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The k-functions in multiplicative number theory, II
Uniform distribution of zeta zeros

by

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1. Introduction and statement of results. The uniform distribution (mod 1) of imaginary parts of zeros of Riemann zeta and Dirichlet L-functions has been considered by E. Hlawka, P. D. T. A. Elliott, A. Fujii, H. Rademacher and others. The theory of k-functions, developed in the first part of this cycle of papers, provides a useful tool for the study of this subject in an alternative way. Instead of the classical uniform distribution (mod 1) in H. Weyl sense we consider one of its generalizations, more directly related to k-functions.

Let $\chi(\bmod q)$, $q \geqslant 1$, denote a primitive Dirichlet character and let $0 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots$ denote positive imaginary parts of non-trivial zeros of the corresponding Dirichlet L-function (each $\gamma = \operatorname{Im}\varrho$ occurs in this sequence according to the multiplicity of ϱ). Let $A = [a_{nk}]$ denote the positive Toeplitz matrix defined by

$$a_{nk} = \frac{1}{n!\,S_n} e^{-\gamma_k}\gamma_k^n, \quad n \geqslant 1,\ k \geqslant 1, \tag{1.1}$$

where

$$S_n = \frac{1}{n!}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n. \tag{1.2}$$

This matrix defines a certain summation method and uniform distribution (mod 1) (matrix method A and A-uniform distribution (mod 1) resp., cf. [7]), which are the focus of attention in this paper.

THEOREM 1. *If a bounded sequence (t_k) of complex numbers has A-limit g, then it is summable by arithmetic means to the value g. In particular, if (t_k) is A-uniformly distributed* (mod 1) *then it is also uniformly distributed* (mod 1) *in H. Weyl sense.*

Suppose S is a summation method. We say that S-uniform distribution (mod 1) is *of type* τ when for every $\tau' < \tau$ and every S-uniformly distributed (mod 1) sequence (t_k), every sequence (t'_k) satisfying

$$\sum_{\substack{1 \leqslant k \leqslant N \\ t_k \neq t'_k}} 1 \ll N^{\tau'} \quad \text{as } N \to \infty$$

is $\dot{S}$-uniformly distributed (mod 1) as well. Moreover, τ is the maximal number with this property. It can easily be seen that always $0 \leqslant \tau \leqslant 1$.

THEOREM 2. *Weyl uniform distribution* (mod 1) *is of type* 1, *whereas A-uniform distribution* (mod 1) *is of type* 1/2.

By this theorem an A-uniformly distributed (mod 1) sequence (t_k), $0 \leqslant t_k < 1$, has to be more "regular" than a sequence uniformly distributed (mod 1) in the classical sense. In particular, not only any number N of its initial terms, but also every subsequence (t_k), $N \leqslant k \leqslant N+N^{1/2+\varepsilon}$, have to fill approximately uniformly the unit interval [0, 1). Theorem 2 implies also that there exist uniformly distributed sequences which are not A-uniformly distributed. The general scheme of construction of such examples is provided by the proof of Theorem 2 (see §6 below).

The basic theorem which justifies the introduction of A-uniform distribution mod 1 is the following.

THEOREM 3. *For every non-zero real number x the sequence $x\gamma_k$, $k = 1, 2, 3, \ldots$, is A-uniformly distributed* (mod 1).

COROLLARY. *Every sequence $x\gamma_k$, $k = 1, 2, 3, \ldots, x \neq 0$, is uniformly distributed* (mod 1) *in H. Weyl sense.*

The assertion of this corollary has been first proved by H. Rademacher (cf. [10]) in a special case (for the Riemann zeta function) and under the Riemann hypothesis. The unconditional result belongs to E. Hlawka [4] (cf. also A. Fujii [2], [3], P. D. T. A. Elliott [1]).

Remarks on notation. Generally we keep the notation used in the first part of this cycle [6].

Non-trivial zeros of $L(s, \chi)$ lying in the upper half-plane are denoted by $\varrho_k = \beta_k + i\gamma_k$, $k = 1, 2, 3, \ldots$ We have $0 < \beta_k < 1$ and $0 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots$ $N(T, \chi)$ denotes the number of ϱ_k satisfying $\gamma_k \leqslant T$.

$$\int_l f(z)\,dz, \qquad l = l(a, b), \qquad a, b \in \mathbf{R}$$

denotes an integral taken along a smooth and simple curve lying on the upper half-plane, joining a to b and such that f is regular on l and also regular on the open domain between l and the real axis.

p always denotes a prime number.

2. k-functions. In order to make this paper as self-contained as possible, we quote explicitly some basic facts on k-functions, proved (in more general form) in [6].

LEMMA 1. *For $z = x+iy$, $y > 0$ and the primitive Dirichlet character $\chi(\mathrm{mod}\, q)$, $q \geqslant 1$, we define the k-function by the formula*

$$k(z, \chi) = \sum_{k=1}^{\infty} e^{\varrho_k z}. \tag{2.1}$$

It is holomorphic on the upper half-plane and can be continued analytically to a meromorphic function on D_1, the complex plane with the slit $(-i\infty, 0]$. More explicitly, for $z \in D_1$ we have

$$2\pi i \cdot k(z, \chi) = N_0(z, \chi) + \frac{e^z E_1(\tfrac{3}{2}z) + e^{2z} E_1(\tfrac{5}{2}z) + H(z, d)}{1-e^{2z}} + h(z, \chi), \tag{2.2}$$

where

$$N_0(z, \chi) = e^{3z/2} \sum_{n \geqslant 2} \frac{\Lambda(n)\chi(n)}{n^{3/2}(z-\log n)} + e^{-z/2} \sum_{n \geqslant 2} \frac{\Lambda(n)\bar{\chi}(n)}{n^{3/2}(z+\log n)} + \frac{1}{z} e^{-z/2} \log\frac{\pi}{q}, \tag{2.3}$$

$$h(z, \chi) = \frac{1}{2\pi i} \int_l \frac{L'}{L}(s, \chi) e^{sz}\, ds, \qquad l = l(-1/2, 3/2), \tag{2.4}$$

d denotes 0 or 1 so that $\chi(-1) = (-1)^d$,

$$H(z, 0) = e^{2z} \int_{l_1} \frac{\Gamma'}{\Gamma}(w) e^{2wz}\, dw + e^{3z} \int_{3/4}^{7/4} \frac{\Gamma'}{\Gamma}(w) e^{-2wz}\, dw, \tag{2.5}$$

$$H(z, 1) = e^{z} \int_{l_2} \frac{\Gamma'}{\Gamma}(w) e^{2wz}\, dw + e^{4z} \int_{5/4}^{9/4} \frac{\Gamma'}{\Gamma}(w) e^{-2wz}\, dw, \tag{2.6}$$

$l_1 = l(-5/4, -1/4)$, $l_2 = l(-3/4, 1/4)$, E_1 denotes that single-valued branch of the modified integral exponential function $-\mathrm{Ei}(-z)$ which for $y > 0$ is given by

$$E_1(z) = \int_z^{\infty} \frac{e^{-w}}{w}\, dw,$$

the path of integration being the half-line $w = z+x$, $x \geqslant 0$. Moreover, for $z \in D_1$ we have

$$k(z, \chi) = \frac{1}{2\pi i} \frac{e^z}{e^z - 1} \log z + \frac{A(\chi)}{2\pi i} \frac{1}{z} + W(z, \chi), \tag{2.7}$$

where W is meromorphic on $\mathbf{C}$ with simple poles at $z = k \log p$, $k \in \mathbf{Z} \setminus \{0\}$, $p \nmid q$ and $A(\chi) = \log(2\pi/q) + C - i\pi/2$, C is the Euler constant.

LEMMA 2. *Let D_2 denote the region of the complex plane consisting of all points $z = x+iy$ satisfying $|x| \geqslant 1/4$, $y \geqslant -1$ or $|x| \leqslant 1/4$, $y \geqslant 1$. Then the inequality*

$$|e^{-z/2} k(z, \chi)| \ll \delta^{-1} e^{|x|} \tag{2.8}$$

holds for $z \in D_2$ whenever $|z - m \log p| \geqslant \delta$ for all integers m, primes $p \nmid q$ and a fixed positive δ, $0 < \delta < 1$.

Proof. For $z \in D_2$, $y \geqslant 1$ the assertion is obvious since by (2.1) we have

$$|e^{-z/2} k(z, \chi)| \leqslant \sum_{k=1}^{\infty} e^{(\beta_k - 1/2)x - \gamma_k y} \leqslant e^{|x|/2} \sum_{k=1}^{\infty} e^{-\gamma_k} \ll e^{|x|/2}.$$

Suppose $|x| \geqslant 1/4$, $|y| \leqslant 1$. Then

$$|E_1(z)| = \left| \int_z^{z+2i} \frac{e^{-w}}{w}\, dw + E_1(z+2i) \right| \ll \int_z^{z+2i} e^{-x} |dw| + \int_{z+2i}^{\infty} \frac{e^{-\mathrm{Re}\, w}}{|w|} |dw| \ll e^{-x}.$$

Hence

$$\left|\frac{e^{z/2}E_1(\tfrac{3}{2}z)+e^{3z/2}E_1(\tfrac{5}{2}z)}{1-e^{2z}}\right| \ll e^{|x|}.$$

Also, by (2.3)–(2.6)

$$|e^{-z/2}N_0(z,\chi)| \ll e^x \sum_{\substack{p,m\\ p\nmid q}} \frac{\log p}{p^{3m/2}|z-m\log p|} + e^{-x}\sum_{\substack{p,m\\ p\nmid q}} \frac{\log p}{p^{3m/2}|z+m\log p|} \ll \delta^{-1}e^{|x|},$$

$$|e^{-z/2}h(z,\chi)| \ll e^{|x|}$$

and

$$\left|\frac{e^{-z/2}H(z,d)}{1-e^{2z}}\right| \ll e^{|x|}.$$

Gathering these estimates and using (2.2) we obtain (2.8), which ends the proof.

3. The sums $P_n(x)$. Let us put for real x and natural n

$$P_n(x) = \frac{1}{n!\,i^n}\sum_{k=1}^{\infty}(\varrho_k-1/2)^n e^{(\varrho_k-1/2)(x+i)}.$$

LEMMA 3. *For $|x| \geqslant 1/2$ we have*

$$P_n(x) = -\frac{1}{2\pi i^{n+1}} \sum_{\substack{p,k\\ |k\log p-x|\leqslant 1/8}} \frac{\chi(p^k)\log p}{\sqrt{p^{|k|}}\,(k\log p-(x+i))^{n+1}} + O(d_0^{-n}e^{2|x|}), \tag{3.1}$$

with an absolute $d_0>1$. For $0<|x|<1/2$ we have

$$P_n(x) = O(d_1^{-n}), \quad d_1>1, \tag{3.2}$$

where d_1 and the implied constant depend on x. Moreover,

$$P_n(0) = \frac{1}{2\pi}\log\frac{qn}{2\pi} + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty. \tag{3.3}$$

Proof. Let us observe that

$$P_n(x) = \frac{1}{2\pi i^{n+1}} \int_{K(r)} \frac{e^{-z/2}k(z,\chi)}{(z-(x+i))^{n+1}}\,dz, \tag{3.4}$$

where $K(r)$ denotes the circle $|z-(x+i)| = r$, $0<r<1$.

Suppose, first, $|x|\geqslant 1/2$. One can find x_1 and a positive number b_1 such that $x+1/8 \leqslant x_1 \leqslant x+1/4$ and $|x_1 \pm \log n| \geqslant b_1 e^{-|x|}$, $|2x-x_1\pm\log n| \geqslant b_1 e^{-|x|}$ for every natural n. Let us put $R=\sqrt{1+(x-x_1)^2}>1$. The function $e^{-z/2}k(z,\chi)$ is meromorphic in the disc $|z-(x+i)|<R$ and holomorphic on the circle $K(R)$. Moreover,

$$|e^{-z/2}k(z,\chi)| \ll e^{2|x|}$$

for $z\in K(R)$ (for $|x|\geqslant 3/2$ we use Lemma 2; for $1/2\leqslant|x|\leqslant 3/2$ the estimate is obvious). Hence

$$\left|\frac{1}{2\pi i^{n+1}}\int_{K(R)}\frac{e^{-z/2}k(z,\chi)}{(z-(x+i))^{n+1}}\,dz\right| \ll e^{2|x|}R^{-n}.$$

By Lemma 1, the integrand in (3.4) has inside $K(R)$ simple poles at $z=k\log p$, $k\in Z$, $p\nmid q$, $|k\log p - x| < x_1 - x$ with residues

$$\frac{\chi(p^k)\log p}{2\pi i\sqrt{p^{|k|}}\,(k\log p-(x+i))^{n+1}}.$$

Hence, by the theorem of residues, we get

$$\begin{aligned} P_n(x) &= -\frac{1}{2\pi i^{n+1}} \sum_{\substack{p,k\\ |k\log p-x|<x_1-x}} \frac{\chi(p^k)\log p}{\sqrt{p^{|k|}}\,(k\log p-(x+i))^{n+1}} + O(e^{2|x|}R^{-n}) \\ &= -\frac{1}{2\pi i^{n+1}} \sum_{\substack{p,k\\ |k\log p-x|\leqslant 1/8}} \frac{\chi(p^k)\log p}{\sqrt{p^{|k|}}\,(k\log p-(x+i))^{n+1}} + O(e^{2|x|}d_0^{-n}), \end{aligned}$$

where $d_0=\sqrt{1+1/64}$; (3.1) therefore follows.

For $0<|x|<1/2$ we can proceed analogously. In this case the integrand in (3.4) is regular for $|z-(x+i)|<d_1$, where $d_1=\sqrt{1+a^2}$, $a=\min(|x|, |x+\log 2|, |x-\log 2|)$. By the Cauchy theorem, the integral (3.4) can be taken round the circle $K(d_1)$. Since $e^{-z/2}k(z,\chi)$ is bounded on $K(d_1)$, (3.2) follows.

Suppose $x=0$. By (2.7) we can write for $z\in D_1$

$$e^{-z/2}k(z,\chi) = W_1(z,\chi) + \frac{A(\chi)}{2\pi iz} + f(z)\log z + \frac{1}{2\pi i}\frac{\log z}{z},$$

where f is an entire function and W_1 is holomorphic for $|z-i|\leqslant\sqrt{1+\log^2 2}$. Hence, the integral (3.4) is split into four parts I_1, I_2, I_3, I_4, say. By the Cauchy theorem I_1 can be taken round the circle $|z-i|=R_1$, $1<R_1<\sqrt{1+\log^2 2}$. This yields

$$|I_1| = O(R_1^{-n}) = O(1/n).$$

Similarly, by the theorem of residues,

$$I_2 = -\frac{1}{2\pi}\log(2\pi/q) - C/2\pi + i/4 + O(1/n).$$

Again by the Cauchy theorem, I_3 can be taken round the contour consisting of the imaginary axis from $z = -i\delta$, $\delta > 0$, to $z = -i(R_1-1)$, the arc of $z = |R_1|$, $-\pi/2 \leqslant \arg z \leqslant 3\pi/2$, the imaginary axis from $-i(R_1-1)$ to $z = -i\delta$ and the arc of $|z| = \delta$ back to the starting-point. The integral along $|z| = \delta$ tends to 0 as $\delta \to 0^+$, and that along $|z| = R_1$ is $O(R_1^{-n}) = O(1/n)$. Hence

$$|I_3| = \left|\int_0^{R_1-1} \frac{f(-it)}{(t+1)^{n+1}}\,dt + O(1/n)\right| = O\left(\int_0^{\infty} \frac{dt}{(t+1)^{n+1}} + \frac{1}{n}\right) = O(1/n).$$

Moreover,

$$I_4 = \frac{1}{2\pi i^{n+1}} \frac{1}{n!} \frac{d^n}{dz^n}\left(\frac{\log z}{z}\right)\bigg|_{z=i} = \frac{1}{2\pi}\sum_{k=1}^{n}\frac{1}{k} + \frac{i}{4} = \frac{1}{2\pi}\log n + \frac{C}{2\pi} - \frac{i}{4} + O\left(\frac{1}{n}\right).$$

Therefore

$$P_n(0) = \sum_{j=1}^{4} I_j = \frac{1}{2\pi}\log\frac{qn}{2\pi} + O\left(\frac{1}{n}\right),$$

which ends the proof.

4. The sums $R_n(x)$. Let for $x \in \mathbf{R}$, $n \geqslant 1$,

$$R_n(x) = \frac{1}{n!}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n e^{(\beta_k - 1/2)x} e^{i\gamma_k x}. \tag{4.1}$$

LEMMA 4. *We have*

$$\left|\frac{1}{n!}\sum_{|\gamma_k - n| > 2\sqrt{n\log n}} (\varrho_k - 1/2)^n e^{(\varrho_k - 1/2)(x+i)}\right| \ll \frac{e^{|x|/2}}{n}, \tag{4.2}$$

$$\left|\frac{1}{n!}\sum_{|\gamma_k - n| > 2\sqrt{n\log n}} e^{-\gamma_k}\gamma_k^n e^{(\beta_k - 1/2)x}\right| \ll \frac{e^{|x|/2}}{n}. \tag{4.3}$$

Proof. We prove (4.2) only; the proof of (4.3) is similar. The parts of (4.2) corresponding to $0 < \gamma_k < n - 2\sqrt{n\log n}$ or $n + 2\sqrt{n\log n} < \gamma_k < 2n$ contribute at most

$$\frac{1}{n!} e^{|x|/2} e^{-n \pm 2\sqrt{n\log n}} (n \mp 2\sqrt{n\log n} + 1)^n N(2n, \chi)$$

$$\ll e^{|x|/2}\sqrt{n}\log n \cdot \exp\left\{\pm 2\sqrt{n\log n} + n\log\left(1 \mp \frac{2\sqrt{n\log n} - 1}{n}\right)\right\}$$

$$\ll e^{|x|/2}\sqrt{n}\log n \exp\{-2\log n\} \ll e^{|x|/2}/n.$$

The estimate of the sum over the range $2n \leqslant \gamma_k \leqslant n^4$ can be performed similarly. The remaining part of the sum can be treated as follows:

$$e^{|x|/2}\frac{1}{n!}\sum_{\gamma_k > n^4} e^{-\gamma_k}(\gamma_k + 1)^n \ll e^{|x|/2} n^2 \sum_{\gamma_k > n^4} \gamma_k^{-2} \ll \frac{e^{|x|/2}}{n}.$$

The proof is complete.

LEMMA 5. *For S_n given by* (1.2) *we have*

$$S_n \ll (\log n)^{3/2} \quad \textit{for } n \geqslant 2.$$

Proof. Lemma 4 yields

$$S_n = \frac{1}{n!}\sum_{|\gamma_k - n| \leqslant 2\sqrt{n\log n}} e^{-\gamma_k}\gamma_k^n + O(1/n)$$

$$\ll n^{-1/2}\left(N(n + 2\sqrt{n\log n}, \chi) - N(n - 2\sqrt{n\log n}, \chi)\right) + O(1/n) \ll (\log n)^{3/2}.$$

LEMMA 6. *For $x \in \mathbf{R}$ we have*

$$P_n(x) = R_n(x) + O(e^{|x|/2} n^{-1/2}\log^2 n) \quad \textit{as } n \to \infty. \tag{4.4}$$

Proof. By Lemmas 4 and 5,

$$|P_n(x) - R_n(x)|$$

$$\ll \frac{1}{n!}\sum_{|\gamma_k - n| \leqslant 2\sqrt{n\log n}} e^{-\gamma_k}\gamma_k^n e^{(\beta_k - 1/2)x}\left|\exp\left\{(\beta_k - 1/2)i + n\log\left(1 - i\frac{\beta_k - 1/2}{\gamma_k}\right)\right\} - 1\right| + O\left(\frac{e^{|x|/2}}{n}\right)$$

$$\ll e^{|x|/2} S_n \sqrt{\frac{\log n}{n}} + \frac{e^{|x|/2}}{n} \ll e^{|x|/2} n^{-1/2}\log^2 n,$$

as required.

By Lemmas 3–6 we get the following result.

LEMMA 7. *For $|x| \geqslant 1/2$,*

$$R_n(x) = -\frac{1}{2\pi i^{n+1}}\sum_{\substack{p,k \\ |k\log p - x| \leqslant 1/8}} \frac{\chi(p^k)\log p}{\sqrt{p^{|k|}}(k\log p - (x+i))^{n+1}} + O(e^{2|x|} n^{-1/2}\log^2 n). \tag{4.5}$$

For $0 < |x| < 1/2$,

$$R_n(x) = O(n^{-1/2}\log^2 n), \tag{4.6}$$

where the implied constant depends on x. In particular, for every $x \neq 0$, we have $R_n(x) = O(1)$ as $n \to \infty$. Moreover,

$$S_n = R_n(0) = \frac{1}{2\pi}\log\frac{qn}{2\pi} + O(n^{-1/2}\log^2 n). \tag{4.7}$$

Let us remark here that the foregoing lemmas provide stronger estimates than those needed just for the proofs of our theorems. They, however, will be used in part III of this cycle.

5. Proof of Theorem 1. Let $(g_n) = A(t_n)$, i.e.

$$g_n = \frac{1}{n!\,S_n}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n t_k.$$

Since $g_n \to g$ as $n \to \infty$ we have also

$$a(T) \to g, \quad \text{as } T \to \infty, \tag{5.1}$$

where

$$a(T) = \frac{1}{T}\sum_{n \leqslant T} g_n.$$

Let us fix a positive ε, $0 < \varepsilon < 1$. By Lemma 7,

$$S_n = \frac{1}{2\pi}\log T + O_\varepsilon(1) \quad \text{for } \varepsilon T \leqslant n \leqslant T.$$

Hence

$$\begin{aligned} a(T) &= \frac{1}{T}\sum_{\varepsilon T \leqslant n \leqslant T} g_n + O(\varepsilon) = \frac{2\pi}{T\log T}\sum_{k=1}^{\infty} f(\gamma_k, T, n)\, t_k + O(\varepsilon) \\ &= \frac{1}{N(T, \chi)}\sum_{k=1}^{\infty} f(\gamma_k, T, n)\, t_k + O(\varepsilon), \end{aligned} \tag{5.2}$$

where

$$f(\gamma, T, n) = e^{-\gamma}\sum_{\varepsilon T \leqslant n \leqslant T}\frac{\gamma^n}{n!}.$$

Since $0 \leqslant f(\gamma, T, n) \leqslant 1$ and for sufficiently large T, we have $N((1+\varepsilon)T, \chi) - N((1-\varepsilon)T, \chi) \ll \varepsilon N(T, \chi)$ and $N(\varepsilon\varepsilon T, \chi) \ll \varepsilon N(T, \chi)$, the terms in the last sum of (5.2), corresponding to $(1-\varepsilon)T \leqslant \gamma_k \leqslant (1+\varepsilon)T$ or $\gamma_k \leqslant e\varepsilon T$, contribute at most $O(\varepsilon)$. For $e\varepsilon T < \gamma_k < (1-\varepsilon)T$ we have, by Taylor's formula,

$$\begin{aligned} f(\gamma_k, T, n) &= 1 - \frac{1}{[T]!}\int_0^{\gamma_k} e^{-u}u^{[T]}\,du + O\big(\sqrt{T}\exp(-\varepsilon T/2)\big) \\ &= 1 + O\big(\sqrt{T}\exp(-\varepsilon^2 T/2)\big) = 1 + O(\varepsilon), \end{aligned}$$

for sufficiently large T. For $\gamma_k > (1+\varepsilon)T$, $T \geqslant T_0(\varepsilon)$, we have

$$f(\gamma_k, T, n) \ll Te^{-\gamma_k}\frac{\gamma_k^{[T]}}{[T]!} \ll \gamma_k^{-2}.$$

Thus for large T we get

$$\begin{aligned} a(T) &= \frac{1}{N(T, \chi)}\sum_{e\varepsilon T < \gamma_k < (1-\varepsilon)T}\big(1 + O(\varepsilon)\big)t_k + \frac{1}{N(T, \chi)}\sum_{\gamma_k > (1+\varepsilon)T} O(\gamma_k^{-2}) + O(\varepsilon) \\ &= \frac{1}{N(T, \chi)}\sum_{\gamma_k \leqslant T} t_k + O(\varepsilon), \end{aligned}$$

which, together with (5.1), ends the proof.

6. Proof of Theorem 2. Let (t_n) be uniformly distributed (mod 1) in the sense of H. Weyl. Let us fix $\tau' < 1$ and let (t_n') satisfy the condition

$$\sum_{\substack{n \leqslant N \\ t_n' \neq t_n}} 1 \ll N^{\tau'} \quad \text{as } N \to \infty. \tag{6.1}$$

Hence we have for every integer $m \neq 0$

$$\frac{1}{N}\sum_{n \leqslant N} e^{2\pi i m t_n'} = \frac{1}{N}\sum_{n \leqslant N} e^{2\pi i m t_n} + O(N^{\tau' - 1}) = o(1),$$

which, by Weyl's criterion, means that (t_n') is uniformly distributed (mod 1). Since $\tau' < 1$ was arbitrary this implies that Weyl's uniform distribution is of type 1.

Let now (t_n) be A-uniformly distributed (mod 1). For every sequence (t_n') satisfying (6.1) with $\tau' < 1/2$ we have, by (4.7) and Lemma 4,

$$\begin{aligned} \frac{1}{n!\,S_n}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n e^{2\pi i m t_k'} &= \frac{1}{n!\,S_n}\sum_{|\gamma_k - n| \leqslant 2\sqrt{n\log n}} e^{-\gamma_k}\gamma_k^n e^{2\pi i m t_k'} + O\left(\frac{1}{n}\right) \\ &= \frac{1}{n!\,S_n}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n e^{2\pi i m t_k} + O\left(\frac{n^{\tau'}e^n}{n^n\sqrt{n}\log n}e^{-n}n^n\right) + O\left(\frac{1}{n}\right) \\ &= o(1) \end{aligned}$$

for every integer $m \neq 0$ and $n \to \infty$. Hence, by the generalized Weyl criterion (cf. [7], Th. 7.11), the sequence (t_n') is A-uniformly distributed (mod 1).

If now $\tau' > 1/2$, (t_k) is A-uniformly distributed (mod 1) and the sequence (t_k') is defined by

$$t_k' = \begin{cases} t_k & \text{if } |\gamma_k - M_n| > M_n^{\tau'}/\log M_n \quad \text{for all } M_n, \\ 0 & \text{otherwise,} \end{cases}$$

where (M_n) is a sequence of natural numbers such that $e^{M_n} \leqslant M_{n+1}$, for $n \geqslant 1$, then

$$\sum_{\substack{k \leqslant N \\ t_k \neq t_k'}} 1 \ll \sum_{M_n \leqslant 2\gamma_N}\frac{M_n^{\tau'}}{\log M_n} \ll N^{\tau'}$$

and

$$\frac{1}{M_n! S_{M_n}} \sum_{k=1}^{\infty} e^{-\gamma_k} \gamma_k^{M_n} e^{2\pi i t_k'}$$
$$= \frac{1}{M_n! S_{M_n}} \sum_{|\gamma_k - M_n| \leqslant 2\sqrt{M_n \log M_n}} e^{-\gamma_k} \gamma_k^{M_n} + O(M_n^{-1}) = 1 + O(M_n^{-1}).$$

This means that (t_k') is not A-uniformly distributed (mod 1). Hence A-uniform distribution is of type $\tau = 1/2$, and the result follows.

7. A density estimate. For real σ, T, H satisfying $1/2 \leqslant \sigma \leqslant 1$, $T > 0$, $0 < H < T$ let us denote by $N(\sigma, T, H, \chi)$ the number of zeros of $L(s, \chi)$ such that $\beta \geqslant \sigma$, $|\gamma - T| \leqslant H$.

LEMMA 8. *Let a denote a real number such that*

(7.1) $$\int_{T-G}^{T+G} |L(\tfrac{1}{2}+it, \chi)|^2 \, dt \ll G \log^2 T \quad \text{for } G \geqslant T^{a+\varepsilon}, \ \varepsilon > 0.$$

Then for every $T \geqslant e$, $H \geqslant T^{a+\varepsilon}$ we have

(7.2) $$N(\sigma, T, H, \chi) \ll H^{4(1-\sigma)/(3-2\sigma)} \log^7 T.$$

It is known that (7.1) is fulfilled for $a = 1/3$ (cf. [8]). For the Riemann zeta function we can even take $a = 35/108$ (see [5]). For our purposes it is sufficient to know that $a < 1/2$ (= the type of A-uniform distribution (mod 1)).

Proof. Since the method depends on Montgomery's technique [9] and is well known, we just indicate the main points. Let

$$Y = H^{2/(3-2\sigma)}, \quad M(s, \chi) = \sum_{n \leqslant H} \mu(n) \chi(n) n^{-s}.$$

For a $(\log T)^{-2}$ proportion of zeros under consideration we have $|\gamma - \gamma'| \geqslant 1$ for $\gamma \neq \gamma'$ and

(7.3) $$\Big| \sum_{U \leqslant n \leqslant 2U} a_n n^{-\varrho} \Big| \gg (\log T)^{-1}$$

or

(7.4) $$Y^{1/2-\sigma} \int_{\gamma - c_0 \log T}^{\gamma + c_0 \log T} |L(1/2+it, \chi) M(1/2+it, \chi)| \, dt \geqslant 1/10,$$

where $|a_n| \leqslant d(n)$, c_0 denotes an absolute constant and $U \in [H, 2Y \log Y]$. Squaring (7.3), summing over zeros and using the classical mean-value inequality for Dirichlet polynomials it can easily be seen that the number of zeros satisfying (7.3) is bounded by

$$Y^{2(1-\sigma)} \log^5 T = H^{4(1-\sigma)/(3-2\sigma)} \log^5 T.$$

Next, summing (7.4) over zeros, using the Cauchy–Schwarz inequality, the mean-value theorem for Dirichlet polynomials and (7.1) we see that the number of zeros satisfying (7.4) is bounded by

$$Y^{1/2-\sigma} H \log^4 T = H^{4(1-\sigma)/(3-2\sigma)} \log^4 T$$

and the result follows.

8. Proof of Theorem 3. By the generalized Weyl criterion (cf. [7], Th. 7.11) it is sufficient to prove that for every $x \neq 0$

(8.1) $$\lim_{n \to \infty} \frac{S_n(x)}{S_n} = 0,$$

where

$$S_n(x) = \frac{1}{n!} \sum_{k=1}^{\infty} e^{-\gamma_k} \gamma_k^n e^{i\gamma_k x} = S_n A(e^{i\gamma_k x}).$$

Let $\varepsilon > 0$. For fixed $x \neq 0$ we can write using, among others, Lemma 4,

(8.2) $$|R_n(x) - S_n(x)| \ll \frac{1}{n!} \sum_{k \in U} e^{-\gamma_k} \gamma_k^n |e^{(\beta_k - 1/2)|x|} - 1| + \frac{1}{\sqrt{n}} \operatorname{card} V + \frac{1}{n},$$

where

$$U = \{k \geqslant 1 : 1/2 \leqslant \beta_k \leqslant 1/2 + \varepsilon\},$$
$$V = \{k \geqslant 1 : |\gamma_k - n| \leqslant 2\sqrt{n \log n}, \ \beta_k \geqslant 1/2 + \varepsilon\}.$$

For $k \in U$ we have $|\exp((\beta_k - 1/2)|x|) - 1| = O(\varepsilon)$; hence the sum in (8.2) is $O(\varepsilon S_n)$. By Lemma 8, the cardinality of V is at most

$$O(n^{(1/2-\varepsilon)/(1-\varepsilon)} \log^{7.5} n);$$

hence the second summand on the right-hand side of (8.2) is bounded by

$$O(n^{-\varepsilon/(2(1-\varepsilon))} \log^{7.5} n).$$

Therefore, $S_n(x) = R_n(x) + o(S_n)$ as n tends to infinity. Since, by Lemma 7, $R_n(x) = O(1)$, we get $S_n(x) = o(S_n)$, which is equivalent to (8.1). The proof is complete.

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Гипотеза Римана и экстремальные значения функции $Z(t)$

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1. Формулировка основных результатов. Пусть $\{\gamma\}$, $\gamma > 0$ — возрастающая последовательность корней уравнения $Z(t) = 0$ (кратность корня не учитывается), где

$$Z(t) = e^{i\vartheta(t)}\zeta(\tfrac{1}{2}+it),$$
$$\vartheta(t) = -\tfrac{1}{2}t\ln\pi + \operatorname{Im}\{\ln\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\} = \vartheta_1(t)+O(1/t), \tag{1}$$
$$\vartheta_1(t) = \tfrac{1}{2}t\ln(t/2\pi) - \tfrac{1}{2}t - \tfrac{1}{8}\pi,$$

(см. [11], стр. 94, 383).

Пусть $\{t_0\}$, $t_0 > 0$ — возрастающая последовательность корней нечетного порядка уравнения $Z'(t) = 0$, удовлетворяющих условию $\gamma' < t_0 < \gamma''$, где γ', γ'' — соседние члены последовательности $\{\gamma\}$. Значит, t_0 — точка локального экстремума функции $Z(t)$, для которой $Z(t_0) \neq 0$. Последовательность $\{t_0\}$ автор начал изучать в работе [2] и продолжал ее изучение, в связи с некоторыми математическими вопросами релятивистской космологии, в работах [3], [4], [9].

В связи с анализом квадратурной формулы П. Л. Чебышева от 1889 г. (см. [12], стр. 249, (12)), автор получил следующий результат.

Теорема. *По гипотезе Римана:*

$$\sum_{T \leqslant t_0 \leqslant T+H} |Z(t_0)| > \frac{2}{\pi}(1-\varepsilon)H\ln P, \quad P \to \infty, \tag{2}$$

где $P = \sqrt{T/(2\pi)}$ *и* $H \in \langle T^{\mu}, \sqrt[4]{T}\,\rangle$ $(0 < \varepsilon, \mu$ — *сколь угодно малые числа*).

Замечание 1. Явно отметим, что в условиях теоремы выступает ограничение $H \leqslant \sqrt[4]{T}$. Справедливость оценки (2) можно расширить и на значения, скажем, $H \in (\sqrt[4]{T}, T\rangle$. Достигается это применением метода ван дер Корпута для оценок тригонометрических сумм, встречающихся в доказательстве леммы А.

Доказательство теоремы опирается на следующие вспомогательные утверждения.

1.1. Пусть $S(a, b)$ обозначает элементарную тригонометрическую сумму:

$$S(a, b) = \sum_{a \leqslant n \leqslant b} n^{it}, \quad 1 \leqslant a < b \leqslant 2a, \ b \leqslant \sqrt{t/(2\pi)}.$$