Acta Math. Hung.
49 (3—4) (1987), 441—453.

OSCILLATORY PROPERTIES OF ARITHMETICAL FUNCTIONS. II

J. KACZOROWSKI (Poznan) and J. PINTZ (Budapest)

1. Introduction and statement of results

In the first paper of this series, [5], we have proved that the real-valued function $f(x)$ has at least $c(f)\log Y$ sign-changes in the interval $(0, Y]$ provided the analytic continuation of its Mellin transform

$$F(s)=\int_0^\infty f(x)x^{-s-1}\,dx \tag{1.1}$$

has all singularities of the form

$$P_\nu(s-\varrho_\nu)\log(s-\varrho_\nu)+F_\nu(s) \tag{1.2}$$

where $F_\nu(s)$ is meromorphic at $s=\varrho_\nu$, and P_ν is a polynomial (or $P_\nu\equiv 0$).

Using the method of [3] it is possible to generalize it for a much wider class of functions $f(x)$. Before we state our theorem let us recall that for any real function $f(x)$, $x>0$ we define the number $V(f, Y)$ of sign-changes in the interval $(0, Y]$ as follows:

$$V(f,Y)=\sup\{N:\ \exists\{x_i\}_{i=1}^N,\ 0<x_1<\ldots<x_N\leqq Y,\\ f(x_i)\neq 0,\ \operatorname{sgn}f(x_i)\neq\operatorname{sgn}f(x_{i+1}),\ 1\leqq i<N\}. \tag{1.3}$$

Moreover, we shall say that $V(f, Y)>h(Y)$ with combined oscillation of size $g(x)$ if there exists a series $\{x_i\}_{i=1}^{h(Y)}$ with $\operatorname{sgn}f(x_i)\neq\operatorname{sgn}f(x_{i+1})$ and

$$|f(x_i)|\geqq g(x_i). \tag{1.4}$$

Under these notations our theorem may be formulated as follows:

THEOREM. *Let $f(x)$ be real for $x>0$ and suppose that $\int_0^\infty f(x)x^{-s-1}\,dx$ converges absolutely for $\sigma\geqq\sigma_1$ and represents in that half-plane a function $F(s)$ having the following properties:*

(1) *$F(s)$ is regular for $\sigma>\Theta$ but not in any half-plane $\sigma>\Theta-\varepsilon$ with $\varepsilon>0$,*

(2) *there exists a denumerable (finite or infinite) set of singularities of $F(s)$, $S=\{\varrho_\nu=\beta_\nu\pm i\gamma_\nu\}$, $\gamma_\nu>0$, without finite limit point satisfying $\Theta-c_0\leqq\beta_\nu\leqq\Theta$ for some $c_0>0$ and such that $F(s)$ can be continued as a meromorphic function in the open set D obtained by making cuts $s=\sigma\pm i\gamma_n$, $\sigma\leqq\beta_n$, in the half-plane $\sigma>\Theta-c_0$.*

(3) *For $|s-\varrho_\nu|\leqq\eta_\nu$, $\eta_\nu>0$, $s\in D$:*

$$F(s)=(s-\varrho_\nu)^{a_\nu}\sum_{t=0}^{k_\nu}g_{\nu t}(s-\varrho_\nu)\log^t(s-\varrho_\nu)+F_\nu(s-\varrho_\nu), \tag{1.5}$$

where $g_{\nu t}$ *and* F_ν *are regular for* $|z|<\eta_\nu$, k_ν *is a nonnegative integer,* a_ν *an arbitrary complex number and* $g_{\nu k_\nu}(0)\neq 0$.

Let $\gamma=\min\limits_{\beta_\nu=\Theta}\gamma_\nu$ *and* $\gamma=\infty$ *if* $\beta_\nu<\Theta$ *for all* $\nu=1,2,\ldots$. *Under these conditions we have*

$$\varliminf_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}, \tag{1.6}$$

and every interval of the form

$$[Y^{1-\varepsilon},Y],\quad Y>Y_0(\varepsilon) \tag{1.7}$$

contains at least one sign-change of $f(x)$. *The sign-changes in* (1.6) *and* (1.7) *are combined with an oscillation of size*

$$x^{\Theta-\varepsilon} \tag{1.8}$$

for arbitrary $\varepsilon>0$.

Our theorem has immediate applications to the theory of distribution of prime numbers. Oscillatory properties of the difference $\pi(x)-\operatorname{li}x$ has been discussed in details in [3]. Further applications are the following.

COROLLARY 1. *If* $(l,q)=1$ *and all L-functions* (mod q) *have no real zeros in* $[1/2,1)$ *then for the difference*

$$\Delta_1(x)=\Pi(x,q,l)-\frac{1}{\varphi(q)}\operatorname{li}x=\sum_{n\equiv l(\mathrm{mod}\ q)}\frac{\Lambda(n)}{\log n}-\frac{\operatorname{li}x}{\varphi(q)} \tag{1.9}$$

we have

$$\varliminf_{Y\to\infty}\frac{V(\Delta_1,Y)}{\log Y}>0 \tag{1.10}$$

with combined oscillation of the size $x^{1/2-\varepsilon}$, $\varepsilon>0$. *Moreover, every interval of the form*

$$[Y^{1-\varepsilon},Y],\quad Y\geqq Y(\varepsilon) \tag{1.11}$$

contains at least one sign-change of $\Delta_1(x)$.

The same assertions are true for the functions

$$\Delta_2(x)=\Pi(x;q,l_1)-\Pi(x;q,l_2) \tag{1.12}$$

where $l_1\not\equiv l_2$ (mod q),

$$\Delta_3(x)=\pi(x;q,l)-\frac{\operatorname{li}x}{\varphi(q)} \tag{1.13}$$

where l is a quadratic nonresidue,

$$\Delta_4(x)=\pi(x;q,l_1)-\pi(x;q,l_2) \tag{1.14}$$

where l_1 and l_2 are both quadratic residues or they are both nonresidues (mod q),

$$\Delta_5(x)=\pi(x;q,1)-\frac{\operatorname{li}x}{\varphi(q)} \tag{1.15}$$

and

$$\Delta_6(x) = \pi(x; q, 1) - \pi(x; q, l) \tag{1.16}$$

for any $l \not\equiv 1 \pmod q$.

Still another field of applications of the Theorem is offered by the theory of factorization in algebraic number fields. Let K denote an arbitrary algebraic number field, R_K its ring of integers and $H(K) = \{X_1 = E, X_2, \ldots, X_h\}$ its classgroup (E denotes the unit class). Let further M denote the set of all irreducible algebraic integers in K and let $M(x)$ and $\zeta(s, M)$ denote the counting function of M and the associated zeta-function respectively, i.e.

$$M(x) = \sum_{\substack{N(aR_k) \leqq x \\ a \in M}} 1, \tag{1.17}$$

where from each set of associated integers only one is counted,

$$\zeta(s, M) = \sum_{\substack{aR_k \\ a \in M}} |N(a)|^{-s}, \tag{1.18}$$

and $N = N_{K/Q}$ denotes the norm-function.

If V denotes the set of all sequences $[d_1, \ldots, d_h]$, $d_i \in N \cup \{0\}$ such that $X_1^{d_1} \ldots X_h^{d_h}$ equals E; moreover, the product $X_1^{e_1} \ldots X_h^{e_h}$, $0 \leqq e_i \leqq d_i$ is equal to E if and only if either all e_i's are zero or $e_i = d_i$ holds for $i = 1, 2, \ldots, h$, then

$$\zeta(s, M) = \sum_{[d_1, \ldots, d_h] \in V} \prod_{i=1}^{h} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{m_1 \geqq 1} \cdots \sum_{m_k \geqq 1} \frac{P_i(m_1 s) \ldots P_i(m_k s)}{m_1 \ldots m_k} \tag{1.19}$$

where

$$P_i(s) = \sum_{\mathfrak{p} \in X_i} N\mathfrak{p}^{-s} \tag{1.20}$$

(compare [4]). The summation in (1.20) is taken over all prime ideals from the class X_i.

We define

$$T(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} \zeta(s, M) \frac{x^s}{s} ds; \tag{1.21}$$

$\mathscr{C}$ denotes the curve of integration consisting of the segment $[1/4, 1 - \varepsilon_0]$ of the lower side of the real axis, the circumference $C(1, \varepsilon_0)$ and the segment $[1 - \varepsilon_0, 1/4]$ of the upper side of real axis, where

$$\varepsilon_0 = 1/2 \min_{\chi \in \widehat{H(K)}} \min_{\substack{\varrho \\ \zeta_K(\varrho, \chi) = 0}} |1 - \varrho|. \tag{1.22}$$

$T(x)$ attains real values only and it was proved in [4] that it is the main term in the asymptotic formula for $M(x)$. $T(x)$ is a rather complicated function of x but for large x its behaviour is described by

$$T(x) \sim x \sum_{r=1}^{\infty} \frac{W_r(\log \log x)}{(\log x)^r} \tag{1.23}$$

(compare [4, Theorem 1]) where W_r are polynomials, deg $W_1 = D-1$ and deg $W_r \leqq D$ for $r \geqq 2$; here D is the Davenport's constant of K (see [6]).

Now we can say more about the difference

$$\Delta_7(x) = M(x) - T(x). \tag{1.24}$$

COROLLARY 2. *If the Dedekind zeta function of the Hilbert class field K_H of K does not vanish in the segment* [1/2, 1) *and has at least one simple zero in the half-plane $\sigma > 1/2$, then*

$$\varlimsup_{Y\to\infty} \frac{V(\Delta_7, Y)}{\log Y} > 0 \tag{1.25}$$

with combined oscillations of size $x^{1/2}$. Moreover, every interval of the form $[Y^{1-\varepsilon}, Y]$, $Y \geqq Y(\varepsilon)$ contains a sign-change of $\Delta_7(x)$.

In the same way as in the proof of Corollary 2 we can treat analogous remainder terms in the asymptotic formulae for the counting functions of certain subsets of R_K (compare [4]). There are some examples of such subsets: the sets F_k, $k=1, 2, \ldots$ of all algebraic integers from K which have at most k factorization into irreducibles, the sets G_k, $k=1, 2, \ldots$ of all algebraic integers from K which have at most k such factorizations of distinct lengths, $F'_k = F_k \cap Z$, $G'_k = G_k \cap Z$, and many others.

Oscillatory properties of the associated remainders depend of course on the analytic properties of the involved zeta-functions. These zeta-functions belong to the ring Ω (see [1]) which is the smallest ring containing all Dirichlet series with abscissas of absolute convergence <1 and also functions of the form $\zeta_K^w(s, \chi)$, $\log^k \zeta_K(a. \chi)$, where K denotes a certain number field, χ is a Hecke character, $w \in C$, Re $w \geqq 0$ if $\chi = \chi_0$ (principal character) and k is a natural number. Hence such zeta functions have analytic continuation into the half-plane $\sigma > \sigma_0$, $\sigma_0 < 1$ and a slightly extended version of our theorem is applicable in all these cases.

It would be interesting to prove a stronger form of Corollary 2 assuming only that $\zeta_{K_H}(s)$ does not vanish in the segment [1/2, 1].

The authors hope to return to this problem on another occasion.

2. Some auxiliary results

Let us introduce the following notations:

$$N = \{1, 2, 3, \ldots\},$$

$$l = \log Y,$$

$$x = e^{\alpha l} = e^{v}, \quad \alpha_1 \leqq \alpha \leqq 1,$$

$$n = bl \ (n \text{ an integer}), \quad 0 < b < \frac{\alpha_1 |\varrho|}{10}, \quad \varrho = \beta + i\gamma, \quad \gamma > 0,$$

$0 < \eta < |\varrho|/10$,

$g(z)$ a regular function for $|z| \leqq \eta$ with $g(0) \neq 0$, $k \in N^* = \{0, 1, 2, \ldots\}$, B an arbitrary complex number, but $B \notin N$ if $k=0$.

Let $L_a(r)$ consist of the segment $[-a, -r]$ on the lower side of the real axis, the circumference $C(0, r)$ and the segment $[-r, -a]$ on the upper side of the real axis, $0<r\leqq a$, r, a real; $L(r)=L_\infty(r)$.

Let $L_a(r;\varrho)=\{s: s-\varrho\in L_a(r)\}$. Let

$$\tilde{\Gamma}(\omega, B)=\frac{1}{2\pi i}\int_{L(r)} e^{z\omega}z^{B-1}\,dz$$

which is convergent for every B and every $\omega\in C$ with $\operatorname{Re}\omega>0$, the value of $\tilde{\Gamma}(\omega, B)$ being independent of r by Cauchy's theorem.

Let

$$\tilde{\Gamma}_j(\omega, B)=\frac{d^j\tilde{\Gamma}(\omega, B)}{dB^j}=\frac{1}{2\pi i}\int_{L(r)} e^{z\omega}z^{B-1}\log^j z\,dz,\quad j\in N^*$$

and let $\tilde{\Gamma}(B)=\tilde{\Gamma}(1, B)=\pi^{-1}(\sin\pi B)\Gamma(B)$, according to Hankel's formula (see [7]).

LEMMA 1. *For $l\to\infty$ we have*

$$I(x)=\frac{1}{2\pi i}\int_{L_\eta(r;\,\varrho)}\frac{x^s g(s-\varrho)\log^k(s-\varrho)(s-\varrho)^{B-1}}{s^n}\,ds= \tag{2.1}$$

$$=\frac{x^\varrho}{\varrho^n l^B}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\log l)^{k-j}A_j(\alpha)$$

with

$$A_j(\alpha)=g(0)\tilde{\Gamma}_j\left(\alpha-\frac{b}{\varrho}, B\right)+ \tag{2.2}$$

$$+\frac{1}{l}\left\{g'(0)\tilde{\Gamma}_j\left(\alpha-\frac{b}{\varrho}, B+1\right)+g(0)\frac{b}{2\varrho^2}\tilde{\Gamma}_j\left(\alpha-\frac{b}{\varrho}, B+2\right)\right\}+O(l^{-2})$$

where the constant in the O-symbol may depend on all parameters α_1, b, η, B, k, j, ϱ and on the function g.

PROOF. Writing $\omega=s-\varrho$ and $z=l\omega$ we have

$$I(x)=\frac{x^\varrho}{\varrho^n}\frac{1}{2\pi i}\int_{L_\eta(r)}\frac{e^{v\omega}g(\omega)\log^k\omega\,\omega^{B-1}}{\left(1+\frac{\omega}{\varrho}\right)^n}\,d\omega= \tag{2.3}$$

$$=\frac{x^\varrho}{\varrho^n l^B}\frac{1}{2\pi i}\int_{L_{\eta l}(r)}\frac{e^{\alpha z}g(z/l)(\log z-\log l)^k z^{B-1}}{\left(1+\frac{z}{l\varrho}\right)^n}\,dz=$$

$$\frac{x}{\varrho^n l^B}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\log l)^{k-j}\frac{1}{2\pi i}\int_{L_{\eta l}(r)}\frac{e^{\alpha z}g(z/l)\log^j z\,z^{B-1}}{\left(1+\frac{z}{l\varrho}\right)^n}\,dz,$$

where we denote the integral by $A(\alpha)$.

To evaluate $A(\alpha)$ we note that for $h\in C$, $|h|\leqq 1/10$:

$$(2.4)\qquad (1+h)^{-1}=e^{-h+h^2/2+O(h^3)},\ (1+h)^{-1}\leqq e^{2|h|},$$

further $|z/l\varrho|\leqq|\eta/\varrho|\leqq 1/10$ for $z\in L_{\eta l}(r)$. Since we can choose r arbitrary with $0<r\leqq\eta l$ we choose $r=1$ and consider the shortened integral $A^*(\alpha)$ on $L^*=L_{\log^2 l}(1)$.

Then by (2.4), $b<\alpha_1|\varrho|/10$, and the regularity of g we have

$$(2.5)\qquad \left|A(\alpha)-\frac{1}{2\pi i}\int_{L^*}\frac{e^{\alpha z}g(z/l)\log^j z z^{B-1}}{\left(1+\frac{z}{l\varrho}\right)^n}dz\right|=$$

$$=O\Big(\int_{\log^2 l}^{\eta l}e^{-\alpha x+2blx/l|\varrho|}\log^j x x^{B-1}dx\Big)=O(e^{-(\alpha_1\log^2 l)/2}).$$

Further again by (2.4) we have

$$(2.6)\qquad A^*(\alpha)=\frac{1}{2\pi i}\int_{L^*}\exp\left(\alpha z-\frac{bz}{\varrho}+\frac{bz^2}{2l\varrho^2}+O\left(\frac{z^3}{l^2}\right)\right)g(z/l)\log^j z z^{B-1}dz=$$

$$=\frac{1}{2\pi i}\int_{L^*}e^{\alpha z-bz/\varrho}\left(1+\frac{bz^2}{2l\varrho^2}+O\left(\frac{z^3}{l^2}\right)\right)\left(g(0)+g'(0)\frac{z}{l}+O\left(\frac{z^2}{l^2}\right)\right)\log^j z z^{B-1}dz=$$

$$=\frac{1}{2\pi i}\int_{L^*}e^{z(\alpha-b/\varrho)}\left\{g(0)+g(0)\frac{bz^2}{2l\varrho^2}+g'(0)\frac{z}{l}\right\}\log^j z z^{B-1}dz+O(1/l^2)=$$

$$=\frac{1}{2\pi i}\int_{L(1)}e^{z\left(\alpha-\frac{b}{\varrho}\right)}\left\{g(0)+\frac{1}{l}\left(g'(0)z+\frac{g(0)bz^2}{2\varrho^2}\right)\right\}\log^j z z^{B-1}dz+O(1/l^2),$$

similarly to (2.5). Now (2.6) yields the required result (2.2).

LEMMA 2. *If* $q\in\mathbf{N}$ *then* $\tilde\Gamma_0(\omega,q)=0$ *for* $\operatorname{Re}\omega>0$. *If* $q\notin\mathbf{N}$ *then for* $\Delta<\Delta_0(b,\varrho,q)$ *we hawe* $|\tilde\Gamma_0(\alpha-b/\varrho,q)|>\Delta$ *for* $\alpha\in\bigcup_{r=1}^{r(\Delta)}(h_r,h_r')\subset[\alpha_1,1]$ *where* $r(\Delta)<\infty$, $h_r'\leqq h_{r+1}(r=1,\dots,r(\Delta)-1)$, $H(\Delta)=1-\alpha_1-\sum_{r=1}^{r(\Delta)}(h_r'-h_r)\to 0$ *as* $\Delta\to 0$. *If* $q\in\mathbf{N}$ *then the above assertion holds for* $\tilde\Gamma_1(\alpha-b/\varrho,)$ *in place of* $\tilde\Gamma_0(\alpha-b/\varrho,q)$.

PROOF. If $q\in\mathbf{N}$ it is sufficient to show that for real $\omega>0$ we have $\tilde\Gamma_0(\omega,q)=0$, since for fixed a $\tilde\Gamma_0(\omega,a)$ is a regular function of ω in the half-plane $\operatorname{Re}\omega>0$. But for $\omega\in R^+$ we have with $w=z\omega$

$$(2.7)\qquad \tilde\Gamma_0(\omega,q)=\frac{\omega^{-q}}{2\pi i}\int_{L(r\omega)}e^w w^{q-1}dw=\frac{\Gamma(q)\sin(q\pi)}{\pi\omega^q}=0.$$

If $q \notin \mathbf{N}$ then in view of (2.7), $\tilde{\Gamma}_0(\omega, q) \neq 0$ for $\omega \in R^+$. Since $\tilde{\Gamma}_0(\omega, q)$ is a regular function for $\operatorname{Re}\omega > 0$ the relation

$$\tilde{\Gamma}_0\left(\alpha - \frac{b}{\varrho}, q\right) = 0 \tag{2.8}$$

holds at most for finitely many $\alpha_1', \ldots, \alpha_k' \in [\alpha_1, 1]$.

Let

$$\xi_0(\alpha) = \left|\tilde{\Gamma}_0\left(\alpha - \frac{b}{\varrho}, q\right)\right|.$$

Now we have clearly for every non-negative continuous real function $\xi(\alpha)$ the relation $\xi(\alpha) \geqq \Delta$ for $\alpha \in \bigcup_{r=1}^{r(\Delta)} (h_1, h_r') \subset [\alpha_1, 1]$ where $h_r' \leqq h_{r+1}$ for $r = 1, \ldots, r(\Delta) - 1$, $r(\Delta) < \infty$, $\lim_{\Delta \to 0} \left\{1 - \alpha_1 - \sum_{r=1}^{r(\Delta)} (h_r' - h_r)\right\} = 0$, if $\xi(\alpha)$ has only finitely many zeros in $[\alpha_1, 1]$. If $q \in \mathbf{N}$ then the same is true for $\tilde{\Gamma}_1\left(\alpha - \frac{b}{\varrho}, q\right)$ since then for $\omega \in R^+$

$$\tilde{\Gamma}_1(\omega, q) = \frac{d\tilde{\Gamma}(\omega, q)}{dq} = \frac{d\left\{\frac{\Gamma(q)\sin(\pi q)}{\pi\omega^q}\right\}}{dq} = \frac{\Gamma(q)\cos(\pi q)}{\omega^q} \neq 0. \tag{2.9}$$

Now Lemmas 1 and 2 clearly imply

LEMMA 3. *Let* $m \in \mathbf{N}^*$, $x = e^{\alpha l}$, $0 < \alpha_1 \leqq \alpha \leqq 1$, $l \to \infty$. *Then there exists a* $\Delta = \Delta(\alpha_1, b, \varrho, B, g, m)$ *independent of* l *such that for* $\alpha \in \bigcup_{r=1}^{r(\Delta)} (h_r, h_r') \subset [\alpha_1, 1]$ *where* $r(\Delta) < \infty$, $1 - \alpha_1 - \sum_{r=1}^{r(\Delta)} (h_r' - h_r) \leqq \frac{\alpha_1}{m}$, $h_r' \leqq h_{r+1}$ *for* $r = 1, \ldots, r(\Delta) - 1$, *the relations*

$$I(x) = \frac{(-1)^k(-k)^j x^{\varrho}(\log l)^{k-j}}{\varrho^n l^B}\left\{g(0)\tilde{\Gamma}_j\left(\alpha - \frac{b}{\varrho}, B\right) + O\left(\frac{1}{l}\right)\right\}, \tag{2.10}$$

$$\left|g(0)\tilde{\Gamma}_j\left(\alpha - \frac{b}{\varrho}, B\right)\right| \geqq \Delta$$

hold, where $j = j(B) = 0$ *if* $B \notin \mathbf{N}$ *and* $j = j(B) = 1$ *if* $B \in \mathbf{N}$.

LEMMA 4. *If*

$$J = \left(x_0, x_0 \exp\left(\frac{2\pi}{\gamma}(1 + c)\right)\right) \subset \bigcup_{r=1}^{r(\Delta)} (Y^{h_r}, Y^{h_r'})$$

where $c > 0$ *is an arbitrary constant then for sufficiently large* Y *there exist* $x_1, x_2, x_3 \in J$ *such that*

$$\operatorname{sgn}\operatorname{Re} x_\mu \neq \operatorname{sgn}\operatorname{Re} x_{\mu+1}\ (\mu = 1, 2), \quad |\operatorname{Re} I(x_\mu)| \gg \frac{x_\mu^\beta(\log l)^{k-j(B)}}{|\varrho|^n j^{\operatorname{Re} B}}\ (1 \leqq \mu \leqq 3). \tag{2.11}$$

PROOF. In order to prove (2.11) we have only to note that for $x \in J$ we have

$$\log x = \log x_0 + O(1), \quad \text{i.e.} \quad \alpha = \frac{\log x_0}{l} + O\left(\frac{1}{l}\right). \tag{2.12}$$

Therefore we have for $x \in J$

(2.13)
$$I(x) = \frac{(-1)^k(-k)^j(\log l)^{k-j}}{\varrho^n l^B} g(0)\tilde{\Gamma}_j\left(\frac{\log x_0}{l} - \frac{b}{\varrho}, B\right) x^{\beta+i\gamma}\left(1+O\left(\frac{1}{l}\right)\right),$$

which implies (2.11).

3. Proof of the Theorem

We are entitled to assume $\Theta > 0$, since otherwise we work with $f'(x) = f(x) \cdot x^C$ with suitably chosen C. Also we can assume that $\gamma > 0$ since otherwise we have nothing to prove.

Since the proof is in many aspect similar to that of Theorem 1 in part I [5], we shall be brief at these places.

Similarly to [5, Section 3] we define for an arbitrary function $h(x)$ the operation δ by

$$\delta h(x) = \int_0^x \frac{h(\xi)}{\xi} d\xi, \tag{3.1}$$

and denote by δ_n the n times iterated operation δ. Then we have

$$\delta_n f(x) = \frac{1}{2\pi i} \int_{(\sigma_1)} F(s) \frac{x^s}{s^n} ds. \tag{3.2}$$

Let us choose an $\eta' > 0$ in such a way that $\eta' < c_0$, $\eta' < \Theta/2$ and that the following region and line, resp.,

$$\sigma > \Theta - \eta', \quad |t| \leqq \gamma \quad \text{and} \quad \sigma = \Theta - \eta' \tag{3.3}$$

should contain no singularity except $\Theta \pm i\gamma$, if $\gamma < \infty$.

If $\gamma = \infty$ let η' be defined so that $\eta' < c_0$, $\eta' < \Theta/2$ and that the segment $[\Theta - \eta', \Theta]$ should be free of singularities of $F(s)$.

We shall choose later on a sufficiently large constant T, so that there should be no singularity ϱ of $F(s)$ on the broken line L' defined by

$$L' = \begin{cases} \sigma = \sigma_1 & \text{if} \quad |t| \geqq T, \\ \Theta - \eta' \leqq \sigma \leqq \sigma_1 & \text{if} \quad |t| = T, \\ \sigma = \Theta - \eta' & \text{if} \quad |t| \leqq T, \end{cases} \tag{3.4}$$

but there should be at least one singularity to the right of L'. Let $\varrho_1 = \beta_1 + i\gamma_1$ denote the lowest singularity above the real axis with $\beta > \Theta - \eta'$. If there are more of them then let us choose that with maximal β. Let for $\nu \geqq 1$ $\varrho_{\nu+1} = \beta_{\nu+1} + i\gamma_{\nu+1}$ denote the singularity with minimal $\gamma > \gamma_\nu$, among those with $\beta \geqq \beta_\nu$, $\gamma < T$. If there are more of them then let us choose that with maximal β.

Let us suppose that we obtain exactly m singularities $\varrho_1, \ldots, \varrho_m$ in such a way.

Let us choose η with $\Theta-\eta'\leqq\beta_1-\eta$ and so that the domains

(3.5) $$\{\beta_\nu-\eta\leqq\sigma<\beta_\nu,\ \gamma_\nu\leqq|t|\leqq\gamma_{\nu+1}\}$$

should be free of singularities of $F(s)$, for $\nu=1,2,\dots,m$, further let

(3.6) $$\eta<\min_{1\leqq\nu\leqq m}\min\left(\eta_\nu,|\varrho_\nu|/20,\ \frac{\gamma_\nu-\gamma_{\nu-1}}{2}\right)$$

where we define $\gamma_0=0$, $\gamma_{m+1}=T$.

Let us choose now with an $r<\eta$, defining $\gamma_{m+1}=T+\eta$, $\beta_0=\Theta+\eta-\eta'$, the following broken line:

(3.7) $$L=\{\sigma=\sigma_1,T\leqq|t|\}\cup\{\sigma_1\geqq\sigma\geqq\beta_m-\eta,|t|=T\}$$
$$\bigcup_{\nu=m}^{1}[\{\sigma=\beta_\nu-\eta,\gamma_{\nu+1}-\eta\geqq|t|\geqq\gamma_\nu\}\cup L_\eta(r,\varrho_\nu)\cup$$
$$\cup\{\sigma=\beta_\nu-\eta,\gamma_\nu\geqq|t|\geqq\gamma_\nu-\eta\}\cup\{\beta_\nu-\eta\geqq\sigma\geqq\beta_{\nu-1}-\eta,|t|=\gamma_\nu-\eta\}]\cup$$
$$\cup\{\sigma=\Theta-\eta',\gamma_1-\eta\geqq|t|\}.$$

Then $F(s)$ is regular on L and to the right of L, so we obtain with the notation $L^+=\{s\in L:\ \mathrm{Re}\,s\geqq0\}$

(3.8) $$\delta_nf(x)=\int_L F(s)\frac{x^s}{s^n}\,ds=2\,\mathrm{Re}\int_{L^+}F(s)\frac{x^s}{s^n}\,ds$$

since L is symmetric to the real axis and $F(\bar s)=\overline{F(s)}$.

Using the notation

(3.9) $$J_\nu(x)=\int_{L_\eta(r,\varrho_\nu)}F(s)\frac{x^s}{s^n}\,ds$$

we have by easy calculation

(3.10) $$\left|\delta_nf(x)-2\sum_{\nu=1}^{m}\mathrm{Re}\,J_\nu(x)\right|\ll\frac{x^{\sigma_1}}{nT^{n-1}}+\frac{x^{\Theta-\eta'}}{(\Theta/2)^n}+\sum_{\nu=1}^{m}\frac{x^{\beta_\nu-\eta}}{(|\varrho_\nu|-2\eta)^n}.$$

Let

(3.11) $$n=[b\log Y],\quad x=Y^\alpha,\quad \sqrt{b}\leqq\alpha\leqq1.$$

If we fix b satisfying

(3.12) $$b<b_0=b_0(\eta')<\min\left(1/100,(|\varrho_1|/10)^2\right)$$

where b_0 is chosen sufficiently small (but independently of T) then with a positive constant $d_1=d_1(\eta',b)$

(3.13) $$\frac{x^{\Theta-\eta'}}{(\Theta/2)^n}\ll\frac{x^{\beta_1}}{|\varrho_1|^n}Y^{-d_1}.$$

Further we have by $|\varrho_1|\leqq\dots\leqq|\varrho_n|$ and (3.6)

(3.14)
$$\frac{x^{\beta_\nu-\eta}}{(|\varrho_\nu|-2\eta)^n}\leqq\frac{x^{\beta_\nu}}{|\varrho_\nu|^n}x^{-\eta}e^{4n\eta/|\varrho_\nu|}\leqq\frac{x^{\beta_\nu}}{|\varrho_\nu|^n}Y^{-\eta\sqrt{b}(1-4\sqrt{b}/|\varrho_1|)}<\frac{x^{\beta_\nu}}{|\varrho_\nu|^n}Y^{-\eta\sqrt{b}/2}.$$

Now we fix T satisfying

$$T = T_0(b, \eta, \eta') \tag{3.15}$$

where T_0 is chosen sufficiently large. Then with a positive constant $d_2=d_2(\eta', \eta, b, T)$

$$\frac{x^{\sigma_1}}{nT^{n-1}} \ll \frac{x^{\beta_1}}{|\varrho_1|^n}\cdot Y^{-d_2}. \tag{3.16}$$

So we have by (3.10)—(3.16) with $d_3=\min(d_1, d_2, \eta\sqrt{b}/2)>0$

$$\left|\delta_n f(x) - 2\sum_{\nu=1}^{m} \operatorname{Re} J_\nu(x)\right| \ll \max_{1\leq\nu\leq m} \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-d_3}. \tag{3.17}$$

From now on we consider b, η, η', T (hence $\varrho_1, \ldots, \varrho_m$) as fixed constants. Further we restrict ourselves for the sake of convenience for those values of Y for which $b \log Y$ is an integer. But it is clearly sufficient to show $V(Y)\geq c\log Y$ $(Y\geq Y_0)$ for these values of Y, since this implies $V(Y)\geq c\log Y+O(1)$ for all Y.

Similarly to [5, (3.12)—(3.17)] there exists a small positive constant $d_4=d_4(b, \varrho_1, \ldots, \varrho_m)$ such that we have disjoint intervals $(e_\nu, e_\nu')\subset[\sqrt{b}, 1]$ (where $e_\nu=e_\nu'$ is possible), $\nu=1, \ldots, m$ with total length at least

$$1-2\sqrt{b} \tag{3.18}$$

so that for $1\leq\nu\leq m$

$$\max_{\substack{1\leq\mu\leq m\\ \mu\neq\nu}} \frac{x^{\beta_\mu}}{|\varrho_\mu|^n} < \frac{x^k}{|\varrho_\nu|^n} Y^{-d_4} \quad \text{if} \quad x\in(Y^{e_\nu}, Y^{e_\nu'}). \tag{3.19}$$

Let ν be fixed and let us consider from now on always $x\in(Y^{e_\nu}, Y^{e_\nu'})$. Then we have with $d_5=\min(d_3, d_4)>0$

$$\left|\delta_n f(x) - 2\sum_{\mu=1}^{m} \operatorname{Re} J_\mu(x)\right| \ll \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-d_5}. \tag{3.20}$$

Denoting the integral $I(x)$ in (2.1) of Lemma 1 by $I_{\mu t}(x)$ in case of $\varrho=\varrho_\mu$, $g=g_{\mu t}$, $B=B_\mu=a_\mu+1$, $k=t$, we have by the regularity of $F_\mu(s)$

$$J_\mu(x) = \sum_{t=0}^{k_\mu} I_{\mu t}(x). \tag{3.21}$$

Taking into account Lemma 3 we obtain

$$\left|\delta_n f(x) - 2\operatorname{Re} I_{\nu k_\nu}(x)\right| \ll \frac{x^{\beta_\nu}}{|\varrho_\nu|^n l^{\operatorname{Re} a_\nu}} (\log l)^{k_\nu-1-j(a_\nu)} \tag{3.22}$$

where as before

$$j(a_\nu) = \begin{cases} 1 & \text{if } a_\nu\in\mathbf{N}^*, \\ 0 & \text{if } a_\nu\notin\mathbf{N}^*. \end{cases}$$

Thus in view of (3.22) and Lemma 3 the oscillatory behaviour of $\delta_n f(x)$ is completely described by that of $I_{\nu k_\nu}(x)$, which is given by Lemma 4 and this clearly yields that $\delta_n f(x)$ has at least

$$(3.23)\qquad 2\left[\frac{\left(e'_\nu - e_\nu - \frac{2\sqrt{b}}{m}\right)\log Y}{(2\pi/\gamma_\nu)(1+b)}\right] \geqq 2\left[\frac{\left(e'_\nu - e_\nu - \frac{2\sqrt{b}}{m}\right)\log Y}{(2\pi/\gamma_1)(1+b)}\right]$$

sign changes in the interval $[Y^{e_\nu}, Y^{e'_\nu}]$, if Y is large enough.

This yields in view of (3.18)

$$(3.24)\qquad V(\delta_n f, Y) \geqq (1-5\sqrt{b})\,\frac{\gamma_1}{\pi}\log Y.$$

Since for any real function $h(x)$ we have

$$(3.25)\qquad V(h, Y) \geqq V(\delta h, Y),$$

inequality (3.24) clearly proves (1.6) if $\gamma<\infty$ since then $\gamma=\gamma_1$ and b can be chosen arbitrarily small. If $\gamma=\infty$ we have only to note that for every constant C we have $\gamma_1>C$ if we choose η' so small at the beginning that the domain $\sigma\geqq\Theta-\eta'$, $|t|\leqq C$ should be free of singularities of $F(s)$.

The assertions (1.7) and (1.8) follow completely in the same way as in [5, Section 3]. To show (1.7) we need, apart from (3.22) and Lemma 4, the trivial observation that if for any function $h(x)$, δh has at least $k+1$ sign changes in $[A, B]$ then h has at least k sign changes.

To show (1.8) we have to work with the functions

$$\bar{f}(x) = \begin{cases} f(x) \pm x^{\Theta-\varepsilon}, & x \geqq 1,\\ 0, & 0 \leqq x \leqq 1,\end{cases}$$

$$\bar{F}(s) = F(s) \pm (s-\Theta-\varepsilon)^{-1}$$

and choose at the beginning $\eta'<\varepsilon/2$.

4. Proof of Corollary 1

We shall prove Corollary 1 in case $\Delta_1(x)=\Pi(x;\ q, l)-\dfrac{\operatorname{li} x}{\varphi(q)}$ only. The other proofs are very similar using the fact that the corresponding generating functions have non-real singularities in $\sigma\geqq 1/2$ (see Grosswald [1]) further that Corollary 1 is true for $\Delta_5(x)$ and $\Delta_6(x)$ if the Generalized Riemann Hypothesis is true which follows from the work [2] of Ingham (although he treated only the special case $q=1$). Let

$$(4.1)\qquad f(x) = \begin{cases} 0 \quad \text{for} \quad 0 \leqq x \leqq 2,\\ \Pi(x;\ q, l) - \dfrac{1}{\varphi(q)}\displaystyle\int_2^x \frac{du}{\log u} \quad \text{for } x>2.\end{cases}$$

Then the corresponding $F(s)$ function is

$$F(s) = \frac{1}{s\varphi(q)} \sum_{\chi(\mathrm{mod}\, q)} \chi(l) \log \{(s-1)^{\varepsilon_\chi} L(s,\chi)\} + h(s) \tag{4.2}$$

where

$$\varepsilon_\chi = \begin{cases} 1 & \text{for } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

and $h(s)$ is holomorphic in the whole complex plane.

In the half-plane $\sigma > 0$ $F(s)$ has singularities of the form

$$\frac{k_\varrho}{s\varphi(q)} \log (s-\varrho) + F_\varrho(s) \tag{4.4}$$

where ϱ denotes a zero of $L(s, \chi)$ for some χ, and k_ϱ is a complex number. Moreover, $k_\varrho > 0$ for at least one zero with $\mathrm{Re}\, \varrho \geqq 1/2$ (compare [1]). The results (1.10) and (1.11) therefore follow from our Theorem.

5. Proof of Corollary 2

Let

$$f(x) = \begin{cases} 0 & \text{for } 0 \leqq x \leqq 1, \\ M(x) - T(x) & \text{for } x > 1. \end{cases} \tag{5.1}$$

Then

$$F(s) = \frac{1}{s} \zeta(s, M) - \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{1}{s-z} \frac{\zeta(z, M)}{z} dz. \tag{5.2}$$

Using (1.5) and (5.2) it is easy to prove that $F(s)$ is regular in the half-plane $\sigma > 1/2$ with the cuts $s = \sigma \pm i\gamma$, $1/2 \leqq \sigma \leqq \beta$ where $\varrho = \beta \pm i\gamma$ is a zero of $\zeta_{K_H}(s)$. At these points $F(s)$ has singularities of the form (1.5) with $a_\nu = 0$. At least one of these singularities is non-trivial. Indeed, (1.19) implies that in the neighbourhood of ϱ_0, the simple zero of $\zeta_{K_H}(s)$ with $\mathrm{Re}\, \varrho_0 > 1/2$, we have

$$F(s) = \frac{R}{h^D s} \log^D (s - \varrho_0) + \sum_{j=0}^{D-1} g_j(s) \log^j (s - \varrho_0), \tag{5.3}$$

where $g_j(s)$, $j = 0, 1, \dots, D-1$ are regular near ϱ_0 and

$$R = \sum_{\substack{[d_1, \dots, d_h] \in V \\ d_1 + \dots + d_h = D}} \frac{1}{d_1! \dots d_h!} > 0. \tag{5.4}$$

This proves Corollary 2.

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(Received May 21, 1985)

INSTITUTE OF MATHEMATICS
A. MICKIEWICZ UNIVERSITY
POZNAŃ
POLAND

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13—15.
H—1053