ACTA ARITHMETICA
XLVIII (1987)

On sign-changes in the remainder-term of the prime-number formula, III

by

J. Kaczorowski (Poznań)

1. In the previous two papers of this cycle, [4], [5], we have proved some estimates from below for the number of sign-changes of the functions

$$\Delta_1(x) = \pi(x) - \operatorname{li} x, \tag{1.1}$$

$$\Delta_2(x) = \Pi(x) - \operatorname{li} x, \tag{1.2}$$

$$\Delta_3(x) = \psi(x) - x, \tag{1.3}$$

$$\Delta_4(x) = \vartheta(x) - x, \tag{1.4}$$

which are the remainder-terms of various versions of the prime-number formula.

In this paper we shall be concerned with the behaviour of the function

$$\Delta_5(x) = \sum_{n=1}^{\infty} \big(\Lambda(n) - 1\big) e^{-n/x}, \tag{1.5}$$

which is substantially the "Abel mean" of Δ_3.

It can be proved that the prime-number theorem is equivalent to the formula

$$\Delta_5(x) = o(x) \quad \text{as} \quad x \to \infty, \tag{1.6}$$

and thus we can call Δ_5 the *fifth remainder-term of the prime-number formula.*

The functions (1.1)–(1.4) and (1.5) have many properties in common. They are closely connected with the distribution of non-trivial zeros of the Riemann zeta function. In particular, the well-known formula

$$\limsup_{x \to \infty} \frac{\log^+ |\Delta_j(x)|}{\log x} = \sup_{\varrho} \operatorname{Re} \varrho, \qquad 1 \leqslant j \leqslant 4, \tag{1.7}$$

($\varrho = \beta + i\gamma$ ranging over non-trivial zeta-zeros) is also true for $j = 5$.

Most of the methods usually applied to Δ_j, $1 \leqslant j \leqslant 4$, work also in the case $j = 5$.

But there are also differences. For instance, the best Vinogradov's zero-free region for $\zeta(s)$ gives

$$\Delta_5(x) = O\big(x\exp(-c_0\log x(\log\log x)^{-2/3-\varepsilon})\big), \quad \varepsilon > 0, \tag{1.8}$$

which is stronger than the known estimates

$$\Delta_j(x) = O\big(x\exp(-c_1(\log x)^{3/5}(\log\log x)^{-1/5})\big) \tag{1.9}$$

obtained for $j = 1, 2, 3, 4$ (c_0 and c_1 denote positive, absolute constants).

Oscillatory properties of Δ_5 were first studied by G. Hardy and J. E. Littlewood in 1918. They proved [1] that, under the Riemann hypothesis,

$$\Delta_5(x) = O(x^{1/2}), \tag{1.10}$$

$$\Delta_5(x) = \Omega_\pm(x^{1/2}) \tag{1.11}$$

as x tends to infinity.

In particular, under the Riemann hypothesis, Δ_5 changes sign infinitely often as $x \to \infty$.

Let $V_5(T)$ denote the number of sign-changes of Δ_5 in the interval $0 \leqslant x \leqslant T$.

From a general theorem of Pólya [8] it follows that

$$\limsup_{T\to\infty} \frac{V_5(T)}{\log T} \geqslant \frac{\gamma}{\pi}, \tag{1.12}$$

where, as usual, γ denotes the imaginary part of the "lowest" zeta-zero on the line

$$\sigma = \theta := \sup_{\zeta(\varrho)=0} \operatorname{Re}\varrho, \tag{1.13}$$

provided there are any. If not, $\gamma = +\infty$.

If we accept the Riemann hypothesis, we get $\gamma = \gamma_0 = 14.13\ldots$ For this case Pólya [9] has proved a result stronger than (1.12), with lim inf in place of lim sup.

The ideas of Ingham's paper [2] from 1936 lead to the following result. If there is at least one zero on the line $\sigma = \theta$ then Δ_5 changes sign in every interval of the form

$$(T, c_2 T) \tag{1.14}$$

for some positive c_2 and T sufficiently large.

Turán's power sum method is also applicable to this problem. S. Knapowski and W. Staś proved [7] that

$$\max_{T^{1/3}\leqslant x\leqslant T} \Delta_5(x) \geqslant T^{1/2}\exp\left(-14\frac{\log T\log\log\log T}{\log\log T}\right) \tag{1.15}$$

and

$$\min_{T^{1/3}\leqslant x\leqslant T} \Delta_5(x) \leqslant -T^{1/2}\exp\left(-14\frac{\log T\log\log\log T}{\log\log T}\right) \tag{1.16}$$

for large T.

This result gave the first effective lower bound for the number of sign-changes:

$$V_5(T) \geqslant c_3\log\log T \quad \text{for} \quad T \geqslant T_2 \tag{1.17}$$

with numerically calculable constants c_3 and T_2.

2. There are some reasons for believing that there exists a positive constant C such that

$$V_5(T) \sim C\log T \quad \text{as} \quad T \to \infty. \tag{2.1}$$

The aim of this paper is to prove certain facts which may support this conjecture.

The first approach to (2.1) is given by the following theorem.

THEOREM 2.1. *There exists an effectively calculable numerical constant T_3 such that*

$$V_5(T) \geqslant \frac{\gamma_0}{4\pi}\log T \quad \textit{for} \quad T \geqslant T_3, \tag{2.2}$$

where $\gamma_0 = 14.13\ldots$ denotes the imaginary part of the "lowest" zero of $\zeta(s)$, and ineffectively,

$$\liminf_{T\to\infty} \frac{V_5(T)}{\log T} \geqslant \frac{\gamma}{\pi}, \tag{2.3}$$

where γ has the same meaning as in (1.12).

The proof of this theorem is essentially the same as in [4], [5] and is therefore omitted.

The problem of finding good estimates from above for $V_5(T)$ appears to be very deep. In particular, we cannot prove any result as sharp as $V_5(T) \ll \log T$ without any hypothesis. In fact, any hypothetical estimate of the form:

$$V_5(T) \leqslant a\log T \quad \text{for} \quad T \geqslant T_0(a), \tag{2.4}$$

or even a weaker statement of type

$$V_5(T_j) \leqslant a\log T_j \quad \text{for a sequence } T_j \to \infty, \tag{2.5}$$

implies that the Riemann zeta-function does not vanish in the half-plane:

$$\sigma > 1-\sigma_0(a) \tag{2.6}$$

where

$$\sigma_0(a) = \frac{c_4}{(\log a)^{2/3}(\log\log a)^{1/3}}.$$

This follows from Theorem 2.1 and Vinogradov's estimate of the zero-free region for $\zeta(s)$.

In view of this observation, our results concerning upper estimates of $V_5(T)$ have to depend on some unproved hypothesis.

3. Let us first discuss the consequences of Ingham's condition which says that there is at least one zero on the line $\sigma = \theta$.

THEOREM 3.1. *The following statements are equivalent*:

1. *Ingham's condition is true*;
2. *there exists an absolute constant n_0 such that in every interval (T, eT), $T > 0$, Δ_5 changes sign at most n_0 times*;
3. $\limsup_{T\to\infty} V_5(T)/\log T < \infty$;
4. $\liminf_{T\to\infty} V_5(T)/\log T < \infty$.

Theorem 3.1 and (1.14) (or Theorem 2.1) immediately imply that, under Ingham's condition,

$$c_5 \log T \leqslant V_5(T) \leqslant c_6 \log T \tag{3.1}$$

for some positive constants c_5 and c_6.

4. It is possible to improve (3.1) under some additional conditions upon the zeta-zeros on the vertical line $\sigma = \theta$. To this end we shall use the Bohr–Weyl method from the theory of almost periodic functions.

Suppose Ingham's condition is true and let

$$0 < \omega_1 < \omega_2 < \ldots \tag{4.1}$$

denote the imaginary parts of zeros on the half-line $\sigma = \theta$, $t > 0$.

Let T denote the one-dimensional torus, i.e. the topological group R/Z with quotient topology. We shall identify T with the interval $[0, 1)$ with addition modulo 1 as group operation.

Let

$$\Omega = T^N, \tag{4.2}$$

where N denotes the number of ω_j's ($N = \infty$ is possible). Then Ω with usual product topology is a compact abelian group.

Let

$$\Omega_0 = \{\xi = (\xi_j)_{j=1}^N \mid \xi_1 = 0\}, \tag{4.3}$$

and

$$\pi\colon \Omega \to \Omega_0 \tag{4.4}$$

be the projection defined by:

$$\pi\big((\xi_j)_{j=1}^N\big) = \left(\left\{\xi_j - \frac{\xi_1}{\omega_1}\omega_j\right\}\right)_{j=1}^N, \tag{4.5}$$

where $\{a\}$ denotes the fractional part of a real number a.

Let further

$$F\colon \Omega \to R \tag{4.6}$$

be defined as follows:

$$F\big((\xi_j)_{j=1}^N\big) = \sum_{j=1}^N |\Gamma(\theta + i\omega_j)| \cos(2\pi\xi_j + b_j), \tag{4.7}$$

where

$$b_j = \arg\Gamma(\theta + i\omega_j), \tag{4.8}$$

and Γ denotes Euler's gamma function.

Let finally

$$M_F = \{\xi \in \Omega \mid F(\xi) = 0\} \tag{4.9}$$

and

$$\nu\colon \Omega_0 \to N \cup \{0\} \tag{4.10}$$

be defined by the formula

$$\nu(\eta) = \#(\pi^{-1}(\eta) \cap M_F). \tag{4.11}$$

The symbol "$\#A$" denotes here the cardinality of a set A. It will be proved that the set $\pi^{-1}(\eta) \cap M_F$ is finite for every $\eta \in \Omega_0$ (see Lemma 8.1 below). Hence, the definition (4.10)–(4.11) is correct.

Using this notation, we have the following result.

THEOREM 4.1. *Suppose that Ingham's condition is true and that the numbers $\omega_1, \omega_2, \ldots$ in (4.1) are linearly independent over Q. Then*

$$V_5(T) \sim \varkappa \log T, \quad \textit{as} \quad T \to \infty \tag{4.12}$$

where

$$\varkappa = \frac{\omega_1}{2\pi} \int_{\Omega_0} \nu(\eta)\, d\mu(\eta) \tag{4.13}$$

and $d\mu$ denotes the normed Haar measure on the group Ω_0.

The assumption that ω_j's are linearly independent is made for the sake of simplicity and Theorem 4.1 is just the simplest result of this type.

Nevertheless, it seems difficult to prove (4.12) by the method presented here without any conditions concerning the algebraic structure of the Z-module generated by the numbers ω_j.

Let $x_1 < x_2 < x_3$ denote three consecutive sign-changes of Δ_5. We say that x_2 is *δ-small* when

$$\max_{x_1 \leqslant x \leqslant x_2} \frac{|\Delta_5(x)|}{x^\theta} < \delta \quad \text{or} \quad \max_{x_2 \leqslant x \leqslant x_3} \frac{|\Delta_5(x)|}{x^\theta} < \delta.$$

Let us denote by $V_5(T, \delta)$ the number of δ-small sign-changes of Δ_5 in the interval $[0, T]$.

THEOREM 4.2. *Under the assumptions of Theorem 4.1, almost all sign-changes of Δ_5 are "big", i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$V_5(T, \delta) \leqslant \varepsilon \log T. \tag{4.14}$$

5. A yet more precise information about the behaviour of Δ_5 can be obtained under the assumption of the Riemann hypothesis. The result is as follows.

THEOREM 5.1. *Suppose that the Riemann hypothesis is true and let*

$$\varrho_0 = \tfrac{1}{2} + i\gamma_0 = \tfrac{1}{2} + i\,14.13\ldots \tag{5.1}$$

denote the "lowest" zero of $\zeta(s)$. Further, let

$$\varphi = \operatorname{Arg} \Gamma(\varrho_0) \tag{5.2}$$

and

$$u_k = (k\pi - \varphi)/\gamma_0, \quad k = 1, 2, \ldots \tag{5.3}$$

Then:

1. *All sing-changes of Δ_5 are "big"; this means that for every two consecutive sign-changes $0 < x_1 < x_2$ we have*

$$\max_{x_1 \leqslant x \leqslant x_2} \frac{|\Delta_5(x)|}{x^{1/2}} \geqslant c_7 \tag{5.4}$$

with an absolute constant $c_7 > 0$.

2. *For sufficiently large k, every interval of the form*

$$(e^{u_k}, e^{u_{k+1}}) \tag{5.5}$$

contains exactly one sign-change of Δ_5.

3. *For $T \to \infty$,*

$$V_5(T) = \frac{\gamma_0}{\pi} \log T + O(1). \tag{5.6}$$

Let us notice that, assuming the Riemann hypothesis, our Theorem 5.1 solves completely the oscillatory problem concerning Δ_5. In particular, (5.6) proves the conjecture (2.1) in a considerably stronger form.

Let us define a real-valued function $K(x)$, $x > 0$, by the formula

$$K(x) = \sum_\varrho \Gamma(\varrho) e^{i\gamma x}, \tag{5.7}$$

where the summation is spread over all non-trivial zeros of $\zeta(s)$.

Let $V(T)$ denote the number of sign-changes of K on the interval $[0, T]$. Then we have the following result.

THEOREM 5.2. *Under Riemann hypothesis we have for $T > 0$*

$$V(T) = \frac{\gamma_0}{\pi} T + \Psi(T), \tag{5.8}$$

where Ψ denotes an almost periodic function belonging to the Stepanov class S^p for every $p \geqslant 1$. Moreover, the function Ψ is bounded.

Recall, [12], that a complex-valued function f belongs to the Stepanov class S^p if for every $\varepsilon > 0$ there exists a relatively dense set of numbers $\tau = \tau(\varepsilon) > 0$ such that

$$\sup_{x \in R} \Big\{ \int_0^1 |f(x+\tau+u) - f(x+u)|^p \, du \Big\}^{1/p} < \varepsilon. \tag{5.9}$$

Let $0 < y_1 < y_2 < \ldots$ denote all sign-changes of K and let $0 < x_1 < x_2 < \ldots$ denote all sign-changes of Δ_5. Then it can be proved that there exists an integer l such that

$$y_j = \log x_{j+l} + o(1) \quad \text{for} \quad j \to \infty. \tag{5.10}$$

Theorem 5.2 shows that the numbers y_j are distributed "almost periodically". Hence, in view of (5.10), the logarithms of the sign-changes of Δ_5 are also in some sense distributed almost periodically.

6. Let us write

$$G(z) = \Delta_5(e^z) \tag{6.1}$$

for complex z satisfying

$$|\operatorname{Im} z| < \pi/2. \tag{6.2}$$

The function $G(z)$ is regular in the horizontal strip (6.2).

Applying the Mellin transform and shifting the line of integration to the left we get

$$\begin{aligned} G(z) &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Big\{ -\frac{\zeta'}{\zeta}(s) - \zeta(s) \Big\} \Gamma(s) e^{sz} \, ds \\ &= -\sum_\varrho \Gamma(\varrho) e^{\varrho z} + h(z), \end{aligned} \tag{6.3}$$

where ϱ runs over all non-trivial zeros of $\zeta(s)$, and

$$h(z) = -\zeta(0) - \frac{\zeta'}{\zeta}(0) - \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \left\{\frac{\zeta'}{\zeta}(s) + \zeta(s)\right\} \Gamma(s) e^{sz} ds. \tag{6.4}$$

This function is regular in the strip (6.2) and

$$|h(z)| \ll_\delta 1 \quad \text{for} \quad |\mathrm{Im}\, z| < \pi/2 - \delta, \quad \delta > 0. \tag{6.5}$$

The foregoing formulae are the basis for most of the considerations in this paper.

7. Proof of Theorem 3.1. It suffices to prove the implication "1 ⇒ 2", because "2 ⇒ 3 ⇒ 4" are trivial, and "4 ⇒ 1" follows from Theorem 2.1.

(6.3) implies that, for $z = x+iy$, $|y| < \pi/4$, $x \geqslant 0$, we have

$$|G(z)| \ll \sum_{\varrho} |\Gamma(\varrho)| e^{\beta x - \gamma y} \ll e^{\theta x}. \tag{7.1}$$

Moreover, for real $x \to \infty$

$$G(x) = -e^{\theta x}\{g(x) + o(1)\}, \tag{7.2}$$

where

$$g(x) = \sum_{\beta = \theta} \Gamma(\varrho) e^{i\gamma x}. \tag{7.3}$$

We shall use the fact that g is an almost periodic function in the sense of Bohr.

Since g is not identically zero, we have

$$\sup_{x \in R} |g(x)| = c_8 > 0. \tag{7.4}$$

It follows that there exist two positive constants T_4 and c_{10} such that for every $T \geqslant T_4$

$$\max_{T \leqslant x \leqslant T + c_{10}} |g(x)| \geqslant c_9 = (1/3) c_8. \tag{7.5}$$

(To prove (7.5) it suffices to consider the set of all translation numbers of g belonging to the number $\varepsilon = c_9$ (compare [3]).)

Hence, owing to (7.2) and (7.5), we have

$$\max_{T \leqslant x \leqslant T + c_{10}} e^{-\theta x} |G(x)| \geqslant c_{11} \tag{7.6}$$

for $T \geqslant T_4$ and a positive constant c_{11}.

Our assertion is equivalent to saying that the number of sign-changes of G in every interval of the form $(T, T+c_{10})$ is bounded for large T. We shall prove it in this form.

Let $x_0 \in [T, T+c_{10}]$ be such that

$$|G(x_0)| \geqslant c_{11} e^{\theta x_0} \tag{7.7}$$

and let

$$\tau\colon K(0, 1) \to \{z = x+iy \mid |y| < \pi/4\} \tag{7.8}$$

be the conformal mapping given by

$$\tau(w) = x_0 + \frac{1}{2} \log \frac{1+w}{1-w}, \quad |w| < 1. \tag{7.9}$$

There exists a real number r_0, $0 < r_0 < 1$, independent of T such that

$$[T, T+c_{10}] \subset \tau(K(0, r_0)). \tag{7.10}$$

We see that the number of sign-changes of G in the interval $(T, T+c_{10})$ is less or equal to the number $n(r_0)$ of zeros of the function

$$G_1(w) = G(\tau(w)) \tag{7.11}$$

in the disc $|w| < r_0$.

The well-known Jensen identity and the estimates (7.1), (7.7) give

$$n(r_0) \ll \log \frac{\max_{|w| \leqslant (r_0+1)/2} |G_1(w)|}{|G_1(0)|} \ll 1 \tag{7.12}$$

and the result follows.

8. We shall prepare in a series of lemmas some auxiliary results needed for the proof of Theorems 4.1 and 4.2. All these results are conditional and we shall not repeat the words "under the assumptions of Theorem 4.1", which apply throughout the section.

The following notation will be used in the sequel.

Let $\lambda\colon R \to \Omega$ be the function defined by the formula

$$\lambda(t) = (\{\omega_j t\})_{j=1}^N. \tag{8.1}$$

Let

$$\chi\colon \Omega \to \Omega \tag{8.2}$$

be defined by

$$\chi((\xi_j)_{j=1}^N) = \left(\xi_1, \left\{\xi_2 + \xi_1 \frac{\omega_2}{\omega_1}\right\}, \left\{\xi_3 + \xi_1 \frac{\omega_3}{\omega_1}\right\}, \ldots\right). \tag{8.3}$$

The function χ is a bijection and

$$\chi^{-1}((\xi_j)_{j=1}^N) = \left(\xi_1, \left\{\xi_2 - \xi_1 \frac{\omega_2}{\omega_1}\right\}, \left\{\xi_3 - \xi_1 \frac{\omega_3}{\omega_1}\right\}, \ldots\right). \tag{8.4}$$

Denote by H, F_χ, H_χ the functions defined on Ω as follows:

(8.5) $$H\big((\xi_j)_{j=1}^N\big) = \sum_{j=1}^N |\Gamma(\theta+i\omega_j)|\,\omega_j \sin(2\pi\xi_j+b_j),$$

(8.6) $$F_\chi = F\circ\chi,$$

(8.7) $$H_\chi = H\circ\chi.$$

We may suppose that the metric in Ω is defined by

(8.8) $$\varrho(\xi,\xi') = \sum_{j=1}^N \frac{1}{\omega_j^2}|e^{2\pi i\xi_j}-e^{2\pi i\xi_j'}|$$

for $\xi = (\xi_j)_{j=1}^N$, $\xi' = (\xi_j')_{j=1}^N$.

LEMMA 8.1. *There exists an absolute constant l_0 such that, for every $\eta\in\Omega_0$ and $t\in[0, 1/\omega_1)$, there exists l, $0\leqslant l\leqslant l_0$, for which*

(8.9) $$\frac{\partial^l}{\partial t^l}F\big(\eta+\lambda(t)\big)\neq 0.$$

Moreover, the function ν in (4.11) is well-defined and bounded.

Proof. Let $\eta\in\Omega_0$ and let $t_0\in[0, 1/\omega_1)$ satisfy

(8.10) $$\big|F\big(\eta+\lambda(t_0)\big)\big| = a > 0.$$

There exists $r_\eta > 0$ such that

(8.11) $$\big|F\big(\eta'+\lambda(t_0)\big)\big| \geqslant a/2 > 0$$

for all $\eta'\in\Omega_0$ with $\varrho(\eta,\eta') < r_\eta$.

Consider the functions $t\mapsto F\big(\eta'+\lambda(t)\big)$ for complex t with $|\mathrm{Im}\, t| < \pi/4$, $-1\leqslant \mathrm{Re}\, t\leqslant 2/\omega_1$. By similar arguments to those used in the proof of Theorem 3.1 we see that each function $F\big(\eta'+\lambda(t)\big)$ has at most $N_0 < \infty$ zeros (counted with multiplicities) in the interval $0\leqslant t < 1/\omega_1$. N_0 depends on a (and thus on η) but not on η'. Since Ω_0 is compact, this implies the existence of such an absolute N_0. The first assertion thus follows.

To finish the proof, let us notice that

$$\pi^{-1}(\eta)\cap M_F = \{\eta+\lambda(t)|\ t\in[0, 1/\omega_1),\ F\big(\eta+\lambda(t)\big) = 0\}.$$

Thus $\nu(\eta)\leqslant N_0$ for every $\eta\in\Omega_0$.

LEMMA 8.2. *Let E denote an arbitrary topological space and let U be an open set in $R^k\times E$, $k\geqslant 1$. Let $(x_0, e_0)\in U$, $x_0 = (x_1^0, \ldots, x_k^0)\in R^k$, $e_0\in E$ and let $f\colon U\to R^k$ be a function such that*

(8.12) $$f(x_0, e_0) = \mathbf{0}.$$

Suppose that f has partial derivatives with respect to each x_i and that they are continuous as functions of $(x, e)\in U$. Moreover, let the Jacobian of f be non-zero at the point (x_0, e_0):

(8.13) $$\frac{\partial f}{\partial(x_1,\ldots,x_k)}\bigg|_{(x,e)=(x_0,e_0)}\neq 0.$$

Then there exist two open sets, A in R^k and B in E, containing x_0 and e_0, respectively, such that for every $e\in B$ there exists exactly one $x(e)\in A$ satisfying

(8.14) $$f\big(x(e), e\big) = \mathbf{0}.$$

The function $B\ni e\mapsto x(e)\in A$ is continuous.

This is the well-known Implicit Function Theorem (see for example [11], Chapter 3).

In analogy to (4.9) we write $M_f = \{\xi\in\Omega|\ f(\xi) = 0\}$ for any real-valued function f on Ω. Moreover, $M_{f|\Omega_0} = M_f\cap\Omega_0$. Thus the meaning of M_H, M_{F_χ}, M_{H_χ} and $M_{F|\Omega_0}$ below is clear.

LEMMA 8.3. *We have*

(8.15) $$\mu\big(\overline{\pi(M_F\cap M_H)}\big) = 0$$

where the bar denotes closure in Ω_0 and μ is the normed Haar measure on this group.

Proof. We have

(8.16) $$\pi(M_F\cap M_H) = \pi_1(M_{F_\chi}\cap M_{H_\chi})$$

where π_1 is the projection

(8.17) $$\pi_1\colon\ \Omega\to\Omega_0$$

defined by

(8.18) $$\pi_1\big((\xi_1, \xi_2, \ldots)\big) = (0, \xi_2, \xi_3, \ldots).$$

We shall examine the set $M_{F_\chi}\cap M_{H_\chi}$ more carefully.

Let us write

(8.19) $$\Omega_1 = \{(\xi_j)_{j=1}^N\in\Omega|\ \xi_1 = \xi_2 = 0\}.$$

Consider F_χ as a function defined on $R^2\times\Omega_1$:

(8.20) $$F_\chi(\xi_1, \xi_2, \eta) = F_\chi(\xi_1, \xi_2, \xi_3, \ldots)$$

for $\eta = (0, 0, \xi_3, \xi_4, \ldots)\in\Omega_1$.

We have

(8.21) $$\frac{\partial}{\partial\xi_1}F_\chi = -\frac{2\pi}{\omega_1}H_\chi$$

and

(8.22) $$F_\chi(\xi) = F(\pi_1(\xi)+\lambda(\xi_1)), \quad \xi\in\Omega.$$

In view of Lemma 8.1,

(8.23) $$M_{F_\chi}\cap M_{H_\chi} \subset \bigcup_{l=1}^{l_0} M_l,$$

where

(8.24) $$M_l = \left\{\xi\in\Omega \,\Big|\, \frac{\partial^j}{\partial\xi_1^j}F_\chi(\xi) = 0 \text{ for } j = 0, \ldots, l,\ \frac{\partial^{l+1}}{\partial\xi_1^{l+1}}F_\chi(\xi)\neq 0\right\}^r.$$

Write

(8.25) $$M_l = M_l^0\cup M_l^1,$$

where

(8.26) $$M_l^0 = \left\{\xi\in M_l \,\Big|\, \frac{\partial}{\partial\xi_2}\frac{\partial^{l-1}}{\partial\xi_1^{l-1}}F_\chi(\xi)\neq 0\right\},$$

(8.27) $$M_l^1 = \left\{\xi\in M_l \,\Big|\, \frac{\partial}{\partial\xi_2}\frac{\partial^{l-1}}{\partial\xi_1^{l-1}}F_\chi(\xi) = 0\right\}.$$

Let us consider M_l^0 first. We apply Lemma 8.2 to $E = \Omega_1$, $k = 2$ and the function

(8.28) $$f = \left(\frac{\partial^{l-1}}{\partial\xi_1^{l-1}}F_\chi,\ \frac{\partial^l}{\partial\xi_1^l}F_\chi\right).$$

For $\xi^0 = (\xi_1^0, \xi_2^0, \eta_0)\in M_l^0$, $\eta_0\in\Omega_1$ the Jacobian of f is non-zero. Indeed,

(8.29) $$\frac{\partial f}{\partial(\xi_1, \xi_2)} = -\frac{\partial}{\partial\xi_2}\frac{\partial^{l-1}}{\partial\xi_1^{l-1}}F_\chi(\xi^0)\frac{\partial^{l+1}}{\partial\xi_1^{l+1}}F_\chi(\xi^0)\neq 0.$$

Lemma 8.2 implies that there exist three open sets $U_1\subset T$, $U_2\subset T$, $U_3\subset\Omega_1$, $\xi_1^0\in U_1$, $\xi_2^0\in U_2$, $\eta_0\in U_3$ and two continuous functions φ_i: $U_3\to U_i$, $i = 1, 2$, such that

(8.30) $$M_l^0\cap U_1\times U_2\times U_3\subset\{(\xi_1, \xi_2, \eta)\in\Omega \mid \xi_i = \varphi_i(\eta),\ i = 1, 2,\ \eta\in U_3\}.$$

If $\xi = (\xi_1, \xi_2, \eta)\in M_l^1$ then

(8.31) $$\cos\left(2\pi\xi_2+b_2+2\pi\frac{\omega_2}{\omega_1}\xi_1\right) = 0$$

or

(8.32) $$\sin\left(2\pi\xi_2+b_2+2\pi\frac{\omega_2}{\omega_1}\xi_1\right) = 0.$$

Thus there exists a finite set of real numbers a_i, a_i', $i = 1, 2, \ldots, i_0$, such

that

(8.33) $$\xi_2 = a_i\xi_1+a_i'$$

for at least one $i\leqslant i_0$.

Moreover, for any such ξ the point $(\xi_1, \eta)\in T\times\Omega_1$, $\eta = (0, 0, \xi_3, \ldots)$, is a zero of the function

(8.34) $$\tilde F(\xi_1, \eta) = \alpha+|\Gamma(\theta+i\omega_1)|\cos(2\pi\xi_1+b_1) + \sum_{j=3}^N|\Gamma(\theta+i\omega_j)|\cos\left(2\pi\xi_j+b_j+2\pi\frac{\omega_j}{\omega_1}\xi_1\right)$$

where α is equal to 0 or $\pm|\Gamma(\theta+i\omega_2)|$, depending on which one of the equalities (8.31) and (8.32) is actually satisfied.

As in the proof of Lemma 8.1, there is a natural number l_0' such that, for every $\eta\in\Omega_1$, there exists l', $0\leqslant l'\leqslant l_0'$, satisfying

(8.35) $$\frac{\partial^j}{\partial\xi_1^j}\tilde F(\xi_1, \eta) = 0, \quad 0\leqslant j\leqslant l',$$

(8.36) $$\frac{\partial^{l'+1}}{\partial\xi_1^{l'+1}}\tilde F(\xi_1, \eta)\neq 0.$$

Applying again Lemma 8.2 to $E = \Omega_1$, $k = 1$, and the function $f = \partial^{l'}F/\partial\xi_1^{l'}$ we see that there exist two open sets $U_1\subset T$ and $U_3\subset\Omega_1$ such that

(8.37) $$M_l^1\cap U_1\times U_2\times U_3\subset\{(\xi_1, a_i\xi_1+a_i', \eta)\in\Omega \mid \xi_1 = \varphi_1(\eta),\ \eta\in U_3\},$$

where φ_1: $U_3\to U_1$ is a continuous function and $U_2 = \{a_i U_1+a_i'\}$.

We have proved that the set $M_{F_\chi}\cap M_{H_\chi}$ is covered by the sets $U_1\times U_2\times U_3$ as in (8.30) and (8.37). Since $M_{F_\chi}\cap M_{H_\chi}$ is compact, it can be covered by a finite number of such sets. Thus there exists a natural number m such that

(8.38) $$M_{F_\chi}\cap M_{H_\chi}\subset\bigcup_{k=1}^m\{(\xi_1, \xi_2, \eta)\in\Omega \mid \xi_i = \varphi_{ik}(\eta),\ i = 1, 2,\ \eta\in U_3^k\}$$

and φ_{ik} are continuous.

Hence,

(8.39) $$\overline{\pi(M_F\cap M_H)} = \overline{\pi_1(M_{F_\chi}\cap M_{H_\chi})} \subset\bigcup_{k=1}^m\{(0, \xi_2, \eta)\in\Omega_0 \mid \xi_2 = \varphi_{2k}(\eta),\ \eta = \overline{U_3^k}\}$$

and such sets have μ-measure zero.

The proof of Lemma 8.3 is complete.

LEMMA 8.4. *For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence of disjoint intervals*

(8.40) $$I_j(\delta) \subset [0, \infty), \quad j = 1, 2, \ldots,$$

such that

(8.41) $$m(I_j(\delta)) = 2\pi/\omega_1,$$

(8.42) $$m\left(\bigcup_{j=1}^{\infty} I_j(\delta) \cap [0, T]\right) \leqslant \varepsilon T \quad \textit{for} \quad T \geqslant T_0(\varepsilon),$$

(8.43) $$|g(t)| + |g'(t)| > \delta \quad \textit{for} \quad t \notin \bigcup_{j=1}^{\infty} I_j(\delta),\ t > 0,$$

where g is defined by (7.3) and m denotes the Lebesgue measure on the line R^1.

Proof. We have

(8.44) $$g(2\pi t) = 2F(\lambda(t)) \quad \text{and} \quad g'(2\pi t) = 2H(\lambda(t))$$

with λ given by (8.1)

Let us partition the interval $[0, \infty)$ into subintervals of the form

(8.45) $$\left[\frac{k}{\omega_1}, \frac{k+1}{\omega_1}\right), \quad k = 0, 1, 2, \ldots$$

For $k/\omega_1 \leqslant t < (k+1)/\omega_1$ we have

(8.46) $$\pi(\lambda(t)) = \left(\left\{t\omega_j - \frac{\{t\omega_1\}\omega_j}{\omega_1}\right\}\right)_{j=1}^{N} = \left(\left\{\frac{[t\omega_1]\omega_j}{\omega_1}\right\}\right)_{j=1}^{N} = \left(\left\{\frac{k\omega_j}{\omega_1}\right\}\right)_{j=1}^{N}.$$

Fix a positive ε. Write

(8.47) $$M_0 = \overline{\pi(M_F \cap M_H)} \cup M_{F|\Omega_0} \cup \left(M_{F|\Omega_0} - \lambda(1/\omega_1)\right),$$

where

(8.48) $$M_{F|\Omega_0} - \lambda(1/\omega_1) = \{\eta - \lambda(1/\omega_1) |\ \eta \in M_{F|\Omega_0}\},$$

and "$-$" denotes subtraction in the group Ω_0.

By Lemma 8.3, $\mu(M_0) = 0$. Since M_0 is closed, there exists a finite sequence of closed (in the topology of Ω_0) cubes $Q_1, \ldots, Q_l$ such that

(8.49) $$M_0 \subset \operatorname{Int} \bigcup_{j=1}^{l} Q_j,$$

(8.50) $$\sum_{j=1}^{l} \mu(Q_j) < \varepsilon.$$

Further, let

(8.51) $$\delta = \inf\left\{|F(\xi)| + |H(\xi)| \ \Big|\ \xi \in \Omega \setminus \pi^{-1}\left(\bigcup_{j=1}^{l} Q_j\right)\right\}.$$

Suppose $\delta = 0$. Then there exists a sequence $\xi_i \in \Omega \setminus \pi^{-1}\left(\bigcup_{j=1}^{l} Q_j\right)$ and two points $\xi_0 \in M_F \cap M_H$, $\eta_0 \in \Omega_0$ such that $\xi_i \to \xi_0$, $\pi(\xi_i) \to \eta_0$.

If $\xi_0 \notin \Omega_0$ then $\eta_0 = \pi(\xi_0)$, since π is continuous on $\Omega \setminus \Omega_0$ and thus

$$\eta_0 \in \pi(M_F \cap M_H) \subset M_0 \subset \operatorname{Int} \bigcup_{j=1}^{l} Q_j.$$

Hence $\xi_i \in \pi^{-1}\left(\bigcup_{j=1}^{l} Q_j\right)$ for sufficiently large i, which is impossible.

If $\xi_0 \in \Omega_0$ then $\eta_0 = \xi_0$ or $\eta_0 = \xi_0 - \lambda(1/\omega_1)$. In both cases $\eta_0 \in M_0 \subset \operatorname{Int} \bigcup_{j=1}^{l} Q_j$ and as before we get a contradiction.

Thus

(8.52) $$\delta > 0.$$

Now we can take for intervals $I_j(\delta)$ those from among the intervals $[2\pi k/\omega_1, 2\pi(k+1)/\omega_1)$ for which

(8.53) $$\left(\left\{k\frac{\omega_j}{\omega_1}\right\}\right)_{j=1}^{N} \in \bigcup_{j=1}^{l} Q_j.$$

Then, by the well-known Kronecker theorem on simultaneous diophantine approximation (or its extended version if $N = \infty$), we get

(8.54) $$m\left(\bigcup_{j=1}^{\infty} I_j(\delta) \cap [0, T]\right) \leqslant \sum_{\substack{\left(\left\{\frac{k\omega_j}{\omega_1}\right\}\right)_{j=1}^{N} \in \bigcup_{j=1}^{l} Q_j \\ k \leqslant \omega_1 T/(2\pi)}} \frac{2\pi}{\omega_1} \sim \mu\left(\bigcup_{j=1}^{l} Q_j\right) T < \varepsilon T.$$

Finally, for $t \notin \bigcup_{j=1}^{\infty} I_j(\delta)$ we have

(8.55) $$\lambda(t/2\pi) \in \Omega \setminus \pi^{-1}\left(\bigcup_{j=1}^{l} Q_j\right).$$

Hence by (8.44), (8.51)

(8.56) $$|g(t)| + |g'(t)| = 2\left|F\left(\lambda(t/2\pi)\right)\right| + 2\left|H\left(\lambda(t/2\pi)\right)\right| > \delta$$

and the lemma follows.

We shall denote by $V(T, g)$ the number of sign-changes of g in the interval $[0, T]$.

LEMMA 8.5. *The asymptotic relation*

$$V_5(T) \sim \varkappa \log T, \quad T \to \infty \tag{8.57}$$

follows from the formula

$$V(T, g) \sim \varkappa T, \quad T \to \infty. \tag{8.58}$$

Proof. Writing

$$g_0(x) = -e^{-\theta x} G(x) - g(x) \tag{8.59}$$

we have

$$|g_0^{(k)}(x)| \to 0 \quad \text{as} \quad x \to \infty \tag{8.60}$$

for every integer $k \geq 0$.

Fix $\varepsilon > 0$ and let δ, $0 < \delta < 1$ and $I_j(\delta)$, $j = 1, 2, \ldots$, have the same meaning as in Lemma 8.4. Write

$$J_j(\delta) = \{t \in R \mid \underset{t' \in I_j(\delta)}{\exists} |t - t'| < \delta\} \tag{8.61}$$

and denote by

$$0 < t_1 < t_2 < \ldots \tag{8.62}$$

all zeros of g in the set $[0, \infty) \setminus \bigcup_{j=1}^{\infty} I_j(\delta)$.

Suppose $(g+g_0)(t_0) = 0$ for $t_0 \notin \bigcup_{j=1}^{\infty} J_j(\delta)$.

If t_0 is sufficiently large then

$$|g(t_0)| = |g_0(t_0)| < \delta^2/(10c_{12}), \tag{8.63}$$

where

$$c_{12} = \max\big(\sup_{t \in R} |g''(t)|, 1\big). \tag{8.64}$$

Hence, in view of Lemma 8.4,

$$|g'(t_0)| \geq \delta - \frac{\delta^2}{10c_{12}} > \tfrac{3}{4}\delta. \tag{8.65}$$

For $|t - t_0| \leq \delta/(4c_{12})$ we have

$$|g'(t)| \geq |g'(t_0)| - \Big|\int_t^{t_0} g''(x)\,dx\Big| \geq \delta/2. \tag{8.66}$$

Therefore

$$g\left(t_0 - \frac{\delta}{4c_{12}}\right) = -(\operatorname{sgn} g'(t_0))\, W(t_0) \tag{8.67}$$

where

$$W(t_0) = \int_{t_0 - \delta/(4c_{12})}^{t_0} |g'(x)|\,dx - g(t_0)\operatorname{sgn} g'(t_0) \geq \delta^2/(4c_{12}) > 0. \tag{8.68}$$

Consequently

$$\operatorname{sgn} g\big(t_0 - \delta/(4c_{12})\big) = -\operatorname{sgn} g'(t_0) \tag{8.69}$$

and similarly

$$\operatorname{sgn} g\big(t_0 + \delta/(4c_{12})\big) = \operatorname{sgn} g'(t_0). \tag{8.70}$$

Since

$$\big(t_0 - \delta/(4c_{12}),\, t_0 + \delta/(4c_{12})\big) \cap \bigcup_{j=1}^{\infty} I_j(\delta) = \emptyset, \tag{8.71}$$

the formulae (8.66), (8.69) and (8.70) imply that there exists exactly one t_j with $|t_0 - t_j| < \delta/(4c_{12})$.

Similarly, it can be proved that for every sufficiently large j the function $g + g_0$ has exactly one sign-change in the interval $\big(t_j - \delta/(4c_{12}),\, t_j + \delta/(4c_{12})\big)$.

Theorem 3.1 and Lemma 8.4 imply that $g + g_0$ has $O(\varepsilon T)$ sign-changes on the set $\bigcup_{j=1}^{\infty} J_j(\delta) \cap [0, T]$, and the same can be said of g. Thus

$$\begin{aligned} V(T, G) = V(T, g+g_0) &= \sum_{t_j \leq T} 1 + O(\varepsilon T) \\ &= V(T, g) + O(\varepsilon T) = \varkappa T + O(\varepsilon T), \end{aligned} \tag{8.72}$$

which is simply another form of (8.57).

LEMMA 8.6. *In the notation of the previous lemmas, there exists a natural number m_0 such that*

$$\pi(M_F) \setminus \bigcup_{j=1}^{l} Q_j = U_1 \cup \ldots \cup U_{m_0}, \tag{8.73}$$

where U_j are open and disjoint sets defined by

$$U_j = \{\eta \in \pi(M_F) \setminus \bigcup_{j=1}^{l} Q_j \mid \nu(\eta) = j\}. \tag{8.74}$$

The function ν is defined by (4.11).

Proof. The representation (8.73) is possible because ν is bounded. We have to prove that U_j's are open.

Let $\eta \in U_r$, $1 \leq r \leq m_0$. Then

$$\pi^{-1}(\eta) \cap M_F = \{\xi_1, \ldots, \xi_r\}. \tag{8.75}$$

There exists a real number $\delta_1 > 0$ depending only on the choice of the cubes Q_j and independent of η, such that

(8.76) $$\xi_i = \eta + \lambda(t_i), \quad 1 \leqslant i \leqslant r,$$

(8.77) $$\delta_1 < t_i < (1/\omega_1) - \delta_1, \quad 1 \leqslant i \leqslant r,$$

(8.78) $$|t_i - t_j| > \delta_1 \quad \text{for} \quad i \neq j.$$

Moreover, we may assume that $\delta_1 < \delta$ and thus

(8.79) $$|F(\xi)| + |H(\xi)| > \delta_1 \quad \text{for} \quad \xi \notin \pi^{-1}\Big(\bigcup_{j=1}^{l} Q_j\Big).$$

Let

(8.80) $$R_\eta = \min\Big\{\varepsilon \inf_{\substack{|t-t_i| \geqslant \varepsilon\delta_1 \\ 1 \leqslant i \leqslant r \\ 0 \leqslant t < 1/\omega_1}} |F(\eta + \lambda(t))|,\ \varepsilon\varrho\Big(\eta, \bigcup_{j=1}^{l} Q_j\Big),\ \varepsilon^2 \delta_1^2\Big\}$$

where $\varepsilon > 0$ is sufficiently small.

Then for $|t - t_i| \geqslant \varepsilon\delta_1$ and $\varrho(\eta, \eta') < R_\eta$ we have

(8.81) $$\begin{aligned} |F(\eta' + \lambda(t))| &= |F(\eta + \lambda(t))| + O(\varrho(\eta, \eta')) \\ &> (1 - O(\varepsilon)) \inf_{\substack{|t-t_i| \geqslant \varepsilon\delta_1 \\ 1 \leqslant i \leqslant r \\ 0 \leqslant t \leqslant 1/\omega_1}} |F(\eta + \lambda(t))| > 0. \end{aligned}$$

Hence

(8.82) $$F(\eta' + \lambda(t)) \neq 0 \text{ for } |t - t_i| \geqslant \varepsilon\delta_1,\ 0 \leqslant t < 1/\omega_1,\ \varrho(\eta, \eta') < R_\eta.$$

On the other hand, we have

(8.83) $$\begin{aligned} F(\eta' + \lambda(t_i \pm \varepsilon\delta_1)) &= F(\eta + \lambda(t_i)) \mp 2\pi\varepsilon\delta_1 H(\eta + \lambda(t_i)) + O(\varrho(\eta, \eta') + \varepsilon^2 \delta_1^2) \\ &= \mp 2\pi\varepsilon\delta_1 |H(\eta + \lambda(t_i))| \{\operatorname{sgn} H(\eta + \lambda(t_i)) + O(\varepsilon)\} \end{aligned}$$

for $\varrho(\eta, \eta') < R_\eta$.

This means that the function $F(\eta' + \lambda(t))$ has at least one zero in the interval $t_i - \varepsilon\delta_1 \leqslant t \leqslant t_i + \varepsilon\delta_1$.

Since $\eta' \notin \bigcup_{j=1}^{l} Q_j$, we have

$$|F(\eta' + \lambda(t))| + |H(\eta' + \lambda(t))| > \delta_1$$

and thus there is exactly one such zero.

This means that $\nu(\eta) = \nu(\eta')$ for $\varrho(\eta, \eta') < R_\eta$ and the set U_r is open.

LEMMA 8.7. *In the notation of previous lemmas we have*

(8.84) $$\mu(\operatorname{bd} U_r) = 0 \quad \text{for} \quad r = 1, 2, \ldots, m_0,$$

where $\operatorname{bd} U_r$ *denotes the boundary of the set* U_r *and* μ *denotes the normed Haar measure on* Ω_0.

Proof. Let $V = \bigcup_{j=1}^{l'} Q'_j$ denote the union of a finite family of closed cubes satisfying

(8.85) $$M_0 \subset \operatorname{Int} V \subset V \subset \operatorname{Int} \bigcup_{j=1}^{l} Q_j.$$

As before, we can prove that for every $\eta = \bar{U}_r$ there exists $R_\eta^* > 0$ depending on V such that

(8.86) $$\nu(\eta') = r \quad \text{for} \quad \varrho(\eta', \eta) < R_\eta^*.$$

This shows that

(8.87) $$\bar{U}_{r_1} \cap \bar{U}_{r_2} = \emptyset \quad \text{for} \quad r_1 \neq r_2.$$

Now it is easy to see that

(8.88) $$\operatorname{bd} U_r \subset \operatorname{bd} \bigcup_{j=1}^{l} Q_j, \quad r = 1, 2, \ldots, m_0,$$

and thus

(8.89) $$\mu(\operatorname{bd} U_r) \leqslant \mu\Big(\operatorname{bd} \bigcup_{j=1}^{l} Q_j\Big) = 0,$$

which ends the proof.

9. Proof of Theorem 4.1. It follows from Lemma 8.5 that it suffices to prove the relation

(9.1) $$V(T, g) \sim \varkappa T \quad \text{for} \quad T \to \infty.$$

As before we use the Kronecker theorem on simultaneous diophantine approximation. By Lemmas 8.6 and 8.7 we get

(9.2) $$\sum_{\substack{k \leqslant x \\ (\{k\frac{\omega_j}{\omega_1}\})_{j=1}^{N} \in U_r}} 1 \sim \mu(U_r)\, x \quad \text{for} \quad x \to \infty,$$

and

(9.3) $$\sum_{\substack{k \leqslant x \\ (\{k\frac{\omega_j}{\omega_1}\})_{j=1}^{N} \in \bigcup_{j=1}^{l} Q_j}} 1 \sim \mu\Big(\bigcup_{j=1}^{l} Q_j\Big)\, x < \varepsilon x.$$

Thus

(9.4) $$V(T, g) = \sum_{k \leqslant \omega_1 T/(2\pi)} \nu\Big(\Big(\Big\{k\frac{\omega_j}{\omega_1}\Big\}\Big)_{j=1}^{N}\Big) + O(1)$$

$$= \sum_{1 \leqslant r \leqslant m_0} r \sum_{\substack{(\{k\frac{\omega_j}{\omega_1}\})_{j=1}^N \in U_r \\ k \leqslant \omega_1 T/(2\pi)}} 1 + O(\varepsilon T)$$

$$\sim \sum_{1 \leqslant r \leqslant m_0} r\mu(U_r)\frac{\omega_1 T}{2\pi} + O(\varepsilon T) = \frac{\omega_1 T}{2\pi}\int_{\Omega_0} \nu(\eta)\, d\mu(\eta) + O(\varepsilon T),$$

which is equivalent to the assertion of our theorem.

10. Proof of Theorem 4.2. It follows from the foregoing analysis that for every $\varepsilon > 0$ there exist $\delta > 0$ and cubes $Q_1, \ldots, Q_l$ such that

(10.1) $$\sum_{j=1}^{l} \mu(Q_j) < \varepsilon,$$

and

(10.2) $$V_5(T, \delta) \ll \sum_{\substack{k \leqslant \omega_1 \log T/(2\pi) \\ (\{k\omega_j/\omega_1\})_{j=1}^N \in \bigcup_{j=1}^l Q_j}} \nu\big((\{k\omega_j/\omega_1\})_{j=1}^N\big)$$

$$\ll \mu\Big(\bigcup_{j=1}^{l} Q_j\Big) \log T \leqslant \varepsilon \log T.$$

The result therefore follows.

11. Before we turn to the proof of Theorems 5.1 and 5.2 we prove some subsidiary estimates.

Let

(11.1) $$0 < \gamma_0 < \gamma_1 < \ldots$$

denote the imaginary parts of the zeros of the Riemann zeta function.

The first γ_j's are approximately

(11.2) $$\gamma_0 = 14.13\ldots,\quad \gamma_1 = 21.02\ldots,\quad \gamma_2 = 25.01\ldots$$

Further, let us write $\varrho_j = 1/2 + i\gamma_j$, $j \geqslant 0$,

(11.3) $$f(z) = \sum_{j=1}^{\infty} \big(\Gamma(\varrho_j)e^{i\gamma_j z} + \Gamma(\bar\varrho_j)e^{-i\gamma_j z}\big),$$

and

(11.4) $$f_0(z) = \Gamma(\varrho_0)e^{i\gamma_0 z} + \Gamma(\bar\varrho_0)e^{-i\gamma_0 z}.$$

The function f is regular for $|\mathrm{Im}\, z| < \pi/2$; f_0 is an integral function.

Lemma 11.1 *For $z = x+iy$, $|y| \leqslant \pi/4$, we have*

(11.5) $$|f(z)| \leqslant 5\sqrt{2\pi}\, e^{-(\pi/2-|y|)\gamma_1}.$$

Proof. The well-known results

(11.6) $$|\Gamma(1/2+iy)| = \sqrt{\frac{\pi}{\cosh \pi y}} \leqslant \sqrt{2\pi}\, e^{-(\pi/2)|y|},$$

(11.7) $$\sum_{j=0}^{\infty} \gamma_j^{-2} < 0.0233 \quad \text{(see [10])},$$

and also the numerical estimates (11.2) furnish the estimate

(11.8) $$|f(z)| \leqslant 2\sqrt{2\pi}\, e^{-\gamma_1(\pi/2-|y|)}\Big(1 + \frac{16}{\pi^2}(1-\gamma_1/\gamma_2)^{-2}\sum_{j=2}^{\infty}\gamma_j^{-2}\Big)$$

$$\leqslant 5\sqrt{2\pi}\, e^{-\gamma_1(\pi/2-|y|)}$$

and the proof is complete.

Let us denote by C_k, $k = 1, 2, \ldots$, the ractangle with vertices

(11.9) $$u_k \pm i\pi/4, \quad u_{k+1} \pm i\pi/4,$$

where the numbers u_k are defined by (5.3).

Lemma 11.2. *We have*

(11.10) $$|f_0(z)| > 2|f(z)| \quad \textit{for} \quad z \in C_k.$$

Proof. As usual we write $z = x+iy$. By (11.6) we get for $|y| = \pi/4$

(11.11) $$|f_0(z)| \geqslant |\Gamma(\varrho_0)|\, e^{(\pi/4)\gamma_0}(1-e^{-\pi\gamma_0/2})$$

$$\geqslant \tfrac{1}{2}\sqrt{\pi}\, e^{-\pi\gamma_0/4} = \tfrac{1}{2}\sqrt{\pi}\, e^{-\gamma_0(\pi/2-|y|)}.$$

Moreover, for $x = u_k$ or $x = u_{k+1}$, $|y| \leqslant \pi/4$, we have

(11.12) $$|f_0(z)| = 2|\Gamma(\varrho_0)|\cosh(\gamma_0 y) > \tfrac{1}{2}\sqrt{\pi}\, e^{-\gamma_0(\pi/2-|y|)}.$$

On the other hand, Lemma 11.1 and (11.2), (11.11), (11.12) give for $z \in C_k$

(11.13) $$|f(z)| < 5\sqrt{2\pi}\, e^{-\gamma_1(\pi/2-|y|)} < \tfrac{1}{2}|f_0(z)|\big(20\sqrt{2}\, e^{-(\pi/4)(\gamma_1-\gamma_0)}\big) < \tfrac{1}{2}|f_0(z)|,$$

the required result.

12. Proof of Theorem 5.1. The formula (6.3) can be written as follows:

(12.1) $$G(z) = -e^{z/2}\big(f_0(z) + f(z)\big) + h(z).$$

It is easy to see that for $z \in C_k$, $k \geq 1$, we have

$$|f_0(z)| \geq c_{13} \tag{12.2}$$

for a certain positive numerical constant c_{13} independent of k.

Using (12.1), (12.2) together with Lemma 11.2, we get

$$\begin{aligned} |-e^{z/2} f_0(z)| &= e^{x/2} |f_0(z)| \\ &> e^{x/2} |f(z)| + \tfrac{1}{2} c_{13} e^{x/2} > |-e^{z/2} f(z) + h(z)|, \end{aligned} \tag{12.3}$$

for $z \in C_k$ and sufficiently large k.

Now, the Rouché theorem implies that for large k the functions f_0 and G have equally many zeros inside C_k. But if $z_0 = x_0 + iy_0$ is a zero of f_0 then

$$|\Gamma(\varrho_0) \exp(i\gamma_0 z_0)| = |\Gamma(\bar{\varrho}_0) \exp(-i\gamma_0 z_0)|;$$

thus $y_0 = 0$. The function f_0 has therefore only real zeros. But for real x we have:

$$f_0(x) = 2|\Gamma(\varrho_0)| \cos(\gamma_0 x + \varphi). \tag{12.4}$$

Hence, f_0 has only one real and simple zero inside C_k. Since $G(\bar{z}) = \overline{G(z)}$, the same statement is true for G. This proves assertion 2 of our theorem.

The first assertion now easily follows. It suffices to note that

$$\begin{aligned} |G(u_k)| &\geq e^{u_k/2} (|f_0(u_k)| - |f(u_k)|) + O(1) \\ &> \tfrac{1}{2} e^{u_k/2} |f_0(u_k)| + O(1) = |\Gamma(\varrho_0)| e^{u_k/2} + O(1), \end{aligned} \tag{12.5}$$

for k large enough.

As regards assertion 3, i.e. the equality (5.6), we have

$$V_5(T) = \sum_{0 < u_k \leq \log T} 1 + O(1) = \left[\frac{\gamma_0 \log T + \varphi}{\pi}\right] + O(1) = \frac{\gamma_0}{\pi} \log T + O(1). \tag{12.6}$$

13. The function K defined by (5.7) is regular for $|\mathrm{Im}\, z| < \pi/2$. Further properties of this function are described by the following lemma (we assume the Riemann hypothesis).

LEMMA 13.1. 1. *The function K is almost periodic in the sense of Bohr, in the strip $|\mathrm{Im}\, z| < \pi/2$. This means that for every real y, $|y| < \pi/2$, the function*

$$K_y(t) = K(t + iy)$$

is almost periodic.

2. *All zeros of K in the strip $|\mathrm{Im}\, z| \leq \pi/4$ are real.*

3. *Let us define the function ω by*

$$\omega(z) = \min_v |z - v|, \quad |\mathrm{Im}\, z| \leq \pi/4 \tag{13.1}$$

where the minimum is taken over all the real zeros of K.

Then for $z = x + iy$, $|y| < \pi/8$, we have

$$|K(z)| \geq c_{14}\, \omega(z), \tag{13.2}$$

for an absolute constant $c_{14} > 0$.

Proof. 1 is obvious.

2. In the same way as in the proof of Theorem 5.1 one can show that K has only real zeros in the region $z = x + iy$, $x > x_0$, $|\mathrm{Im}\, z| \leq \pi/4$, for a sufficiently large x_0.

Suppose $K(z_1) = K(x_1 + iy_1) = 0$ with $0 < |y_1| \leq \pi/4$. Then in view of the almost periodicity, K has infinitely many zeros in the horizontal strip $|y - y_1| < \varepsilon$ for every arbitrarily small $\varepsilon > 0$. But this is impossible.

3 follows from the analysis of [3], Chapter II.

14. Proof of Theorem 5.2. Analogously to Section 12 we can assert that the function

$$\Psi(T) = V(T) - \frac{\gamma_0}{\pi} T \tag{14.1}$$

is bounded.

So it remains to prove that Ψ is the Stepanov almost periodic function from the class S^p, for every $p \geq 1$.

From Lemma 13.1, 3 it follows that for every y, $0 < |y| < \pi/8$, we have

$$\inf_{x \in R} |K(x + iy)| > 0. \tag{14.2}$$

Hence, by the well-known Bohr theorem (compare [3]),

$$\mathrm{Arg}\, K(x + iy) = c_0(y) x + \tilde{\Psi}(x, y), \tag{14.3}$$

where $\tilde{\Psi}$ is again an almost periodic function of x.

Let us denote by L the polygonal line with vertices

$$0, \quad -iy, \quad T - iy, \quad T \quad (0 < y < \pi/8), \tag{14.4}$$

and let us assume that $K(T) \neq 0$. Then

$$\begin{aligned} V(T) &= \frac{1}{\pi} \Delta_L \mathrm{Arg}\, K(z) \\ &= \frac{c_0(y)}{\pi} T + \frac{1}{\pi} \tilde{\Psi}(T, y) + O\Big(\Big|\overset{iy}{\underset{0}{\Delta}} \mathrm{Arg}\, K(z)\Big| + \Big|\overset{T}{\underset{T-iy}{\Delta}} \mathrm{Arg}\, K(z)\Big|\Big). \end{aligned} \tag{14.5}$$

This formula can be also used as a definition of Ψ for $T < 0$.

Since $V(T) \sim (\gamma_0/\pi)T$, we have

(14.6) $$c_0(y) = \gamma_0 \quad \text{for every } 0 < y < \pi/8.$$

Moreover, Lemma 13.1, 3 yields

(14.7) $$\Big|\mathop{\Delta}_{T-iy}^{T} \operatorname{Arg} K(z)\Big| \ll \int_{-y}^{0} \Big|\frac{K'}{K}(T+it)\Big| dt \ll \frac{y}{\omega(T)}.$$

Similarly, since $K(0) \neq 0$ (compare [6], page 165), we have

(14.8) $$\Big|\mathop{\Delta}_{0}^{-iy} \operatorname{Arg} K(z)\Big| \ll y.$$

Let $\varepsilon > 0$ be fixed.

Using (14.5)–(14.8) we get

(14.9) $$\Psi(T) = \Psi_\varepsilon(T) + O(\varepsilon) \quad \text{for} \quad \omega(T) \geqslant \varepsilon^p,$$

where

(14.10) $$\Psi_\varepsilon(T) = \frac{1}{\pi}\tilde{\Psi}(T, \varepsilon^{p+1}).$$

Since Ψ_ε is almost periodic, there exists a real number $l > 0$ such that every interval of length l contains a number τ such that

(14.11) $$\sup_T |\Psi_\varepsilon(T+\tau) - \Psi_\varepsilon(T)| < \varepsilon.$$

Let

(14.12) $$X_1 = \{t \in [0, 1] \mid \omega(T+\tau+t) \geqslant \varepsilon^p \text{ and } \omega(T+t) \geqslant \varepsilon^p\}$$

and

(14.13) $$X_2 = [0, 1] \setminus X_1.$$

Then

(14.14) $$m(X_2) \ll \varepsilon^p.$$

Owing to (14.9)–(14.14), we get

(14.15) $$\int_{X_1} |\Psi(T+t+\tau) - \Psi(T+t)|^p dt \ll \varepsilon^p,$$

(14.16) $$\int_{X_2} |\Psi(T+t+\tau) - \Psi(T+t)|^p dt \ll \varepsilon^p.$$

Hence

(14.17) $$\Big\{\int_0^1 |\Psi(T+t+\tau) - \Psi(T+t)|^p dt\Big\}^{1/p} \ll \varepsilon$$

and the result follows.

15. We now prove (5.10). Each one of the functions G and K has exactly one zero inside the rectangle C_k for large k. Let y_i be such a zero of K. Then for $|z - y_i| = c_{15}\exp(-y_i/2)$ with sufficiently large c_{15} we have

(15.1) $$|K(z)| \geqslant c_{14} c_{15} e^{-y_i/2} > |e^{-z/2} h(z)|,$$

where h is defined by (6.4). Thus, by Rouché's theorem, G has a zero inside the circle $|z - y_i| = c_{15}\exp(-y_i/2)$.

Since the sequences x_i and y_i may differ significantly only in a finite number of initial terms, there exists an integer l such that

(15.2) $$\log x_{j+l} = y_i + O(e^{-y_j/2})$$

for large j; (5.10) therefore follows.

References

[1] G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. 41 (1918), pp. 119–196.

[2] A. E. Ingham, *A note on the distribution of primes*, Acta Arith. 1 (1936), pp. 201–211.

[3] B. Jessen and H. Tornehave, *Mean motions and zeros of almost periodic functions*, Acta Math. 77 (1945), pp. 137–279.

[4] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. I*, Acta Arith. 44 (1984), pp. 365–377.

[5] — *On sign-changes in the remainder-term of the prime-number formula, II*, ibid. 45 (1985), pp. 65–74.

[6] S. Knapowski and W. Staś, *A note on a theorem of Hardy and Littlewood*, ibid. 7 (1962), pp. 161–166.

[7] — — *Another note on Hardy–Littlewood's theorem*, ibid. 8 (1963), pp. 205–212.

[8] G. Pólya, *Über das Vorzeichen des Restgliedes im Primzahlsatz*, Gött. Nachrichten, 1930, pp. 19–27.

[9] — *On polar singularities of power series and of Dirichlet series*, Proc. London Math. Soc. (2), 33 (1932), pp. 81–101.

[10] J. B. Rosser, *The n-th prime is greater that n log n*, Proc. London Math. Soc. (1939), pp. 21–44.

[11] L. Schwartz, *Analyse Mathématique*, Hermann 1967.

[12] W. Stiepanov, *Ueber einige Verallgemeinerungen der fastperiodischen Funktionen*, Acta Math. 95 (1926), pp. 473–498.

INSTITUTE OF MATHEMATICS OF THE ADAM MICKIEWICZ UNIVERSITY
Poznań, Poland

Received on 6.9.1985
and in revised form on 19.3.1986 (1539)