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OSCILLATORY PROPERTIES OF ARITHMETICAL FUNCTIONS. I

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1. Introduction

The first general theorem concerning sign changes of partial sums of arithmetical functions has been proved by E. Landau [8] in 1905 and sounds slightly reformulated as follows (we shall use the notation $s=\sigma+it$ throughout this paper):

THEOREM (Landau). *Let $f(x)$ be real for $x\geqq x_0$; suppose $F(s)=\int\limits_{x_0}^{\infty} f(x)x^{-s-1}\,dx$ is regular for $\sigma>\theta$ but not regular in any half-plane $\sigma>\theta-\varepsilon$ with $\varepsilon>0$. If $F(s)$ is regular at $s=\theta$ then $f(x)$ changes sign infinitely often as $x\to\infty$.*

Unfortunately, this very beautiful and general theorem does not yield any information about the frequency of sign changes. For any real function $f(x)$ defined for $x>0$ we may define the number $V(f, Y)$ of sign changes in the interval $(0, Y]$ as follows:

$$V(f,Y)=\sup\bigl\{N;\ \exists\{x_i\}_{i=1}^{N},\quad 0<x_1<\ldots<x_N\leqq Y,\quad f(x_i)\neq 0,\quad \operatorname{sgn} f(x_i)\neq \operatorname{sgn} f(x_{i+1}),\quad 1\leqq i<N\bigr\}. \tag{1.1}$$

We shall say, that $V(f, Y)>h(Y)$ with combined oscillation of size $g(x)$ if there exists a series $\{x_i\}_{i=1}^{h(Y)}$ with $\operatorname{sgn} f(x_i)\neq\operatorname{sgn} f(x_{i+1})$ and $|f(x_i)|>g(x_i)$.

Imposing more conditions on the function f, Pólya [11] was able to deduce another general theorem concerning the behaviour of the function $V(f, Y)$.

THEOREM (Pólya). *Let $f(x)$ and $F(s)$ satisfy the conditions of Landau's theorem, further let $F(s)$ be meromorphic in some half-plane $\sigma\geqq\theta-c_0$, $c_0>0$. Let $\gamma=\inf\{|t|;$ $F(s)$ is not regular at $s=\theta+it\}$ and let $\gamma=\infty$ if $F(s)$ is regular on $\sigma=\theta$. Then*

$$\varlimsup_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}. \tag{1.2}$$

Finally, Grosswald [3] succeeded in generalizing the theorem of Pólya, for the case when logarithmic singularities with principal part $P_n(s-s_n)\log(s-s_n)$ are allowed in the strip $\theta-c_0\leqq\sigma\leqq\theta$ too, where $P_n(u)$ are polynomials with $\sup\limits_n \deg P_n(u)<\infty$.

The aim of this work is to show that the ideas of the first named author [5] which led to the proof of

$$\varliminf_{Y\to\infty}\frac{V(\psi(x)-x,Y)}{\log Y}>0,\quad \varliminf_{Y\to\infty}\frac{V(\Pi(x)-\operatorname{li}x,Y)}{\log Y}>0 \tag{1.3}$$

can be extended as to show Grosswald's theorem with $\underline{\lim}$ instead of $\overline{\lim}$. This will be our Theorem 1. The corresponding sharpening of Pólya's theorem will be formulated as Corollary 1. Corollary 1 enables us to prove at least $c \log Y$ sign changes for the partial sums of many number theoretic functions, including

$$\psi(x, q, l_1)-\psi(x, q, l_2), \quad (l_1, q) = (l_2, q) = 1, \quad l_1 \not\equiv l_2(q), \tag{1.4}$$

$$M(x) = \sum_{n \leqq x} \mu(n), \tag{1.5}$$

$$R_k(x) = Q_k(x)-\frac{x}{\zeta(k)} = \sum_{n \leqq x} \sum_{d^k | n} \mu(d)-\frac{x}{\zeta(k)}, \tag{1.6}$$

in case of (1.4) using the hypothesis that there are no real positive zeros of L-functions mod q. This condition is necessary in some sense, due to the explicit formula

$$\psi(x, q, l_1)-\psi(x, q, l_2) = \sum_{\chi(q)} \left(\bar{\chi}(l_2)-\bar{\chi}(l_1)\right) \sum_{\varrho=\varrho_\chi} \frac{x^\varrho}{\varrho}+O(\log x). \tag{1.7}$$

In the case of (1.4) Knapowski and Turán proved [7] a very weak lower bound for the number of sign changes under the same condition as our Corollary 2 (in part V). Further they showed (in part VI) $V(f, Y)>c \log\log Y$ under the additional condition that the domain $\sigma>\frac{1}{2}$, $|t| \leqq cq^{10}$ is free of zeros of $L(s, \chi, q)$ functions. In case of (1.5) and (1.6) the best known lower bounds for $V(f, Y)$ were $c \log Y/\log_2^{3/2} Y$ (Pintz [10]) and $c \log\log Y$ (Kátai [6]), resp.

The only general theorem, existing in the literature, which yields concrete lower estimate of $V(f, Y)$ for every value of Y, seems to be Theorem 2 of Kátai [6] which ensures the inequality $V(f, Y) \gg \log\log Y$ for a wide class of functions. It is too long to quote exactly his theorem; however, we may remark that this class includes the functions (1.5) and (1.6) (but not (1.4), for general q). His theorem, although it refers to a smaller class of functions and it gives a weaker lower bound for $V(f, Y)$ has two advantages over our Theorem:

(i) it usually yields effective lower bounds for $V(f, Y)$;

(ii) it ensures a larger (in some cases, apart from a constant factor, optimal) size of oscillation.

Due to some theoretic reasons the method presented does not allow to obtain optimal oscillation. However, it is possible to prove a restricted version of it, Theorem 2, which leads to $V(f, Y)>c \log Y$, as an effective estimate for a rather wide class of functions. The conditions imposed for $F(s)$ are similar to that of Kátai's Theorem 2 [6].

Unfortunately the type of singularities, as required in (3) of our Theorem 1 are not general enough to cover the most important applications with logarithmic singularities as $\pi(x)-\mathrm{li}\, x$, e.g. (which was dealt with in Kaczorowski [5] using more complicated arguments). Thus, we have to remark that it is stated erroneously in Grosswald [3] that his Theorem 6 follows from Theorem 2. (Similarly Theorem D of his paper [4] does not imply Theorems 3, 5a, 18, 20, 22, 24.) Another extension of Pólya's theorem for the case of functions having logarithmic singularities is due to

Levinson [9], although his theorem needs also some modifications to yield the needed applications.

In the 2nd part of this work, we shall extend Theorem 1 for a larger class of functions (including $\pi(x)-\mathrm{li}\,x$, $\pi(x, q, l_1)-\pi(x, q, l_2)$ and some other important arithmetical functions).

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2. Statement of results

We shall prove first a general theorem which, however, yields in most cases ineffective results.

THEOREM 1. *Let $f(x)$ be real for $x>0$ and suppose that the integral $\int_0^\infty f(x)x^{-s-1}dx$ converges absolutely for $\sigma \geqq \sigma_1$ and represents in that half plane a function $F(s)$ having the following properties:*

(1) *$F(s)$ is regular for $\sigma>\theta$ but not in any half plane $\sigma>\theta-\varepsilon$ with $\varepsilon>0$;*

(2) *there exists a denumerable (finite or infinite) set $S=\{\varrho_\nu=\beta_\nu \pm i\gamma_\nu\}$ $(\gamma_\nu \geqq 0)$ without finite limit point satisfying $\theta-c_0 \leqq \beta_\nu \leqq \theta$ for some $c_0>0$ and such that $F(s)$ can be continued as a meromorphic function in the open set D obtained by making the cuts $s=\sigma \pm i\gamma_n$ $(\sigma \leqq \beta_n)$ in the half-plane $\sigma>\theta-c_0$;*

(3) *for $s \to \varrho_\nu$ $(s \in D)$ $F(s)=P_\nu(s-s_\nu)\log(s-s_\nu)+F_\nu(s)$ where $F_\nu(s)$ is meromorphic at $s=\varrho_\nu$, and P_ν is a polynomial ($P_\nu \equiv 0$ is possible too). Let $\gamma=\min_{\beta_\nu=\theta} \gamma_\nu$ and $\gamma=\infty$ if $\beta_\nu<\theta$ for all $\nu=1, 2, \ldots$.*

Under these conditions we have

$$\varliminf_{Y\to\infty} \frac{V(f, Y)}{\log Y} \geqq \frac{\gamma}{\pi}, \tag{2.1}$$

and every interval of the form

$$[Y^{1-\varepsilon}, Y], \quad Y > Y_0(\varepsilon) \tag{2.2}$$

contains at least one sign change of $f(s)$. The sign changes in (2.1) *and* (2.2) *are combined with an oscillation of size (cf. the definition following* (1.1)*)*

$$x^{\theta-\varepsilon}, \tag{2.3}$$

for arbitrary $\varepsilon>0$.

Theorem 1 yields the following sharpening of Pólya's theorem.

COROLLARY 1. *If $f(x)$ is real for $x>0$, $F(s)=\int_0^\infty f(x)x^{-s-1}dx$ converges absolutely for $\sigma \geqq \sigma_1$ and*

(1′) *$F(s)$ is regular for $\sigma>\theta$ but not in any half-plane $\sigma>\theta-\varepsilon$ with $\varepsilon>0$;*

(2′) *$F(s)$ is meromorphic for $\sigma\geqq\theta-c_0$ with some $c_0>0$.*

Then relations (2.1) to (2.3) holds.

We remark that Grosswald [3] needs additionally the condition that $\sup_{1\leqq\nu<\infty}\deg P_\nu<\infty$. His result is with the additional condition

$$\varliminf_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}. \tag{2.4}$$

The example $f(x)=0$ for $x<1$, $f(x)=x^{\beta+i\gamma}+x^{\beta-i\gamma}(x\geqq 1)$ (with the corresponding $F(s)=(s-\beta-i\gamma)^{-1}+(s-\beta+i\gamma)^{-1}$) shows that inequality (2.1) (unlike (2.2)) is best possible, since in this case $V(f,Y)\sim\frac{\gamma}{\pi}\log Y$.

Since in the proof of Theorem 1 many singularities of $F(s)$ may occur and in concrete applications we do not have enough information about the distribution of them (this being the case in the most important number theoretic problems when singularities of $F(s)$ are zeros of the Riemann zeta or Dirichlet's L-functions) we shall prove a second theorem which yields effective results as well. Here only one singularity of $F(s)$ occurs and therefore the conditions might be checked in concrete cases (although they are stronger in some sense than in Theorem 1).

For the aim of concrete applications we give the formulation of Theorem 2 only for meromorphic functions but this can be extended in the same way for the case of logarithmic singularity as Theorem 1.

THEOREM 2. *If $f(x)$ is real for $x>0$, $F(s)=\int_0^\infty f(x)x^{-s-1}dx$ is absolutely convergent for $\sigma>\sigma_1$ and*

(1) *$F(s)$ has a pole at $\varrho_0=\beta_0+i\gamma_0$, $\gamma_0>0$, $\beta_0>0$ with principal part $\sum_{j=1}^{k+1}h_j(s-\varrho_0)^{-j}$;*

(2) *apart from the poles ϱ_0 and $\overline{\varrho_0}$, $F(s)$ is regular on and to the right of the broken line L defined by*

$$L=\begin{cases}|t|\geqq\Gamma, & \sigma=\sigma_1+a_0\\ \beta_0+a_1\leqq\sigma\leqq\sigma_1+a_0, & |t|=\Gamma\\ H\leqq|t|\leqq\Gamma, & \sigma=\beta_0+a_1\\ \beta_0-a_2\leqq\sigma\leqq\beta_0+a_1, & |t|=H\\ |t|\leqq H, & \sigma=\beta_0-a_2,\end{cases} \tag{2.5}$$

where $a_0>0$, $0<a_2<\beta_0$, $-a_2\leqq a_1\leqq\sigma_1+a_0-\beta_0$, $\Gamma\geqq H>\gamma_0$ further

$$d_1=\max\left(\frac{\sigma_1+a_0-\beta_0}{\log\frac{|\sigma_1+a_0+i\Gamma|}{|\varrho_0|}},\ \frac{a_1}{\log\frac{|\beta_0+a_1+iH|}{|\varrho_0|}}\right)<\frac{a_2}{\log\frac{|\varrho_0|}{\beta_0-a_2}}=d_2; \tag{2.6}$$

(3) *$|F(s)|\leqq M$ for $s\in L$.*

Then for every $\varepsilon>0$ *and* $Y>Y_0=Y_0(\varepsilon, a_0, a_1, a_2, \sigma_1, \beta_0, \gamma_0, H, M, \Gamma, h_1, \ldots$ $\ldots, h_{k+1})$, *effective constant, we have*

$$V(f, Y) > \left(1-\frac{d_1}{d_2}-\varepsilon\right)\frac{\gamma_0}{\pi}\log Y. \tag{2.7}$$

We remark that if the singularities of $F(s)$ are the zeros of $\zeta(s)$ then by the calculations of Brent, van de Lune, te Riele and Winter [1] (see also the remark in Zbl. 486 10027), the first 300 million zeros are on the critical line and therefore we may choose

$$\sigma_1 = 1, \beta_0 = \frac{1}{2}, \quad \gamma_0 = 14.13\ldots, \quad H = 21, \quad \Gamma = 10^8, \tag{2.8}$$

and arbitrary values a_i with $a_0>0$, $0<a_1<\frac{1}{2}$, $0<a_2<\frac{1}{2}$.

The most important applications of Theorem 1 (more precisely of Corollary 1), apart from those discussed in [5] (cf. (1.3)) are the following.

COROLLARY 2. *If* $(l_1, q)=(l_2, q)=1$, $l_1 \not\equiv l_2(q)$ *and the L-functions* mod q *have no real zeros in* $\left[\frac{1}{2}, 1\right]$ *then for*

$$\Delta_3(x, q, l_1, l_2) = \Delta_3(x) = \sum_{\substack{n\equiv l_1(q)\\ n\leqq x}} \Lambda(n) - \sum_{\substack{n\equiv l_2(q)\\ n\leqq x}} \Lambda(n) \tag{2.9}$$

(where $\Lambda(n)$ *is von Mangoldt's function) we have*

$$\varlimsup_{Y\to\infty} \frac{V(\Delta_3, Y)}{\log Y} > 0, \tag{2.10}$$

with combined oscillation of size $x^{1/2-\varepsilon}$.

COROLLARY 3. *If* $(l_1, q)=(l_2, q)=1$, $l_1 \not\equiv l_2(q)$, *both* l_1 *and* l_2 *are quadratic non-residues or both are quadratic residues and the L-functions* mod q *have no real zeros in* $\left[\frac{1}{2}, 1\right]$ *then for*

$$\Delta_4(x, q, l_1, l_2) = \Delta_4(x) = \sum_{p\equiv l_1(q)} \log p - \sum_{p\equiv l_2(q)} \log p \tag{2.11}$$

we have

$$\varlimsup_{Y\to\infty} \frac{V(\Delta_4, Y)}{\log Y} > 0, \tag{2.12}$$

with combined oscillation of size $x^{1/2-\varepsilon}$.

We remark that the condition concerning the absence of real zeros of L-functions was verified by Spira [12] for all $q<25$. So for these moduli (2.9)—(2.12) hold without any unproved hypotheses.

COROLLARY 4. *Let* $\mu(n)$ *denote the Möbius-function,* $\theta = \sup_{\zeta(\varrho)=0} \operatorname{Re} \varrho$, $\gamma_1 = \min_{\operatorname{Im} \varrho > 0} \operatorname{Im} \varrho = 14.13...$, *a arbitrary real number and*

(2.13)
$$M_a(x) = \begin{cases} 0 \quad \text{for} \quad x < 1 \\ \sum_{n \leqq x} \mu(n) n^{-a} - \frac{1}{\zeta(a)} \quad \text{for} \quad x \geqq 1. \end{cases}$$

Then we have

(2.14)
$$\varlimsup_{Y \to \infty} \frac{V(M_a, Y)}{\log Y} \geqq \frac{\gamma_1}{\pi}$$

with combined oscillation of size

(2.15)
$$x^{\theta - a - \varepsilon}.$$

Hence in case of $a < \theta$ the constant $1/\zeta(a)$ can be deleted from the definition of $M_a(x)$ or can be substituted by an arbitrary other constant.

By the aid of Theorem 2 we are able to prove a similar but effective theorem for a restricted range of a, however. It is possible to prove e.g., the following version:

COROLLARY 5. *If* $-10^{-8} < a < 1/4$ *and*

(2.16)
$$\overline{M_a}(x) = \sum_{n \leqq x} \mu(n) n^{-a}$$

then for $Y > c_1(a)$, *effective constant, we have*

(2.17)
$$V(M_a, Y) > \frac{1}{3} \log Y.$$

Further, Theorem 2 yields

COROLLARY 6. *Let* $Q_k(x)$ *denote the number of k-free numbers not exceeding x, where* $k \geqq 2$ *is a natural number and let* $R_k(x) = 0$ *for* $x < 1$ *and*

(2.18)
$$R_k(x) = Q_k(x) - \frac{x}{\zeta(k)} = \sum_{\substack{n \leqq x \\ p|n \to p^k \nmid n}} 1 - \frac{x}{\zeta(k)} \quad \text{for} \quad x \geqq 1.$$

Then we have for $Y > c_2(k)$, *effective constant,*

(2.19)
$$V(R_k, Y) > \frac{5}{2k} \log Y.$$

3. Proof of Theorem 1

We are entitled to assume $\theta > 0$ since otherwise we can work with $f(x) \cdot x^c$. Let us define the operation δ by

(3.1)
$$\delta f(x) = \int_0^x \frac{f(\xi)}{\xi} d\xi,$$

and let δ_k be the k times iterated operation, i.e. $\delta_1=\delta$, $\delta_n=\delta\delta_{n-1}$. It is easy to see that

$$\delta_2(f(x)) = \int_0^x \frac{1}{t}\int_0^t \frac{f(u)}{u}\,du\,dt = \int_0^x \frac{f(u)}{u}\log\frac{x}{u}\,du = \tag{3.2}$$

$$= \int_0^\infty \frac{f(u)}{u}\cdot\frac{1}{2\pi i}\int_{(\sigma_1)}\frac{(x/u)^s}{s^2}\,ds = \frac{1}{2\pi i}\int_{(\sigma_1)} F(s)\frac{x^s}{s^2}\,ds$$

and we obtain by induction according to n that

$$\delta_n f(x) = \frac{1}{2\pi i}\int_{(\sigma_1)} F(s)\frac{x^s}{s^n}\,ds. \tag{3.3}$$

Let us consider first the case when $F(s)$ is meromorphic for $\sigma\geqq\theta-c_0$ (or at least for $\sigma\geqq\theta-\eta$, $|t|\leqq\Gamma$ with some $\eta>0$ and with a sufficiently large Γ).

We may suppose $\gamma>0$ otherwise we have nothing to prove. Let us choose $\eta>0$ in such a way that $\eta<c_0$, $\eta<\frac{\theta}{2}$ and that the following region and line,

$$\sigma > \theta-\eta,\quad |t|\leqq\gamma \quad\text{and}\quad \sigma=\theta-\eta,\quad -\infty<|t|<\infty, \tag{3.4}$$

resp., should contain no singularity of $F(s)$ except $\theta\pm i\gamma$, if $\gamma<\infty$. If $\gamma=\infty$, let η be defined so that $\eta<c_0$, $\eta<\frac{\theta}{2}$ and that the segment $[\theta-\eta,\theta]$ should be free of singularities of $F(s)$.

Later on we shall choose a sufficiently large constant Γ so that there should be no singularity ϱ of $F(s)$ on the broken line L defined by

$$L = \begin{cases} |t|\geqq\Gamma, & \sigma=\sigma_1\\ \theta-\eta\leqq\sigma\leqq\sigma_1, & |t|=\Gamma\\ |t|\leqq\Gamma, & \sigma=\theta-\eta\end{cases} \tag{3.5}$$

but there should be at least one singularity to the right of L. Then we have

$$\delta_n f(x) = 2\sum_{\nu=1}^m \operatorname{Re}\left\{\operatorname{Res}\left(F(s)\frac{x^s}{s^n}\right)_{s=\varrho_\nu}\right\} + O\left(\frac{x^{\theta-\eta}}{(\theta/2)^n}\right) + O\left(\frac{x^{\sigma_1}}{n\Gamma^{n-1}}\right) =: \tag{3.6}$$

$$=: 2\sum_{\nu=1}^m \operatorname{Re}\left\{A_\nu(x)\frac{x^{\varrho_\nu}}{\varrho_\nu^n}\right\} + R_1 + R_2,$$

where $\varrho_1, \dots, \varrho_m$ $(m\geqq1)$ are the singularities of $F(s)$ above the real axis and right of L, numerated according to $0<\gamma_1\leqq\dots\leqq\gamma_m$.

Let

$$n = [b\log Y],\quad x = Y^\alpha,\quad \sqrt{b}\leqq\alpha\leqq1. \tag{3.7}$$

If we fix b satisfying

$$b < b_0 = b_0(\eta) < \min\left(\frac{1}{100}, \left(\frac{\theta}{4}\right)^2\right) \tag{3.8}$$

where b_0 is chosen sufficiently small (but independently of Γ) then with a positive d_1

$$(3.9)\qquad R_1 \ll \frac{x^{\beta_1}}{|\varrho_1|^n} Y^{-d_1}.$$

Now we fix Γ satisfying

$$(3.10)\qquad \Gamma > \Gamma_0(b, \eta)$$

where Γ_0 is chosen sufficiently large. Then with a positive d_2 we have

$$(3.11)\qquad R_2 \ll \frac{x^{\beta_1}}{|\varrho_1|^n} Y^{-d_2}.$$

Later on we shall choose Δ sufficiently small with

$$(3.12)\qquad \Delta < b \min_{\substack{1 \leqq \nu < \mu \leqq n \\ \beta_\nu = \beta_\mu}} \big|\log|\varrho_\nu| - \log|\varrho_\mu|\big| = \Delta_0.$$

Then it is easy to see that the inequality

$$(3.13)\qquad \big|(\beta_\nu \alpha - b \log|\varrho_\nu|) - (\beta_\mu \alpha - b \log|\varrho_\mu|)\big| > \Delta$$

holds for all ν, μ with $1 \leqq \nu \leqq \mu \leqq m$ and for all $\alpha \in [0, 1]$ apart from finitely many intervals of total length at most

$$(3.14)\qquad \Delta D, \quad D = D(\Gamma, \eta) = 2 \sum_{\substack{1 \leqq \nu < \mu \leqq m \\ \beta_\nu \neq \beta_\mu}} \frac{1}{|\beta_\nu - \beta_\mu|}.$$

Let us choose

$$(3.15)\qquad \Delta = \min\left(\frac{\sqrt{b}}{D}, \Delta_0\right).$$

In such a way we obtain disjoint intervals of the form $(e_\nu, e'_\nu) \subset [\sqrt{b}, 1]$ ($e_\nu = e'_\nu$ is possible) of total length at least

$$(3.16)\qquad 1 - \Delta D - \sqrt{b} \geqq 1 - 2\sqrt{b}$$

such that for $1 \leqq \nu \leqq m$

$$(3.17)\qquad \max_{\substack{1 \leqq \mu \leqq m \\ \mu \neq \nu}} \frac{x^{\beta_\mu}}{|\varrho_\mu|^n} < \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-\Delta} \quad \text{if} \quad x \in (Y^{e_\nu}, Y^{e'_\nu}).$$

Taking into account that with the notation in (3.16)

(3.18)
$$\left(\frac{d}{ds^j} \frac{x^s}{s^n}\right)_{s=\varrho_\mu} = \frac{x^{\varrho_\mu}}{\varrho_\mu^n} \sum_{l=0}^{j} \binom{j}{l} \log^{j-l} x \cdot (-n) \dots (-n-l+1) \varrho_\mu^{-l} \ll_j \frac{x^{\beta_\mu}}{|\varrho_\mu|^n} \left(\frac{\log Y}{\theta/2}\right)^j$$

we obtain by (3.9), (3.11) and (3.17) with a positive d_3

$$(3.19)\qquad \delta_n f(x) = 2 \operatorname{Re}\left\{A_\nu(x) \frac{x^{\varrho_\nu}}{\varrho_\nu^n}\right\} + O\left(\frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-d_3}\right) \quad \text{if} \quad x \in (Y^{e_\nu}, Y^{e'_\nu}).$$

If the principal part of $F(s)$ at $s=\varrho_\nu$ has the form $(k\geqq 0,\ h_{k+1}\neq 0)$ $\sum_{j=1}^{k+1} h_j(s-\varrho_\nu)^j$ then we have by (3.18)

$$A_\nu(x)=\frac{h_{k+1}}{k!}\sum_{l=0}^{k}\binom{k}{l}\log^{k-l}x\cdot(-n)\dots(-n-l+1)\varrho_\nu^{-l}+O(\log^{k-1}Y)=$$

$$(3.20)\qquad =\frac{h_{k+1}}{k!}\sum_{l=0}^{k}\binom{k}{l}\log^{k-l}x(-n)^l\varrho_\nu^{-l}+O(\log^{k-1}Y)=$$

$$=\frac{h_{k+1}}{k!}\left(\log x-\frac{n}{\varrho_\nu}\right)^k+O(\log^{k-1}Y)=\frac{h_{k+1}}{k!}\left(\log x-\frac{n}{\varrho_\nu}\right)^k\left(1+O\left(\frac{1}{\log Y}\right)\right),$$

owing to (3.7)—(3.8). If $v=\log x$ runs over an interval

$$(3.21)\qquad v\in I=\left(v_0,\ v_0+\frac{2\pi}{\gamma_\nu}(1+b)\right)\subset(Y^{e_\nu},\ Y^{e'_\nu})$$

then by (3.7)—(3.8)

$$(3.22)\qquad \left(v-\frac{n}{\varrho_\nu}\right)^k=\left(v_0-\frac{n}{\varrho_\nu}\right)^k\left(1+O\left(\frac{1}{\log Y}\right)\right).$$

Therefore we have for $J=\left(X_0,\ X_0\exp\left(\frac{2\pi}{\gamma_\nu}(1+b)\right)\right)\subset(Y^{e_\nu},\ Y^{e'_\nu})$

$$(3.23)\qquad x_1,x_2,x_3\in J,\quad x_1<x_2<x_3$$

such that for $j=1,2$ and $j=1,2,3$, resp.

$$(3.24)\qquad \operatorname{sgn}\delta_n f(x_j)\neq\operatorname{sgn}\delta_n f(x_{j+1}),\quad |\delta_n f(x_j)|\gg\frac{x_j^{\beta_\nu}}{|\varrho_\nu|^n}\geqq\frac{x_j^{\beta_1}}{|\varrho_1|^n}.$$

This implies that the number of sign changes of $\delta_n f(x)$ in the interval $(Y^{e_\nu},\ Y^{e'_\nu})$ is at least

$$(3.25)\qquad 2\left[\frac{(e'_\nu-e_\nu)\log Y}{(2\pi/\gamma_\nu)(1+b)}\right]\geqq 2\left[\frac{(e'_\nu-e_\nu)\log Y}{(2\pi/\gamma_1)(1+b)}\right].$$

Taking into account (3.16) we obtain for Y sufficiently large

$$(3.26)\qquad V(\delta_n f,Y)>(1-3\sqrt{b})\frac{\gamma_1}{\pi}\log Y.$$

Now we have only to note that if $\gamma<\infty$ then by (3.4) we have $\gamma_1=\gamma$. If $\gamma=\infty$ then for every constant C we have $\gamma_1>C$ if we choose η so small that the domain $\sigma\geqq\theta-\eta$, $|t|\leqq C$ is free of singularities of $F(s)$. Remarking further that for an arbitrary function g

$$(3.27)\qquad V(g,Y)\geqq V(\delta g,Y)$$

we see that

$$(3.28)\qquad \varliminf_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}.$$

We remark further that if for an arbitrary g the function δg has at least $k+1$ sign changes in an interval $[A, B]$ then g has at least k sign changes in $[A, B]$. Since owing to (3.16) and (3.23)—(3.24) the interval

$$\left[Y^{1-3\sqrt{b}}\exp\left(-n\frac{2\pi}{\gamma_1}\right),\ Y\right]\subset\left[Y^{1-3\sqrt{b}-(2\pi b/\gamma_1)},\ Y\right] \tag{3.29}$$

contains at least $n+1$ sign changes of $\delta_n f$, the function $f(x)$ has at least one sign change in the interval

$$[Y^{1-4\sqrt{b}}, Y],\quad Y>Y_0 \tag{3.30}$$

if b was chosen sufficiently small.

What concerns the order of magnitude of the oscillation of $f(x)$ we obtain the assertion of our Theorem if we can show the same assertion with $f(x)$ replaced by

$$\bar{f}(x)=\begin{cases} f(x), & 0\leqq x<1\\ f(x)\pm x^{\theta-\varepsilon}, & x\geqq 1.\end{cases} \tag{3.31}$$

But we have obviously for the corresponding function

$$\bar{F}(s)=F(s)\pm\frac{1}{s-\theta+\varepsilon} \tag{3.32}$$

and since we do not use in the proof any properties of $F(s)$ in the halfplane $\sigma<\theta-\eta$, everything remains actually unchanged if at the beginning we choose

$$\eta\leqq\varepsilon/2. \tag{3.33}$$

If $F(s)$ has logarithmic singularities too, then we proceed similarly with the choice of the parameters and broken line L. But, using the idea of Grosswald [3] we consider now the functions

$$F^{(K)}(s)=\int_0^\infty(-1)^K\log^K x\cdot f(x)x^{-s}dx,\quad K=\max_{1\leqq\nu\leqq m}(\deg P_\nu+1) \tag{3.34}$$

instead of $f(s)$ and $F(s)$. Thus, similarly to [3, p. 215] we obtain that $F^{(K)}(s)$ is meromorphic on L and to the right of L and therefore the argumentation (3.1)—(3.33) can be applied to $F^{(K)}(s)$. So we have the same conclusions for the function $\tilde{f}(x)=(-1)^K\log^K x\cdot f(x)$ in place of f. Since we have obviously

$$|V(f,Y)-V(\tilde{f},Y)|\leqq 1 \tag{3.35}$$

and

$$f(x)\gg\log^{-K}x\cdot\tilde{f}(x), \tag{3.36}$$

all assertions of Theorem 1 hold for $f(x)$ in this case too.

4. Proof of Theorem 2

Since the proof of Theorem 2 is very similar to that of Theorem 1, we shall be brief. We obtain similarly to (3.1)—(3.6)

$$\delta_n f(x) = 2\,\mathrm{Re}\left\{A_0(x)\frac{x^{\varrho_0}}{\varrho_0^n}\right\}+O\left(M|\sigma_1+a_0+i\Gamma|\left(\frac{x^{\beta_0-a_2}}{(\beta_0-a_2)^n}+\right.\right.$$
$$\left.\left.+\frac{x^{\beta_0+a_1}}{|\beta_0+a_1+iH|^n}+\frac{x^{\sigma_1+a_0}}{|\sigma_1+a_0+i\Gamma|^n}\right)\right) \tag{4.1}$$

with absolute constants in the O symbols.

If we choose now

$$n=\left[d_1\log Y+\sqrt{\log Y}\right],\quad x\in\left[Y^{d_1/d_2}\exp(\log^{3/4}Y),\,Y\right] \tag{4.2}$$

then easy calculation shows that the three error terms in (4.1) are all

$$\ll \exp(-\log^{1/5}Y)\frac{x^{\beta_0}}{|\varrho_0|^n}\quad\text{if}\quad Y>Y(\beta_0,\sigma_1,a_i,d_j,M,\Gamma). \tag{4.3}$$

Further we obtain, similarly to (3.18)—(3.24), at least two sign changes of $\delta_n f$ in every interval of the form ($\varepsilon>0$ is arbitrary)

$$J=\left(X_0,\,X_0\exp\left(\frac{2\pi}{\gamma_0}(1+\varepsilon)\right)\right)\subset[Y^{d_1/d_2}\exp(\log^{3/4}Y),\,Y], \tag{4.4}$$

if $Y>Y_0$. This gives the desired inequality (2.7) similarly to (3.25)—(3.28).

5. Proofs of Corollaries 2 to 6

In case of Corollary 2 the corresponding function $F(s)$ is, as well known,

$$F(s)=\frac{1}{s}\sum_{\substack{n\equiv l_1(q)\\ n\le x}}\frac{\Lambda(n)}{n^s}-\sum_{\substack{n\equiv l_2(q)\\ n\le x}}\frac{\Lambda(n)}{n^s}=\frac{1}{\varphi(q)s}\sum_{\chi(q)}\left(\bar\chi(l_2)-\bar\chi(l_1)\right)\frac{L'}{L}(s,\chi), \tag{5.1}$$

which is meromorphic in the whole plane and has singularities in the half-plane $\sigma\ge\frac{1}{2}$ (see Grosswald [2]). This proves Corollary 2. If l_1 and l_2 are both quadratic non-residues then we have

$$\Delta_4(x)=\Delta_3(x)+O(x^{1/3}) \tag{5.2}$$

whilst the oscillation of $\Delta_3(x)$, ensured by Corollary 2 is at least $x^{1/2-\varepsilon}$ and therefore Corollary 3 is true in this case. If l_1 and l_2 are both quadratic residues then let $\alpha_1',\dots,\alpha_N'$ and $\alpha_1'',\dots,\alpha_N''$ denote the solutions of the congruences $x^2\equiv l_1 \pmod q$ and $x^2\equiv l_2 \pmod q$. (The number of solutions of the two congruences is equal.) If we define

$$\bar\Delta_4(x)=\sum_{\substack{n\equiv l_1(q)\\ n\le x}}\Lambda(n)-\sum_{\substack{n\equiv l_2(q)\\ n\le x}}\Lambda(n)-\sum_{j=1}^{N}\Big\{\sum_{\substack{n\equiv\alpha_j'(q)\\ n^2\le x}}\Lambda(n^2)-\sum_{\substack{n\equiv\alpha_j''(q)\\ n^2\le x}}\Lambda(n^2)\Big\} \tag{5.3}$$

then we have clearly

$$\bar{\Delta}_4(x) = \Delta_4(x) + O(x^{1/3}). \tag{5.4}$$

On the other hand, the function

$$\int_0^\infty \frac{\bar{\Delta}_4(x)}{x^{s+1}}\,dx = \frac{1}{\varphi(q)s}\sum_{\chi(q)}\big(\bar{\chi}(l_2)-\bar{\chi}(l_1)\big)\frac{L'}{L}(s,\chi)- \tag{5.5}$$

$$-\frac{1}{\varphi(q)s}\sum_{j=1}^{N}\sum_{\chi(q)}\big(\bar{\chi}(\alpha_j'')-\bar{\chi}(\alpha_j')\big)\frac{L'}{L}(2s,\chi)$$

is also meromorphic in the whole plane and has also singularities in the half plane $\sigma\geqq\frac{1}{2}$, since the second summand is regular for $\sigma\geqq\frac{1}{2}$. Therefore Corollary 1, applied to $\bar{\Delta}_4(x)$ and (5.4) prove Corollary 3.

To prove Corollary 4 we have only to note that the function

$$\int_0^\infty \frac{M_a(x)}{x^{s+1}}\,dx = \frac{1}{s}\sum_{n=1}^{\infty}\frac{\mu(n)n^{-a}}{n^s}-\frac{1}{s\zeta(a)} = \frac{1}{s\zeta(s+a)}-\frac{1}{s\zeta(a)} \tag{5.6}$$

is regular for real $s>-a-2$, meromorphic in the whole plane and has its "lowest" non-real singularity at $s=\frac{1}{2}-a+i\gamma_1$, $\gamma_1=14.13\ldots$ so Corollary 4 follows from Corollary 1.

In the proof of Corollary 5 we have the identity

$$\int_0^\infty \frac{\bar{M}_a(x)}{x^{s+1}}\,dx = \frac{1}{s\zeta(s+a)}. \tag{5.7}$$

Thus, in view of (2.8) we may choose

$$\sigma_1 = 1-a,\quad \beta_0 = \frac{1}{2}-a,\quad \gamma_0 = 14.13\ldots,\quad H = 21,\quad \Gamma = 10^8, \tag{5.8}$$

$$a_0 = 10^{-3},\quad a_1 = 10^{-20},\quad a_2 = \min\left(\frac{4}{5}\left(\frac{1}{2}-a_0\right), 2\right)$$

and with some calculations this leads to $d_1/d_2<0.9$. Hence we obtain (2.17) by Theorem 2.

In case of Corollary 6 we have

$$\int_0^\infty \frac{R_k(x)}{x^{s+1}}\,dx = \frac{\zeta(s)}{s\zeta(ks)}-\frac{1}{(s-1)\zeta(k)}, \tag{5.9}$$

which is regular for real $s>0$, meromorphic in the whole plane and has simple poles at $\frac{1}{k}\left(\frac{1}{2}\pm i\gamma_0\right)$, $\gamma_0=14.13\ldots$. Further, in view of (2.8) we can choose

$$\sigma_1=\frac{1}{k},\quad H=\frac{21}{k},\quad \Gamma=\frac{10^8}{k},\quad a_0=a_1=\frac{10^{-3}}{k},\quad a_2=\frac{2}{5k} \tag{5.10}$$

which leads to $d_1/d_2<2/5$. Thus Theorem 2 implies Corollary 6.

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