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The k-functions in multiplicative number theory, I
On complex explicit formulae

by

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1. Introduction. The celebrated explicit formula due to B. Riemann and H. von Mangoldt states that if

$$\psi(x) = \sum_{n\leqslant x} \Lambda(n), \quad \tilde{\psi}(x) = \sum_{n\leqslant x} \Lambda(n)/n, \quad x > 1,$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m,\ p \text{ prime},\ m \geqslant 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi_0(x) = \tfrac{1}{2}(\psi(x+0)+\psi(x-0)), \quad \tilde{\psi}_0(x) = \tfrac{1}{2}(\tilde{\psi}(x+0)+\tilde{\psi}(x-0)),$$

then

$$\psi_0(x) = x - \sum_{\varrho} x^{\varrho}/\varrho - \log(2\pi) - \tfrac{1}{2}\log(1-x^{-2}), \tag{1.1}$$

$$\tilde{\psi}_0(x) = \log x + \sum_{\varrho} \frac{(1/x)^{\varrho}}{\varrho} - C - \frac{1}{x} - \frac{1}{2}\log\frac{x-1}{x+1}, \tag{1.2}$$

where ϱ runs over all complex zeros of the Riemann zeta function, $\sum_{\varrho} x^{\varrho}/\varrho$ denotes the limit of $\sum_{|\mathrm{Im}\varrho|\leqslant T} x^{\varrho}/\varrho$ as $T \to \infty$ and C is the Euler constant (compare e.g. [5]). The generalizations of (1.1) and (1.2) have proved to be very useful in the theory of primes. Generally speaking such formulae express the values of a function depending on primes in terms of poles and zeros of zeta functions. But we can look at (1.1) and (1.2) also in another way. Considering the right-hand side of (1.1) as a given function f,

$$f(x) = x - \sum_{\varrho} x^{\varrho}/\varrho - \log(2\pi) - \tfrac{1}{2}\log(1-x^{-2}), \quad x > 1, \tag{1.3}$$

formula (1.1) proves that $f(x) = 0$ for $1 < x < 2$ and f is a piecewise constant function with discontinuities of size $\log p$ at the prime powers $x = p^m$, $m \geqslant 1$. In other words (1.1) describes the basic analytic properties of f. This function is defined in terms of non-trivial zeros of the zeta function but its "singularities" have a sharply outlined arithmetic character and are closely related to prime numbers. Similar remarks on (1.2) are also true. In the following we consider expressions analogous to f in (1.3) and to the left-hand side of (1.2) which, however, are functions of a complex variable. To make the results more suitable for intended applications we do not restrict ourselves to the Riemann zeta function, but, more generally, we consider L-functions corresponding to an arbitrary primitive Dirichlet character $\chi \pmod q$, $q \geqslant 1$. For a complex number z from the upper half-plane $H = \{z \in C\colon \operatorname{Im} z > 0\}$ let us define the functions k and K as follows:

(1.4) $$k(z, \chi) = \sum_{\gamma > 0} e^{\varrho z},$$

(1.5) $$K(z, \chi) = \sum_{\gamma > 0} \frac{1}{\varrho} e^{\varrho z}, \qquad z \in H,$$

where the summation is over all non-trivial $L(s, \chi)$ zeros $\varrho = \beta + i\gamma$ with positive imaginary parts. It is easy to see that k and K are holomorphic on H. Our principal aim is to describe the analytic character of these functions. We show, in particular, that k can be continued analytically to a meromorphic function on a Riemannian surface M of logarithmic type and it satisfies certain functional equations (see § 3). From this the complete description of the basic analytic properties of K follows easily. It turns out that these results can be considered as a complex form of the well-known explicit formulae for Chebyshev's functions $\psi(x, \chi)$. We prove that the phrase "complex explicit formulae" used in the title of this paper is fully justified, i.e. we show how the formulae of Riemann–Mangoldt type in their classical form can be derived from the just developed theory of k-functions.

The main results of this paper can be considered as a continuation of works by H. Cramér [2] and P. Guinand [4].

The significance of k-functions lies in the fact that they permit the presentation of a remarkable part of multiplicative theory of numbers in a uniform way. A systematic study of these functions leads not only to the refinements of the known results but it provides some new theorems as well. The author intends to discuss some applications of k-functions in the subsequent parts of this work which shall appear under the following subtitles.

Part II: Uniform distribution of zeta zeros;

Part III: Uniform distribution of zeta zeros, discrepancy;

Part IV: On a method of A. E. Ingham;

Part V: Changes of sign of some arithmetical error terms.

2. Notation. For a Dirichlet character χ let $q = q(\chi)$ denote its conductor and let $d = d(\chi)$ denote 0 or 1 so that $\chi(-1) = (-1)^d$.

We write also

$$e(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0 \text{ (principal character)}, \\ 0 & \text{otherwise}; \end{cases}$$

$$e_1(\chi) = \begin{cases} 1 & \text{if } d = 0,\ \chi \neq \chi_0, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, for $x > 1$

$$\psi(x, \chi) = \sum_{n \leqslant x} \Lambda(n)\chi(n), \qquad \psi_0(x, \chi) = \tfrac{1}{2}(\psi(x+0, \chi) + \psi(x-0, \chi)),$$

$$\tilde{\psi}(x, \chi) = \sum_{n \leqslant x} \frac{\Lambda(n)\chi(n)}{n}, \qquad \tilde{\psi}_0(x, \chi) = \tfrac{1}{2}(\tilde{\psi}(x+0, \chi) + \tilde{\psi}(x-0, \chi)).$$

p always denotes a prime number.

$C = 0.577\ldots$ is the Euler constant.

Let D_0 denote the complex plane with the slit along the non-positive real axis $(-\infty, 0]$. For $z \in D_0$ we define the function

$$E_1(z) = \int_z^\infty \frac{e^{-w}}{w}\, dw,$$

where the path of integration is the half-line $w = z + u$, $u \geqslant 0$. E_1 is closely related to the integral exponential function Ei (see [1]):

$$E_1(z) = -\operatorname{Ei}(-z) = -\operatorname{li}(e^{-z})$$

and thus

(2.1) $$E_1(z) = -\log z - C - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n! \cdot n}.$$

This formula gives analytic continuation of E_1 to a multi-valued analytic function. Of course we can make it single-valued on the Riemannian surface of the function $\log z$. Let us denote this surface by M. As is well known, it consists of infinite many complex planes cut along the half-line $(-\infty, 0]$ with the following identifications of lower and upper edges of the slits. If we number the planes with integers then, for every n, we identify the upper edge of the slit on the nth plane with the lower edge of the slit on the $(n+1)$th plane. We can identify the zero plane with $D_0 \cup \{\text{upper side of the slit } (-\infty, 0]\}$. Hence, the argument on this plane varies between $-\pi$ and π (π included). More generally on the mth plane the argument lies between $(2m-1)\pi$ and $(2m+1)\pi$. Every point $z \in M$ can be uniquely written as

$$z = re^{ia}, \qquad r > 0,\ a \in \mathbf{R},$$

$z \in H$ if and only if $0 < a < \pi$. The following mappings play an important role in this paper:

$$M \ni z = re^{ia} \mapsto z^* = re^{i(a-\pi)} \in M,$$

$$M \ni z = re^{ia} \mapsto z^c = re^{-ia} \in M.$$

Since we have the natural projection $M \to C \setminus \{0\}$, any meromorphic, single-valued function F on C can be considered as a function on M. For such F we have $F(z^*) = F(-z)$, $F(z^c) = F(\bar{z})$; $\bar{z}$ denotes the complex conjugation of z.

For any two real numbers a and b let us denote by $l(a, b)$ a simple and smooth curve $\tau\colon [0, 1] \to C$ such that $\tau(0) = a$, $\tau(1) = b$ and $\operatorname{Im} \tau(t) > 0$ for $t \in (a, b)$. Moreover, the notation

$$\int_{l(a,b)} f(z)\,dz$$

for a meromorphic function f means that f is regular on the curve $l(a, b)$ and also regular in the open domain bounded by $l(a, b)$ and the interval $[a, b]$. But, of course, it can have poles in (a, b).

For $z \in C$ and $d \in \{0, 1\}$ we define $H(z, d)$ as follows:

$$H(z, 0) = e^{2z} \int_{l(-5/4,\,-1/4)} \frac{\Gamma'}{\Gamma}(w) e^{2wz}\,dw + e^{3z} \int_{3/4}^{7/4} \frac{\Gamma'}{\Gamma}(w) e^{-2wz}\,dw, \tag{2.2}$$

$$H(z, 1) = e^{z} \int_{l(-3/4,\,1/4)} \frac{\Gamma'}{\Gamma}(w) e^{2wz}\,dw + e^{4z} \int_{5/4}^{9/4} \frac{\Gamma'}{\Gamma}(w) e^{-2wz}\,dw, \tag{2.3}$$

where Γ denotes the Euler gamma function. Moreover, for a Dirichlet character χ we write

$$h(z, \chi) = \int_{l(-1/2,\,3/2)} \frac{L'}{L}(w, \chi) e^{wz}\,dw, \tag{2.4}$$

$$h^-(z, \chi) = \overline{\int_{l(3/2,\,-1/2)}} \frac{L'}{L}(w, \chi) e^{wz}\,dw. \tag{2.5}$$

Of course H, h and h^- are entire functions of z.

We also need the following function defined for $x > 0$ and $d \in \{0, 1\}$:

$$R(x, d) = \begin{cases} \frac{1}{2}\log(1 - e^{-2x}) & \text{if } d = 0, \\ \frac{1}{2}\log \dfrac{e^x - 1}{e^x + 1} & \text{if } d = 1. \end{cases} \tag{2.6}$$

We denote the generic $L(s, \chi)$ zero by $\varrho = \beta + i\gamma$. When it is necessary we add the subscript χ: $\varrho_\chi = \beta_\chi + i\gamma_\chi$. $N(T) = N(T, \chi)$, $T > 0$, denotes the number of $\varrho = \beta + i\gamma$ with $0 < \beta < 1$, $0 < \gamma < T$. The only information about N needed in this paper is the following almost trivial estimate:

$$N(T+1) - N(T) \ll \log T, \quad T \geq 2.$$

3. Analytic character of k.

THEOREM 3.1. *For $z \in H$ and a primitive Dirichlet character $\chi \pmod q$, $q \geq 1$, we have*

$$2\pi i \cdot k(z, \chi) = N_0(z, \chi) + \frac{e^z E_1(\frac{3}{2}z) + e^{2z} E_1(\frac{5}{2}z) + H(z, d)}{1 - e^{2z}} + h(z, \chi), \tag{3.1}$$

where H and h are defined by (2.2)–(2.4) and N_0 denotes the meromorphic function defined by

$$N_0(z, \chi) = e^{3z/2} \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{n^{3/2}(z - \log n)} + e^{-z/2} \sum_{n \geq 2} \frac{\Lambda(n)\bar{\chi}(n)}{n^{3/2}(z + \log n)} - \frac{1}{z} e^{-z/2} \log \frac{q}{\pi}. \tag{3.2}$$

Hence k can be continued analytically to the meromorphic function on M and for $z \in M$ we have

$$k(z, \chi) = \frac{1}{2\pi i} \frac{e^z}{e^z - 1} \log z + N_1(z, \chi), \tag{3.3}$$

where N_1 is meromorphic and single-valued on C.

THEOREM 3.2. *The function k (meromorphic on M) satisfies the following functional equations:*

$$k(z, \chi) + e^z k(z^*, \bar{\chi}) = D(z, \chi), \tag{3.4}$$

$$k(z, \chi) + \overline{k(z^c, \bar{\chi})} = e^z D(-z, \chi), \tag{3.5}$$

where

$$D(z, \chi) = e(\chi) - e_1(\chi) e^z + e^{(3 - d(\chi))z}/(e^{2z} - 1) - \sum_{\gamma = 0} e^{\beta z}. \tag{3.6}$$

The summation in (3.6) is taken over all non-trivial real zeros β of $L(s, \chi)$ (if there are any).

As a corollary to the foregoing theorems we get the complete list of singularities of the k-function.

THEOREM 3.3. *The only singularities of k (meromorphic on M) are simple poles at the points:*

$$z = k \log p, \quad \arg z = 2m\pi, \quad k \geq 1, \quad m \in Z, \quad p \nmid q, \tag{3.7}$$

with residue

$$\frac{1}{2\pi i} \chi(p^k) \log p; \tag{3.8}$$

$$z = -k \log p, \quad \arg z = \pi + 2m\pi, \quad k \geq 1, \quad m \in Z, \quad p \nmid q \tag{3.9}$$

with residue

$$\frac{1}{2\pi i} \bar{\chi}(p^k) p^{-k} \log p; \tag{3.10}$$

(3.11) $z = 2k\pi i,\quad \arg z = \pi/2+2m\pi,\quad k \geq 1,\quad m\in Z,\quad m \neq 0$

with residue m;

(3.12) $z = -2k\pi i,\quad \arg z = -\pi/2+2m\pi,\quad k \geq 1,\quad m\in Z$

with residue $-1/2+m$;

(3.13) $z = -(2k+1)\pi i,\quad \arg z = -\pi/2+2m\pi,\quad k \geq 0,\quad m\in Z$

with residue $\frac{1}{2}(-1)^{d+1}$.

4. Classical explicit formulae. We derive these formulae from the theorems of Section 3 using K-functions defined by (1.5). For $z\in H$ we have

$$K(z, \chi) = \int_{i\infty}^{z} k(s, \chi)\, ds,$$

the path of integration being the half-line $s = z+iy$, $\infty \geq y \geq 0$. Hence K can be continued analytically along every curve lying on M and not passing through the singularities of k. K becomes a multivalued function on M. In fact, every pole of k becomes a logarithmic branch point for K. In particular for $|z-k\log p| < r_0$, $p\not| q$, $k\in Z$, $k \neq 0$, $r_0 > 0$ sufficiently small, we can write

$$K(z, \chi) = \frac{\chi(p^k)\log p}{2\pi i p^{bk}}\log(z-k\log p)+G(z, \chi), \tag{4.1}$$

where G is holomorphic in the disc $|z-k\log p| < r_0$ and the number b equals 0 for $k > 0$ and 1 for $k < 0$.

For a real x let us write

$$F(x, \chi) = \lim_{y\to 0^+} \{K(x+iy, \chi)+\overline{K(x+iy, \overline{\chi})}\}. \tag{4.2}$$

It is obvious that this limit does exist for every x which is a regular point of K. For $x = k\log p$, $k \neq 0$, $p\not| q$ its existence follows easily from (4.1). The remaining case of $x = 0$ can be settled using (3.3). It implies that for $y\to 0^+$

$$K(iy, \chi)+\overline{K(iy, \overline{\chi})} = \left(\frac{1}{2}+\frac{1}{2\pi i}(A(\chi)-\overline{A(\overline{\chi})})\right)\log y+c_0+o(1)$$

where c_0 is a constant and

$$A(\chi) = 2\pi i \operatorname*{Res}_{z=0} N_1(z, \chi). \tag{4.3}$$

But $A(\chi)-\overline{A(\overline{\chi})} = -\pi i$ (compare (8.11)) and the limit $F(0, \chi)$ really exists and is finite.

Moreover, (4.1) yields

$$F(x, \chi) = \tfrac{1}{2}(F(x+0, \chi)+F(x-0, \chi)) \tag{4.4}$$

for every $x \neq 0$.

Using Theorems 3.1–3.3 and the theorem of residues we get the following result.

THEOREM 4.1. *For* $x > 0$ *we have*

$$F(x, \chi)+\sum_{\gamma=0} e^{\beta x}/\beta = -\psi_0(e^x, \chi)+e(\chi)e^x-e_1(\chi)x-R(x, d)+B(\chi), \tag{4.5}$$

where the constant $B(\chi)$ *is given by*

$$B(\chi) = \sum_{\gamma=0} 1/\beta-e(\chi)+d\log 2-C/2-\tfrac{1}{2}\log(\pi/q)+F(0, \chi). \tag{4.6}$$

For $x < 0$ *we have*

$$F(x, \chi)+\sum_{\gamma=0} e^{\beta x}/\beta = \tilde{\psi}_0(e^{|x|}, \chi)+e(\chi)e^x+e(\chi)x+R(|x|, 1-d)+C(\chi), \tag{4.7}$$

where

$$C(\chi) = B(\chi)+C+\log(2\pi/q). \tag{4.8}$$

This theorem is very close to classical explicit formulae except for the left-hand sides of (4.5) and (4.7). Indeed, it is not quite obvious that they are equal to

$$\sum_{\varrho} e^{\varrho x}/\varrho = \lim_{T\to\infty} \sum_{|\gamma|\leq T} e^{\varrho x}/\varrho. \tag{4.9}$$

Moreover, we cannot assume here that the last limit does exist at all. To clarify the situation we need more information about the behaviour of a generalized Dirichlet series with complex exponents on the boundary of its half-plane of convergence.

THEOREM 4.2. *Let* $w_n = a_n+ib_n$, $n = 1, 2, 3, \ldots$, *denote complex numbers such that* $|a_n| \leq A$, $n \geq 1$, $b_1 \leq b_2 \leq b_3 \leq \ldots$, $\lim_{n\to\infty} b_n = \infty$. *Moreover, let the series*

$$f(z) = \sum_{n=1}^{\infty} A_n e^{w_n z},\quad A_n\in C,$$

converge for $y = \operatorname{Im} z > 0$ *and satisfy the conditions*

$$\Big|\sum_{n=N+1}^{\infty} A_n e^{(w_n-w_N)z}\Big| = o(|y|^{-1}),\quad N\to\infty, \tag{4.10}$$

for $y\to 0^+$ *almost uniformly with respect to* $x = \operatorname{Re} z$, *and*

$$\Big|\sum_{n=1}^{N} A_n e^{(w_n-w_N)z}\Big| = o(|y|^{-1}),\quad N\to\infty, \tag{4.11}$$

for $y\to 0^-$ *also almost uniformly with respect to* $x = \operatorname{Re} z$.

If f *is holomorphic at the boundary point* $x_0\in \mathbf{R}$ *then the series* $\sum_n A_n \exp(w_n x_0)$ *is convergent to* $f(x_0)$. *Moreover, the convergence is uniform on every compact real interval consisting of regular points of* f *only.*

This result is a generalization of the classical theorem of M. Riesz [6]. It says that if the coefficients of the Dirichlet series

$$g(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s), \qquad \lambda_n \in \mathbf{R},\ \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots,$$

satisfy the condition

$$(4.12) \qquad \sum_{k=1}^{n} a_k e^{\lambda_k c} = o(e^{\lambda_n c}) \quad (c > 0, \text{ fixed})$$

then the conclusions as in Theorem 4.2 hold: The series $\sum a_n \exp(-i\lambda_n t_0)$ converges to $f(s_0)$ for every regular point $s_0 = it_0$ of f. The convergence is uniform in every compact interval of the imaginary axis containing regular points of f only. Let us remark that this theorem follows from Theorem 4.2 by putting $w_n = i\lambda_n$, $s = iz$; conditions (4.10) and (4.11) follow from (4.12) by partial summation. In fact our proof of Theorem 4.2 is a modification of Riesz' method.

It is not difficult to see that K-functions satisfy (4.10) and (4.11). Let us number the complex zeros $L(s, \chi)$ lying on H according to increasing imaginary parts: $\varrho_1, \varrho_2, \varrho_3, \ldots$ For zeros with the same γ's the order of ϱ's can be arbitrarily fixed. For $y > 0$ and $N \to \infty$ we split the sum

$$\sum_{n=N+1}^{\infty} \frac{1}{\varrho_n} e^{(\varrho_n - \varrho_N)z}$$

into three sums S_1, S_2 and S_3 according to the following ranges of γ_n: $\gamma_N < \gamma_n \leqslant \gamma_N + \sqrt{\gamma_N}$, $\gamma_N + \sqrt{\gamma_N} < \gamma_n \leqslant 2\gamma_N$, $\gamma_n > 2\gamma_N$. Their contributions can be (trivially) estimated as follows:

$$|S_1| \ll \frac{1}{\gamma_N}\left(N(\gamma_N + \sqrt{\gamma_N}) - N(\gamma_N)\right) \ll \frac{\log \gamma_N}{\sqrt{\gamma_N}},$$

$$|S_2| \ll \frac{1}{y} \frac{1}{\gamma_N \sqrt{\gamma_N}} N(2\gamma_N) \ll \frac{\log \gamma_N}{y\sqrt{\gamma_N}},$$

$$|S_3| \ll \frac{1}{y} \sum_{\gamma_n \geqslant 2\gamma_N} \frac{1}{\gamma_n^2} \ll \frac{1}{y} \frac{\log \gamma_N}{\gamma_N}$$

almost uniformly with respect to x. Hence K satisfies (4.10). Similarly one can prove that for $y \to 0^-$

$$\left|\sum_{n=1}^{N} \frac{1}{\varrho_n} e^{(\varrho_n - \varrho_N)z}\right| \ll \frac{1}{|y|} \frac{\log^2 N}{\sqrt{N}}$$

almost uniformly with respect to x, which implies (4.11).

Hence we get the following corollary to Theorem 4.2.

Corollary 4.1. *For $x \neq k \log p$, $k \in Z$, $p \nmid q$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{\varrho_n} e^{\varrho_n x}$$

is convergent to $K(x, \chi)$. The convergence is uniform in every closed interval not containing points of the form $k \log p$, $p \in Z$, $p \nmid q$. In particular

$$F(x, \chi) + \sum_{\gamma_\chi = 0} \frac{e^{\beta_\chi x}}{\beta_\chi} = \sum_{n=1}^{\infty} \frac{1}{\varrho_{n\chi}} e^{\varrho_{n\chi} x} + \overline{\sum_{n=1}^{\infty} \frac{1}{\varrho_{n\bar{\chi}}} e^{\varrho_{n\bar{\chi}} x}} + \sum_{\gamma_\chi = 0} \frac{1}{\beta_\chi} e^{\beta_\chi x} = \sum_{\varrho_\chi} \frac{e^{\varrho_\chi x}}{\varrho_\chi}.$$

In the case of a logarithmic singularity on the real axis we have the following result.

Theorem 4.3. *Let f be such as in Theorem 4.1. Suppose for certain $x_0 \in \mathbf{R}$, $g \in C$ and $r_0 > 0$ we have*

$$f(z) = g \log(z - x_0) + h(z),$$

for $|z - x_0| < r_0$, $\operatorname{Im} z > 0$, where h is holomorphic in the whole disc $|z - x_0| < r_0$. Then for T tending to infinity

$$\sum_{\operatorname{Im} w_n \leqslant T} A_n e^{w_n x_0} = -g \log T - gC + h(x_0) + g\pi i/2 + o(1).$$

In view of (4.1) we get

Corollary 4.2. *For $T \to \infty$ and $x_0 = k \log p$, $k \neq 0$, $p \nmid q$ we have*

$$\sum_{0 < \gamma \leqslant T} \frac{1}{\varrho} e^{\varrho x_0} = -\frac{\chi(p^k) \log p}{2\pi i p^{kb}} (\log T + C - \pi i/2) + G(x_0, \chi) + o(1).$$

In particular

$$\sum_{\varrho} \frac{1}{\varrho} e^{\varrho x_0} = \lim_{T \to \infty} \left\{ \sum_{0 < \gamma_\chi \leqslant T} \frac{1}{\varrho_\chi} e^{\varrho_\chi x_0} + \overline{\sum_{0 < \gamma_{\bar{\chi}} \leqslant T} \frac{1}{\varrho_{\bar{\chi}}} e^{\varrho_{\bar{\chi}} x_0}} \right\} + \sum_{\gamma = 0} \frac{1}{\beta} e^{\beta x_0}$$

$$= \frac{1}{2} \frac{\chi(p^k) \log p}{p^{kb}} + G(x_0, \chi) + \overline{G(x_0, \bar{\chi})} + \sum_{\gamma = 0} \frac{1}{\beta} e^{\beta x_0}$$

$$= F(x_0, \chi) + \sum_{\gamma = 0} \frac{1}{\beta} e^{\beta x_0}.$$

The foregoing results provide Riemann–von Mangoldt type formulae in their classical form. In conclusion let us remark that the constants $B(\chi)$ and $C(\chi)$ can sometimes be explicitly computed. For instance in case $\chi = \chi_0$ (principal character), using (4.6) we get

$$B(\chi_0) = -1 - C/2 - (\log \pi)/2 + F(0, \chi_0).$$

But

$$F(0, \chi_0) = 2 \sum_{\gamma > 0} \beta / |\varrho|^2$$

and this sum can be computed using the Weierstrass product formula for the Riemann zeta function (see [3], § 12). It equals $C/2 + 1 - (\log 4\pi)/2$. Hence $B(\chi_0) = -\log(2\pi)$ and $C(\chi_0) = C$, which (of course) confirms (1.1) and (1.2).

5. Proof of Theorem 3.1. We have

$$2\pi i\cdot k(z,\chi)=k_1(z,\chi)+k_2(z,\chi)+h(z,\chi), \tag{5.1}$$

where

$$k_1(z,\chi)=\int_{-1/2+i\infty}^{-1/2}\frac{L'}{L}(s,\chi)e^{sz}\,ds,\qquad k_2(z,\chi)=\int_{3/2}^{3/2+i\infty}\frac{L'}{L}(s,\chi)e^{sz}\,ds \tag{5.2}$$

and h is defined by (2.4).

Firstly let us consider k_1. From the functional equation for $L(s,\chi)$ we get

$$\frac{L'}{L}(s,\chi)=-\log\frac{q}{\pi}-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s+d}{2}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+d}{2}\right)-\frac{L'}{L}(1-s,\bar\chi).$$

Hence we can split the integral (5.2) into four integrals

$$k_1(z,\chi)=-k_{11}(z,\chi)-\tfrac{1}{2}k_{12}(z,\chi)-\tfrac{1}{2}k_{13}(z,\chi)-k_{14}(z,\chi), \tag{5.3}$$

say. Direct integration yields

$$k_{11}(z,\chi)=\log\frac{q}{\pi}\int_{-1/2+i\infty}^{-1/2}e^{sz}ds=\log\frac{q}{\pi}\,\frac{1}{z\cdot e^{z/2}}. \tag{5.4}$$

Substituting in k_{12}, $w=(1-s+d)/2$, $w=2t_0+it$, $-\infty<t\leqslant 0$, $t_0=(3+2d)/4$, applying the functional equation for the Euler gamma function and using the Cauchy integral theorem we get

$$k_{12}(z,\chi)=2e^{(1+d)z}g_1(z,d)-2e^{(1+d)z}\int_{t_0-i\infty}^{t_0}\frac{\Gamma'}{\Gamma}(w+1)e^{-2wz}dw$$
$$=2e^{(1+d)z}g_1(z,d)-2e^{(3+d)z}\int_{l(t_0,t_0+1)}\frac{\Gamma'}{\Gamma}(w)e^{-2wz}dw+e^{2z}k_{12}(z,\chi),$$

where

$$g_1(z,d)=\int_{t_0-i\infty}^{t_0}\frac{e^{-2wz}}{w}dw.$$

Since $\frac{\partial}{\partial z}g_1(z,d)=\frac{1}{z}e^{-2t_0z}$ and for $y=\operatorname{Im}z>0$, $|g_1(x+iy,d)|\to 0$ as $x\to\infty$ thus $g_1(z,d)=-E_1(2t_0z)$. Hence we have

$$k_{12}(z,\chi)=-\frac{2e^{(1+d)z}}{1-e^{2z}}\{E_1(2t_0z)+e^{2z}H_1(z,d)\}, \tag{5.5}$$

where

$$H_1(z,d)=\int_{l(t_0,t_0+1)}\frac{\Gamma'}{\Gamma}(w)e^{-2wz}dw.$$

We can treat k_{13} similarly. The result is

$$k_{13}(z,\chi)=-\frac{2e^{(2-d)z}}{1-e^{2z}}\{E_1(-2t_1z)+H_2(z,d)\}, \tag{5.6}$$

where

$$H_2(z,d)=\int_{l(t_1,t_1+1)}\frac{\Gamma'}{\Gamma}(w)e^{2wz}dw$$

and $t_1=(2d-5)/4$.

To compute k_{14} it is enough to apply the definition of $L(s,\bar\chi)$ in the half-plane $\operatorname{Re}s>1$. Indeed, we have

$$k_{14}(z,\chi)=\int_{-1/2+i\infty}^{-1/2}\frac{L'}{L}(1-s,\bar\chi)e^{sz}ds \tag{5.7}$$
$$=-\sum_{n\geqslant 2}\frac{\Lambda(n)\bar\chi(n)}{n}\int_{-1/2+i\infty}^{-1/2}e^{s(z+\log n)}ds=-e^{-z/2}\sum_{n\geqslant 2}\frac{\Lambda(n)\bar\chi(n)}{n^{3/2}(z+\log n)}.$$

Similarly

$$k_2(z,\chi)=e^{3z/2}\sum_{n\geqslant 2}\frac{\Lambda(n)\chi(n)}{n^{3/2}(z-\log n)}. \tag{5.8}$$

Collecting (5.1)–(5.8) we get (3.1) and (3.2). Finally, let us observe that (3.3) follows from (2.1) and (3.1). Therefore Theorem 3.1 is proved.

6. Proof of Theorem 3.2. Let us consider the function

$$k^-(z,\chi)=\sum_{\gamma<0}e^{\varrho z} \tag{6.1}$$

defined for $z\in H^-=\{z\in\mathbf{C}\colon \operatorname{Im}z<0\}$. The summation is taken over all non-trivial zeros of $L(s,\chi)$ lying on H^-. We have

$$2\pi i\cdot k^-(z,\chi)=k_1^-(z,\chi)+k_2^-(z,\chi)+h^-(z,\chi), \tag{6.2}$$

where

$$k_1^-(z,\chi)=\int_{3/2-i\infty}^{3/2}\frac{L'}{L}(s,\chi)e^{sz}ds,\qquad k_2^-(z,\chi)=\int_{-1/2}^{-1/2-i\infty}\frac{L'}{L}(s,\chi)e^{sz}ds \tag{6.3}$$

and h^- is defined by (2.5).

Expanding L'/L in (6.3) into Dirichlet series and using (5.8) as the definition of $k_2(z,\chi)$ for $\operatorname{Im}z<0$, it can easily be seen that

$$k_1^-(z,\chi)=-k_2(z,\chi). \tag{6.4}$$

Let us consider k_2^- next. Suppose $z = x+iy$, $x > 0$, $y < 0$. Applying the Cauchy integral theorem we get

$$(6.5)\qquad k_2^-(z, \chi) = \int_{-1/2}^{-1/2-i} \frac{L'}{L}(s, \chi)e^{sz}ds + \int_{-1/2-i}^{-\infty-i} \frac{L'}{L}(s, \chi)e^{sz}ds.$$

The first term defines an entire function, the second defines a function holomorphic in the half-plane $x > 0$. Similarly, for $y > 0$ we have

$$(6.6)\qquad k_1(z, \chi) = \int_{-\infty+i}^{-1/2+i} \frac{L'}{L}(s, \chi)e^{sz}ds + \int_{-1/2+i}^{-1/2} \frac{L'}{L}(s, \chi)e^{sz}ds$$

and, as before, (6.6) gives the analytic continuation of k_1 to the half-plane $x > 0$.

Comparing (6.5) and (6.6) we get

$$(6.7)\qquad -k_1(z, \chi)-k_2^-(z, \chi) = 2\pi i \sum_w \operatorname*{Res}_{s=w}\left\{\frac{L'}{L}(s, \chi)e^{sz}\right\},$$

where the summation in (6.7) is taken over all zeros of $L(s, \chi)$ lying on the half-line $(-\infty, -1/2]$. In case $d = 0$ we have $w = -2m$, $m = 1, 2, 3, \ldots$ and in case $d = 1$, $w = -2m-1$, $m = 0, 1, 2, \ldots$ Hence

$$(6.8)\qquad k_2^-(z, \chi) = -k_1(z, \chi)-2\pi i\frac{e^{dz}}{e^{2z}-1}.$$

Moreover,

$$(6.9)\qquad -h^-(z, \chi)-h(z, \chi) = 2\pi i \sum_w \operatorname*{Res}_{s=w}\left\{\frac{L'}{L}(s, \chi)e^{sz}\right\},$$

where the summation in (6.9) is taken over all singularities of $(L'/L)(s, \chi)\exp(sz)$ in the interval $[0, 1]$. They are simple poles at the points: $w = 0$ (when $\chi \neq \chi_0$, $d = 1$) with residue 1; $w = \beta$ (a real zero of $L(s, \chi)$, if there are any) with residue $e^{\beta z}$; $w = 1$ (when $\chi = \chi_0$) with residue $-e^z$. Hence

$$(6.10)\qquad h^-(z, \chi) = -h(z, \chi)+2\pi i\Big(e(\chi)e^z-e_1(\chi)-\sum_{\gamma=0} e^{\beta z}\Big).$$

Formulae (6.2), (6.4), (6.8) and (6.10) imply

$$(6.11)\qquad k^-(z, \chi) = -k(z, \chi)+e(\chi)e^z-e_1(\chi)-\frac{e^{dz}}{e^{2z}-1}-\sum_{\gamma=0} e^{\beta z}.$$

In view of the known analytic properties of k this formula gives the analytic continuation of k^- to a meromorphic function on the surface M.

To prove (3.4) let us observe that, owing to the functional equation of Dirichlet L-series, if ϱ is a zero of $L(s, \chi)$ then $1-\varrho$ is a zero of $L(s, \bar\chi)$. Thus for

$$(6.12)\qquad z \in M, \quad 0 < \arg z < \pi,$$

we have by (6.11)

$$(6.13)\qquad k(z, \chi) = \sum_{\gamma_\chi > 0} e^{\varrho_\chi z} = \sum_{\gamma_{\bar\chi} < 0} e^{(1-\varrho_{\bar\chi})z} = e^z k^-(z^*, \bar\chi) = -e^z k(z^*, \bar\chi)+D(z, \chi),$$

where D is given by (3.6). Therefore the functional equation (3.4) follows from (6.13) by the principle of analytic continuation.

To prove (3.5) let us observe that if ϱ is a zero of $L(s, \chi)$ then $\bar\varrho$ is a zero of $L(s, \bar\chi)$. Hence for z satisfying (6.12) we have

$$k(z, \chi) = \sum_{\gamma_\chi > 0} e^{\varrho_\chi z} = \overline{\sum_{\gamma_{\bar\chi} < 0} e^{\varrho_{\bar\chi} z^c}} = \overline{k^-(z^c, \bar\chi)} = -\overline{k(z^c, \bar\chi)}+e^z D(-z, \chi),$$

which ends the proof.

7. Proof of Theorem 3.3. The fact that the only singularities of k on M are simple poles at the points (3.7), (3.9), (3.11), (3.12) and (3.13) follows easily from Theorem 3.1 and the functional equation (3.4). Hence it remains to compute the residues. Let us observe that (3.8) and (3.10) are obvious in view of (3.1) and (3.2). The residue at (3.11) can be computed using (3.3),

$$\operatorname*{Res}_{\substack{z=2k\pi i\\ \arg z=\pi/2+2m\pi}} k(z, \chi) = \operatorname*{Res}_{\substack{z=2k\pi i\\ \arg z=\pi/2}}\left\{k(z, \chi)+m\frac{e^z}{e^z-1}\right\} = m.$$

To compute the residue at (3.12) suppose, first, that $m = 0$. Then by (3.4)

$$\operatorname*{Res}_{\substack{z=-2k\pi i\\ \arg z=-\pi/2}} k(z, \chi) = -\operatorname*{Res}_{\substack{z=2k\pi i\\ \arg z=\pi/2}}\{k(z, \bar\chi)+e^z k(z^*, \chi)\} = -\operatorname*{Res}_{z=2k\pi i} D(z, \chi) = -1/2.$$

Hence, by (3.3)

$$\operatorname*{Res}_{\substack{z=-2k\pi i\\ \arg z=-\pi/2+2m\pi}} k(z, \chi) = \operatorname*{Res}_{\substack{z=-2k\pi i\\ \arg z=-\pi/2}}\left\{k(z, \chi)+m\frac{e^z}{e^z-1}\right\} = -\frac{1}{2}+m.$$

The residue at (3.13) can be treated much in the same way. One should note only that $e^z/(e^z-1)$ is regular at $z = -(2k+1)\pi i$ and

$$\operatorname*{Res}_{z=(2k+1)\pi i} D(z, \chi) = \tfrac{1}{2}(-1)^{d+1}.$$

8. Proof of Theorem 4.1. Suppose, first, that $x > 0$. In view of (4.4) we assume without loss of generality that $x \neq k\log p$, $p \nmid q$. Hence k is regular at $z = x$. Let $0 < a < \min(x, \log 2)$. We have

$$(8.1)\qquad K(x, \chi) = K(a, \chi)+\int_l k(s, \chi)\,ds,$$

where $l = l(a, x)$. By the theorem of residues and our Theorem 3.3,

$$(8.2)\qquad \int_l k(s, \chi)\,ds-\int_{l^c} k(s, \chi)\,ds = -2\pi i \sum_{\substack{k\log p\leq x\\ p\nmid q,\, k\geq 1}} \operatorname*{Res}_{s=k\log p} k(s, \chi) = -\psi(e^x, \chi).$$

Moreover, using the functional equation (3.5), we get

$$\int_{l^c} k(s,\chi)\,ds = \int_{l} k(s^c,\chi)\,ds^c = -\overline{\int_{l} k(s,\overline{\chi})\,ds} + \int_{a}^{x} e^t D(-t,\chi)\,dt \tag{8.3}$$

$$= -\overline{K(x,\overline{\chi})} + \overline{K(a,\overline{\chi})} + \int_{a}^{x} e^t D(-t,\chi)\,dt.$$

Also

$$\int_{a}^{x} e^t D(-t,\chi)\,dt = \int_{a}^{x} \{e(\chi)e^t - e_1(\chi) - \sum_{m=0}^{\infty} e^{-(2m+2-d)t} - \sum_{\gamma=0} e^{\beta t}\}\,dt$$

$$= e(\chi)e^x - e_1(\chi)x - \sum_{m=1}^{\infty} \frac{e^{(d-2m)x}}{d-2m} - \sum_{\gamma=0} \frac{e^{\beta x}}{\beta}$$

$$- e(\chi)e^a + e_1(\chi)a + \sum_{m=1}^{\infty} \frac{e^{(d-2m)a}}{d-2m} + \sum_{\gamma=0} \frac{e^{\beta a}}{\beta}$$

and moreover

$$\sum_{m=1}^{\infty} \frac{e^{(d-2m)x}}{d-2m} = R(x,d).$$

Combining the above equalities and letting $a \to 0^+$ we arrive at

$$F(x,\chi) + \sum_{\gamma=0} \frac{e^{\beta x}}{\beta} = -\psi(e^x,\chi) + e(\chi)e^x - e_1(\chi)x - R(x,d) + B(\chi), \tag{8.4}$$

where

$$B(\chi) = \sum_{\gamma=0} \frac{1}{\beta} - e(\chi) + \lim_{a\to 0^+} \{F(a,\chi) + R(a,d)\}. \tag{8.5}$$

This proves the case $x > 0$ of Theorem 4.1 except for (4.6). We have to show that

$$\lim_{a\to 0^+} \{F(a,\chi) + R(a,d)\} = -d\log 2 - \tfrac{1}{2}\log(\pi/q) - \tfrac{1}{2}C + F(0,\chi). \tag{8.6}$$

By (3.3) we can write

$$k(s,\chi) = \frac{1}{2\pi i}\frac{e^s}{e^s-1}\log s + \frac{A(\chi)}{2\pi i}\frac{1}{s} + W(s), \tag{8.7}$$

where W is holomorphic and single-valued at $s = 0$. Therefore,

$$K(a,\chi) = K(ia,\chi) + \int_{ia}^{a} k(s,\chi)\,ds \tag{8.8}$$

$$= K(ia,\chi) + \frac{1}{2\pi i}\int_{ia}^{a} \frac{\log s}{s}\,ds + \frac{A(\chi)}{2\pi i}\int_{ia}^{a} \frac{ds}{s} + o(1)$$

$$= K(ia,\chi) - \frac{1}{4}\log a - \frac{\pi}{16}i - \frac{1}{4}A(\chi) + o(1)$$

as $a \to 0^+$. Hence

$$F(a,\chi) = K(ia,\chi) + \overline{K(ia,\overline{\chi})} - \tfrac{1}{2}\log a - \tfrac{1}{4}(A(\chi) + \overline{A(\overline{\chi})}) + o(1). \tag{8.9}$$

Next we observe that

$$R(a,d) = \tfrac{1}{2}\log a + (\tfrac{1}{2} - d)\log 2 + o(1), \quad a \to 0^+. \tag{8.10}$$

Moreover, by (8.7), (3.1), (3.2) and (2.1),

$$A(\chi) = \operatorname*{Res}_{s=0}\left(2\pi i\cdot k(s,\chi) - \frac{e^s}{e^s-1}\log s\right)$$

$$= \log(\pi/q) + \operatorname*{Res}_{s=0} \frac{\log(4/15) - 2C + H(0,d)}{1-e^{2z}}$$

$$= \log(\pi/q) + \tfrac{1}{2}\log(15/4) + C - \tfrac{1}{2}H(0,d).$$

But (2.2) and (2.3) imply that $H(0,d) = \log(15/16) + \pi i$ and hence

$$A(\chi) = \log(2\pi/q) + C - (\pi i/2). \tag{8.11}$$

Gathering (8.9), (8.10) and (8.11) we get (8.6), and formulae (4.5) and (4.6) follow.

Since the proof of formulae (4.7) and (4.8) is very similar we only indicate the main points. Again we can assume that $x \neq k\log p$, $p \nmid q$ and let $\max(-\log 2, x) < a < 0$. Formula (8.1) still holds. Let $l' \subset M$ denote the curve symmetrical to l upon the real axis and such that $\pi \leqslant \arg z < 3\pi/2$ for $z \in l'$. Instead of (8.2) we have

$$\int_{l} k(s,\chi)\,ds - \int_{l'} k(s,\chi)\,ds = \tilde{\psi}(e^{|x|},\overline{\chi}).$$

Moreover, by (3.3)

$$\int_{l'} k(s,\chi)\,ds = \int_{l} k(s^c e^{2\pi i},\chi)\,ds^c = \int_{l} k(s^c,\chi)\,ds^c + \int_{a}^{x} \frac{e^t}{e^t-1}\,dt.$$

Hence using the functional equation (3.5) and arguing similarly to the first part of the proof we arrive at

$$F(x,\chi) + \sum_{\gamma=0} e^{\beta x}/\beta = \tilde{\psi}(e^{|x|},\overline{\chi}) + e(\chi)e^x + e(\chi)x + R(|x|, 1-d) + C(\chi),$$

where

$$C(\chi) = \sum_{\gamma=0} (1/\beta) - e(\chi) + \lim_{a\to 0^-} \{F(a,\chi) - R(|a|, 1-d)\}.$$

By a reasoning similar to (8.8) we get $F(a,\chi) = F(|a|,\chi) + \log|a| + \operatorname{Re} A(\chi) + o(1)$ as $a \to 0^-$. Moreover, (8.10) implies $\log|a| - R(|a|, 1-d) = R(|a|, d) + o(1)$ for $a \to 0^-$. Hence

$$C(\chi) = B(\chi) + \operatorname{Re} A(\chi) = B(\chi) + \log(2\pi/q) + C$$

and (4.8) follows. The proof is complete.

9. Proof of Theorem 4.2. Suppose f is holomorphic inside the rectangle Q with the vertices $x_1 \pm ib$, $x_2 \pm ib$, where $0 < b < 1$ and $x_1 < x_0 < x_2$. Consider the auxiliary function

$$g_N(z) = e^{-w_N z}(z-x_1)(z-x_2)\Big[f(z) - \sum_{n=1}^{N} A_n e^{w_n z}\Big]. \tag{9.1}$$

Let us put $H_0 = \sqrt{(x_1-x_2)^2+b^2}$, $M_0 = \max\{|f(z)|: z \in Q\}$ and $X = |x_1|+|x_2|$. Fix a positive ε. By (4.10) and (4.11) there exists $N_0 = N_0(\varepsilon)$ such that for $N \geqslant N_0$ and $z \in Q$ we have

$$b_N \geqslant 2\varepsilon^{-1} H_0 M_0 e^{AX}, \quad e^{b_N b} \geqslant 2\varepsilon^{-1} H_0^2 M_0 e^{AX}, \tag{9.2}$$

$$\Big|\sum_{n=N+1}^{\infty} A_n e^{(w_n - w_N)z}\Big| \leqslant \varepsilon y^{-1}(H_0 + H_0^2 b^{-1})^{-1}, \quad 0 < y < 1. \tag{9.3}$$

$$\Big|\sum_{n=1}^{N} A_n e^{(w_n - w_N)z}\Big| \leqslant \tfrac{1}{2}\varepsilon |y|^{-1}(H_0 + H_0^2 b^{-1})^{-1}, \quad -1 < y < 0. \tag{9.4}$$

If now z belongs to the segment $(x_1, x_1+ib]$ or $(x_2, x_2+ib]$ then by (9.1) and (9.3),

$$|g_N(z)| \leqslant H_0 y \Big|\sum_{n=N+1}^{\infty} A_n e^{(w_n - w_N)z}\Big| \leqslant \varepsilon.$$

Since g_N is zero at x_1 and x_2 the last inequality holds for these two points as well. Similarly for $z \in [x_1+ib, x_2+ib]$ we have

$$|g_N(z)| \leqslant H_0^2 \Big|\sum_{n=N+1}^{\infty} A_n e^{(w_n - w_N)z}\Big| \leqslant \varepsilon.$$

For $z \in [x_1-ib, x_1) \cup [x_2-ib, x_2)$ we have by (9.1), (9.2) and (9.4)

$$|g_N(z)| \leqslant e^{AX} e^{-b_N|y|} H_0|y| M_0 + H_0|y| \tfrac{1}{2}\varepsilon |y|^{-1}(H_0 + H_0^2 b^{-1})^{-1}$$
$$\leqslant b_N^{-1} H_0 M_0 e^{AX} + \tfrac{1}{2}\varepsilon \leqslant \varepsilon.$$

Finally, for $z \in [x_1-ib, x_2-ib]$,

$$|g_N(z)| \leqslant e^{AX} e^{-b_N b} H_0^2 M_0 + H_0^2 \Big|\sum_{n=1}^{N} A_n e^{(w_n - w_N)z}\Big| \leqslant \varepsilon.$$

By what we have already shown and by the maximum modulus principle, the inequality $|g_N(z)| \leqslant \varepsilon$ holds for every z inside Q and every $N \geqslant N_0(\varepsilon)$. Hence

$$\Big|f(x_0) - \sum_{n=1}^{N} A_n e^{w_n x_0}\Big| \leqslant \varepsilon \frac{e^{AX}}{|x_0-x_1||x_0-x_2|},$$

which implies that the series $\sum A_n \exp(-w_n x_0)$ converges to $f(x_0)$ and moreover, the convergence is uniform in every closed interval $[x_1', x_2'] \subset (x_1, x_2)$. The result therefore follows.

10. Proof of Theorem 4.3. On the upper half-plane H let us consider the subsidiary function F defined by the formula

$$F(z) = f(z) - gl(z-x_0),$$

where

$$l(z) = \log(1-e^{iz}) = -\sum_{n=1}^{\infty} \frac{1}{n} e^{inz}.$$

F satisfies the hypotheses of Theorem 4.2 and it is regular at $z = x_0$. Hence the limit

$$\lim_{T\to\infty} \Big\{\sum_{\operatorname{Im} w_n \leqslant T} A_n e^{w_n x_0} + g \sum_{n \leqslant T} 1/n\Big\}$$

exists and is equal to

$$F(x_0) = \lim_{y\to 0^+} \{f(x_0+iy) - gl(iy)\}$$
$$= h(x_0) + g \lim_{y\to 0^+} \{\log(iy) - l(iy)\} = h(x_0) + g\pi i/2.$$

Since it is well known that

$$\sum_{n \leqslant T} 1/n = \log T + C + o(1) \quad (T \to \infty),$$

the proof is complete.

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