writing*

$$Q(x) = \frac{1}{4}\left(\log 5 \left[\frac{x}{\log 5}\right]_0 - x - e^x\right),$$

we have for $x > 0$

$$P_1(x) = e^{-x/2}(\psi_0(e^x, 5, 1) + Q(x) + \tfrac{1}{2}\log(e^x - 1) + B_1),$$
$$P_2(x) = e^{-x/2}(\psi_0(e^x, 5, 2) + Q(x) + B_2),$$
$$P_3(x) = e^{-x/2}(\psi_0(e^x, 5, 3) + Q(x) + B_3),$$
$$P_4(x) = e^{-x/2}(\psi_0(e^x, 5, 4) + Q(x) + \tfrac{1}{2}\log(e^x + 1) + B_4),$$

with B_j*'s as in Lemma 2.*

Proof. This follows at once from (2) and (15).

COROLLARY 1. *For* $x \to \infty$ *we have*

$$P_j(x) = e^{-x/2}\psi_0(e^x, 5, j) - \tfrac{1}{4}e^{x/2} + O(xe^{-x/2}).$$

Proof. We have $Q(x) = -\tfrac{1}{4}e^x + O(x)$.

4. PROOF OF THEOREM 2

The main argument we want to use in this proof is a general result concerning boundary values of generalized Dirichlet series proved in [3]. To make its formulation simpler let us denote by $\mathscr{B}$ the set of all functions

$$F(z) = \sum_{n=1}^{\infty} \alpha_n e^{iw_n z}, \qquad z = x + iy, \quad y > 0, \tag{16}$$

satisfying the following conditions.

1. $0 \leqslant w_1 \leqslant w_2 \leqslant \cdots$ are real numbers.
2. $\alpha_n \in C$, $n = 1, 2, 3, \ldots$.
3. The series in (16) converges absolutely for $y > 0$.
4. The limit

$$P(x) = \lim_{y \to 0^+} P(x + iy),$$

where $P(x + iy) = \Re F(x + iy)$, $y > 0$, exists for almost all real x. (Putting $P(x) = 0$ for the remaining x we get P well defined on the closed upper half-plane $\bar{H} = \{z \in C : \Im z \geqslant 0\}$.)

5. We have

$$\lim_{y \to 0^+} \sup_{x \in R} \int_{-1/2}^{1/2} |P(x + t) - P(x + t + iy)|^2 \, dt = 0.$$

LEMMA 4 (see [3, Lemma 3]). *Let* $F_j \in \mathscr{B}$ *for* $j = 1, 2, \ldots, n$ *and let* $x_0 \in R$ *be a continuity point of the mapping*

$$P : \bar{H} \ni z \mapsto (P_1(z), P_2(z), \ldots, P_n(z)) \in R^n, \qquad P_j = \Re F_j, j = 1, 2, \ldots, n.$$

Then for every open neighborhood $U \subset R^n$ *of* $P(x_0)$ *there exist constants* $b_0 = b_0(U) > 0$ *and* $l_0 = l_0(U) > 0$ *such that*

$$\mu(P^{-1}(U) \cap \mathscr{J}) > b_0,$$

for an arbitrary interval $\mathscr{J} \subset R$ *of length* $\geqslant l_0$, μ *being the Lebesgue measure on the real axis.*

We apply this lemma to the functions F_j, $j = 1, 2, 3, 4$, defined in (14). The F_j's belong to $\mathscr{B}$, as has been proved in [3].

Let

$$\tilde{P}_j(x) = e^{x/2} P_j(x) - Q(x).$$

Let $\sigma = (a_1, a_2, a_3, a_4)$ be any permutation of (1, 2, 3, 4). We say that $x_0 \in R$ *belongs to the permutation* σ, when

$$\tilde{P}_{a_1}(x_0) > \tilde{P}_{a_2}(x_0) > \tilde{P}_{a_3}(x_0) > \tilde{P}_{a_4}(x_0).$$

According to Lemmas 2 and 3, for positive x we have

$$\tilde{P}_1(x) = \psi_0(t, 5, 1) + \tfrac{1}{2}\log(t-1) + 0.777858\ldots,$$
$$\tilde{P}_2(x) = \psi_0(t, 5, 2) + 0.420807\ldots,$$
$$\tilde{P}_3(x) = \psi_0(t, 5, 3) + 0.509143\ldots,$$
$$\tilde{P}_4(x) = \psi_0(t, 5, 4) + \tfrac{1}{2}\log(t+1) + 0.130067\ldots,$$

where $t = e^x$. Using a simple computer program the author computed the values shown in Table I.

We see by Table I that for each of $24 = 4!$ permutations of (1, 2, 3, 4) there exists a real number that belongs to this permutation and is a continuity point of all $\tilde{P}_j$, $j = 1, 2, 3, 4$. Given a permutation $\sigma = (a_1, a_2, a_3, a_4)$ we denote this point by $x_0(\sigma)$. Let us observe that for all σ we have

$$\min_{\substack{1 \leqslant i, j \leqslant 4 \\ i \neq j}} |\tilde{P}_i(x_0(\sigma)) - \tilde{P}_j(x_0(\sigma))| > \tfrac{1}{10}.$$

Hence for arbitrary $\sigma = (a_1, a_2, a_3, a_4)$,

$$P_{a_1}(x_0(\sigma)) > P_{a_2}(x_0(\sigma)) > P_{a_3}(x_0(\sigma)) > P_{a_4}(x_0(\sigma))$$

TABLE I

t	$\tilde{P}_1(x)$	$\tilde{P}_2(x)$	$\tilde{P}_3(x)$	$\tilde{P}_4(x)$	Permutation
337.5	90.5918...	87.7236...	84.6079...	73.7340...	(1, 2, 3, 4)
41.5	12.8671...	11.2957...	8.0013...	10.1083...	(1, 2, 4, 3)
43.5	12.8912...	11.2957...	11.7625...	10.1313...	(1, 3, 2, 4)
16.5	5.2393...	3.0598...	4.8658...	3.3529...	(1, 3, 4, 2)
191.5	50.4697...	48.5365...	44.7444...	48.8203...	(1, 4, 2, 3)
62.5	17.1868...	15.1459...	15.7328...	16.3325...	(1, 4, 3, 2)
18.5	5.3000...	5.8930...	4.8658...	3.4070...	(2, 1, 3, 4)
49.5	12.9572...	15.1459...	11.7625...	12.1404...	(2, 1, 4, 3)
353.5	90.6150...	93.5729...	92.4203...	79.6121...	(2, 3, 1, 4)
8.5	1.7853...	3.0598...	2.3009...	1.9488...	(2, 3, 4, 1)
37.5	9.1015...	11.2957...	8.0013...	10.0588...	(2, 4, 1, 3)
617.5	149.5666...	162.8079...	154.2089...	156.4094...	(2, 4, 3, 1)
1164.5	291.8992...	291.5804...	296.7735...	287.1499...	(3, 1, 2, 4)
13.5	4.4386...	3.0598...	4.8658...	3.2589...	(3, 1, 4, 2)
53.5	12.9968...	15.1459...	15.7328...	12.1785...	(3, 2, 1, 4)
28.5	5.5259...	6.9916...	8.0013...	6.5584...	(3, 2, 4, 1)
121.5	29.8974...	28.5981...	33.8042...	30.9036...	(3, 4, 1, 2)
23.5	5.4256...	5.8930...	8.0013...	6.4656...	(3, 4, 2, 1)
151.5	39.9011...	38.3623...	34.4974...	40.9515...	(4, 1, 2, 3)
32.5	9.0278...	7.6848...	8.0013...	9.9893...	(4, 1, 3, 2)
19.5	5.3277...	5.8930...	4.8658...	6.3764...	(4, 2, 1, 3)
179.5	39.9864...	48.5365...	44.7444...	48.7882...	(4, 2, 3, 1)
89.5	22.7301...	19.3505...	24.4421...	26.0609...	(4, 3, 1, 2)
29.5	5.5438...	6.9916...	8.0013...	9.9424...	(4, 3, 2, 1)

and

$$\min_{\substack{1 \leqslant i,j \leqslant 4 \\ i \neq j}} |P_i(x_0(\sigma)) - P_j(x_0(\sigma))| \geqslant 1/(10\sqrt{\max_\sigma x_0(\sigma)}) = 4b_0.$$

Applying Lemma 4 to the open set

$$U_\sigma = \{(x_1, x_2, x_3, x_4) \in R^4 : \max_{1 \leqslant j \leqslant 4} |x_j - P_j(x_0(\sigma))| < b_0\},$$

we see that there exist two constants $l_0 > 0$, $b_1 > 0$ such that

$$\mu\{T \leqslant x \leqslant T + l_0 : (P_1(x), P_2(x), P_3(x), P_4(x)) \in U_\sigma\} \geqslant b_1. \tag{17}$$

By Corollary 1

$$(P_1(x), P_2(x), P_3(x), P_4(x)) \in U_\sigma$$

implies

$$\psi(e^x, 5, a_1) > \psi(e^x, 5, a_2) > \psi(e^x, 5, a_3) > \psi(e^x, 5, a_4)$$

and

$$\min_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} |\psi(e^x, 5, i) - \psi(e^x, 5, j)| \geq b_0 e^{x/2}$$

for sufficiently large x. Hence Theorem 2 follows from (17) by the change of variable $t = e^x$ and the obvious remark that $\psi(t, 5, a) = \psi([t], 5, a)$.

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