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Irregularities in the distribution of primes in arithmetic progressions II.

By

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1. In the present work we shall continue our investigations of the oscillatory properties of the functions (p always runs through the primes)

$$(1.1)\qquad \begin{aligned} \Delta_1(x, q, l_1, l_2) &= \sum_{p \leqq x} \varepsilon(p, q, l_1, l_2) \\ \Delta_2(x, q, l_1, l_2) &= \sum_{n \leqq x} \varepsilon(n, q, l_1, l_2) \frac{\Lambda(n)}{\log n} \\ \Delta_3(x, q, l_1, l_2) &= \sum_{p \leqq x} \varepsilon(p, q, l_1, l_2) \log p \\ \Delta_4(x, q, l_1, l_2) &= \sum_{n \leqq x} \varepsilon(n, q, l_1, l_2) \Lambda(n) \end{aligned}$$

where we define

$$(1.2)\qquad \begin{aligned} &\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \\ 0 & \text{otherwise}, \end{cases} \\ &\varepsilon(n, q, l_1, l_2) = \varepsilon(n) = \varepsilon_1(n) - \varepsilon_2(n), \qquad \varepsilon_i(n) = \begin{cases} 1 & \text{if } n \equiv l_i (\text{mod } q) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and we assume the trivial condition

$$(1.3)\qquad (l_1, q) = (l_2, q) = 1, \quad l_1 \not\equiv l_2 (\text{mod } q).$$

Knapowski showed the inequality [1]

$$(1.4)\qquad \frac{1}{Y} \int\limits_{Y \exp(-\log^{3/4} Y)}^{Y} |\Delta_i(x, q, l_1, l_2)| \, dx > Y^{1/2} \exp\left(-7 \frac{\log Y}{\log_2 Y}\right)$$

for $Y > \max(c_1, e^{e^q})$ in case of $i = 1$ and for $Y > \max(c_2, \exp(q^{40}))$ in case of $i = 2$ and 4, under the assumption that

$$(1.5)\qquad L(s, \chi, q) \neq 0 \quad \text{for } \sigma > \tfrac{1}{2}, \ |t| \leqq \max(c_3, q^7).$$

Extending the results of part I [2] we shall now prove

Theorem. *Assume*

(1.6) $$L(s,\chi,q)\neq 0 \quad \text{for } \sigma>\tfrac{1}{2},\ |t|\leqq D=c_4 q^2\log^6 q.$$

Let (with the notation $\log_2 Y=\log\log Y$*)*

(1.7) $$Y>\exp(c_5 q^{10}),\qquad \frac{\sqrt{\log Y}}{\sqrt{q}\log_2 Y}<\lambda<\frac{c_6\log Y}{q\log_2^2 Y}.$$

Then for $1\leqq i\leqq 4$ *we have*

(1.8) $$\frac{1}{Y}\int_{Y^{1-7/\lambda}}^{Y}|\Delta_i(x)|\,dx\geqq\sqrt{Y}\exp\left(-\frac{9\log Y}{\lambda}-c_7 q\lambda\log_2^2 Y\right).$$

Choosing $\lambda=7q^{-1/2}\log^{1/2}Y\log_2^{-1}Y$ and $\lambda=7\log Y\log_2^{-3}Y$, resp., we obtain the following corollaries.

Corollary 1. *If* (1.6) *holds and if* $Y>\exp(c_5 q^{10})$ *then for* $1\leqq i\leqq 4$ *we have*

(1.9) $$\frac{1}{Y}\int_{A(Y)}^{Y}|\Delta_i(x)|\,dx\geqq\sqrt{Y}\exp(-c_8\sqrt{q}\sqrt{\log Y}\log_2 Y)$$

with $A(Y)=Y\exp(-\sqrt{q}\sqrt{\log Y}\log_2 Y)$.

Corollary 2. *If* (1.6) *holds and* $Y>\exp(\exp(c_9 q))$ *then for* $1\leqq i\leqq 4$ *we have the inequality*

(1.10) $$\frac{1}{Y}\int_{A'(Y)}^{Y}|\Delta_i(x)|\,dx\geqq\sqrt{Y}\exp\left(-c_{10}q\frac{\log Y}{\log_2 Y}\right)$$

where

$$A'(Y)=Y\exp(-\log_2^3 Y).$$

2. Since in part I [2] we proved the theorem for $i=2,4$ and in case of quadratic non-residues l_1, l_2 also for $i=1,3$ we can assume now $i=1$ or 3. First we shall treat the case when both l_1 and l_2 are quadratic residues. As the proof is similar to the case l_1, l_2 being quadratic non-residues dealt with in [2], a sketch of the proof will suffice and we shall point out only the necessary changes. (What concerns the case l_1 is a quadratic residue and l_2 a non-residue, we shall be even more brief.)

Let us denote the solutions of the congruences

(2.1) $$x^2\equiv l_1\ (\mathrm{mod}\ q),\qquad x^2\equiv l_2\ (\mathrm{mod}\ q)$$

by $\alpha_1^{(1)},\dots,\alpha_N^{(1)}$ and $\alpha_1^{(2)},\dots,\alpha_N^{(2)}$ resp. (their number being equal, $N=N(q)$) and let ($j=1,2$)

(2.2) $$F(s)=\sum_n\frac{\varepsilon(n)\Lambda(n)}{n^s}=\frac{1}{\varphi(q)}\sum_\chi(\bar\chi(l_2)-\bar\chi(l_1))\frac{L'}{L}(s,\chi),$$

(2.3) $$F_j(s)=-\sum_n\frac{\varepsilon_j(N^2)\Lambda(n^2)}{n^{2s}}=\frac{1}{\varphi(q)}\sum_{i=1}^{N}\sum_\chi\bar\chi(\alpha_i^{(j)})\frac{L'}{L}(2s,\chi).$$

Then using the integral formula ($A \in \mathbb{R}^+$, $B \in \mathbb{C}$)

$$(2.4) \qquad \frac{1}{2\sqrt{\pi A}} \exp\left(-\frac{B^2}{4A}\right) = \frac{1}{2\pi i} \int_{(2)} e^{As^2+Bs}\,ds$$

we can show similarly to Section 3 of [22] that with $k \geqq \lambda^{-1}$, $|\mu - k\lambda^2| \leqq 1$ we have

$$S \overset{\text{def}}{=} \frac{1}{2\sqrt{\pi K}} \left\{ \sum_n \varepsilon(n) \Lambda(n) \exp\left(-\frac{(\mu - \log n)^2}{4K}\right) - \sum_n \varepsilon(n^2) \Lambda(n^2) \exp\left(-\frac{(\mu - \log n^2)^2}{4K}\right) \right\}$$

$$= \frac{1}{2\pi i} \int_{(2)} (F(s) + F_1(s) - F_2(s))\, e^{Ks^2+\mu s}\,ds = \sum_\varrho \alpha_\varrho e^{K\varrho^2+\mu\varrho}$$

$$(2.5) \qquad + \sum_\varrho a_\varrho^* e^{K\left(\frac{\varrho}{2}\right)^2+\mu\left(\frac{\varrho}{2}\right)} + O(1) \overset{\text{def}}{=} \sum + \sum{}^* + O(1),$$

where, denoting the multiplicity of ϱ as a zero of $L(s, \chi)$ by $m_\chi(\varrho)$,

$$(2.6) \qquad a_\varrho = \frac{1}{\varphi(q)} \sum_{\substack{\chi \\ L(\varrho,\chi)=0}} (\bar\chi(l_2) - \bar\chi(l_1))\, m_\chi(\varrho)$$

$$(2.7) \qquad a_\varrho^* = \frac{1}{2\varphi(q)} \sum_{\substack{\chi \\ L(\varrho,\chi)=0}} \sum_{i=1}^{N} (\bar\chi(\alpha_i^{(1)}) - \bar\chi(\alpha_i^{(2)}))\, m_\chi(\varrho)$$

and in the summation in (2.5) ϱ runs through all zero of $L(s, \chi, q)$ with $\operatorname{Re}\varrho \geqq 0$.

In view of (1.6) the contribution of zeros $\varrho = \beta + i\gamma$ with $|\gamma| < D$ to Σ^* is

$$(2.8) \qquad \ll N \sum_{1 \leqq m \leqq D} e^{K\left(\frac{1}{16} - \frac{(m-1)^2}{4}\right) + \frac{\mu}{4}} \log(qm) \ll e^{\mu/3}.$$

Further we have

$$(2.9) \qquad \sum_{|\varrho| \geqq 2\lambda} a_\varrho e^{K\varrho^2+\mu\varrho} \ll \sum_{n \geqq [2\lambda]-1} e^{\mu + K(1-n^2)} \log(qn) \ll 1$$

and similarly

$$(2.10) \qquad \sum_{|\varrho| \geqq 2\lambda} a_\varrho^* e^{K\left(\frac{\varrho}{2}\right)^2+\mu\left(\frac{\varrho}{2}\right)} \ll \sum_{n \geqq [2\lambda]-1} e^{\frac{\mu}{2} + \frac{K}{4}(1-n^2)} \log(qn) \ll 1.$$

So we are led to consider the finite power-sum

$$(2.11) \qquad \Sigma_1 = \sum_{|\varrho| \geqq 2\lambda} a_\varrho e^{K\varrho^2+\mu\varrho} + \sum_{D < |\varrho| < 2\lambda} a_\varrho^* e^{K\left(\frac{\varrho}{2}\right)^2 + \mu\frac{\varrho}{2}}.$$

We quote Lemma 5 of [2] as

Lemma 1. *There exist real numbers*

$$(2.12) \qquad K_0 = \frac{1}{P^2 \log^2 P}, \quad \mu_0 = \log P, \quad \frac{D}{2} < P \log^2 P < D$$

and an absolute constant $c_{11} > 0$ (*independent of* c_4) *such that*

$$|\sum_{\varrho} a_\varrho e^{K_0 \varrho^2 + \mu_0 \varrho}| \geqq c_{11} D. \tag{2.13}$$

Let

$$L = \left(1 + \frac{3}{\lambda}\right)^{-1} \log Y, \quad B = \frac{1}{L^2}. \tag{2.14}$$

Further let v be an integer to be chosen later with

$$v \in \left[\frac{L - \mu_0}{B} - c_{12} q \lambda \log \lambda, \frac{L - \mu_0}{B}\right] \tag{2.15}$$

and

$$w = \frac{L/\lambda^2 - K_0}{L - \mu_0} \tag{2.16}$$

$$K = K_0 + B w v, \quad \mu = \mu_0 + B v. \tag{2.17}$$

The above choice of parameters assures similarly to (4.2) of [2]

$$K \in \left[\frac{L}{\lambda^2}\left(1 - \frac{1}{L}\right), \frac{L}{\lambda^2}\right], \quad \mu \in [L - 1, L]. \tag{2.18}$$

Now Σ_1 can be written as

$$\Sigma_1 = \sum_{|\varrho| < 2\lambda} b_\varrho z_\varrho^v + \sum_{D < |\varrho| < 2\lambda} b_\varrho^* (z_\varrho^*)^v \tag{2.19}$$

where

$$b_\varrho = a_\varrho e^{K_0 \varrho^2 + \mu_0 \varrho} \qquad b_\varrho^* = a_\varrho^* e^{K\left(\frac{\varrho}{2}\right)^2 + \mu_0 \frac{\varrho}{2}}$$

$$z_\varrho = e^{B w \varrho^2 + B \varrho} \qquad z_\varrho^* = e^{B w \left(\frac{\varrho}{2}\right)^2 + B \frac{\varrho}{2}}. \tag{2.20}$$

In the lower estimation of Σ_1, Lemma 1 of [2], essentially due to Knapowski [1], plays a crucial role. We formulate this power-sum theorem as

Lemma 2. *Let* b_j, z_j $(j = 1, 2, \ldots, n)$ *be complex numbers with*

$$|z_1| \geqq |z_2| \geqq \ldots \geqq |z_n| \tag{2.21}$$

and let $m > 0$, $1 \leqq h \leqq n$. *Then there exists a* $v \in [m, m + n]$ *such that*

$$\left|\sum_{j=1}^{n} b_j z_j^v\right| \geqq \left(\operatorname{Min}_{l \geqq h} \left|\sum_{j=1}^{l} b_j\right|\right) |z_h|^v \left|\frac{z_h}{z_1}\right|^n \left(\frac{n}{16 e^2 (m + n)}\right)^n. \tag{2.22}$$

First we note that by (2.12)

$$\sum_{|\varrho| > D} |b_\varrho| + \sum_{|\varrho| > D} |b_\varrho^*| \leqq c_{13} \sum_{n \geqq [D] - 1} \log(qn) \cdot D e^{-K_0 n^2/4} \leqq c_{14} \tag{2.23}$$

(where the constants c_{13} and c_{14} are independent of c_4), and so we have by Lemma 1

$$\operatorname*{Min}_{l \geqq h} \left(\left| \sum_{j=1}^{l} b_j \right| \right) > c_{11} D - c_{14} > 1 \tag{2.24}$$

if we choose h as the maximal index which corresponds to a zero z_ϱ with $|\varrho| \leqq D$. Inequality (2.24) settles the most critical estimation in (2.22). Since (1.6) and (1.7) imply $\lambda \geqq D^2$ (if c_5 is chosen sufficiently large compared to c_4) we obtain

$$\begin{aligned} |z_h| &> e^{B/2 - B|w| D^2} > e^{B/2 - B|w| \lambda} \\ |z_1| &< e^{B + B|w| D^2} < e^{B + B|w| \lambda}. \end{aligned} \tag{2.25}$$

Further we choose $m = \dfrac{L - \mu_0}{B}$ and note that by Jensen's inequality we have

$$n < c_{12} q \lambda \log \lambda . \tag{2.26}$$

So we obtain by Lemma 2 a v with (2.15) such that, similarly to Section 3 of [2]

$$\begin{aligned} |\Sigma_1| > \exp\left\{ \frac{Bv}{2} - L\lambda |w| - c_{12} q \lambda \log \lambda \left(\frac{B}{2} + 2B\lambda |w| \right.\right. \\ \left.\left. + c_{15} + \log \frac{L}{Bq\lambda \log \lambda} \right) \right\} > \exp\left\{ \frac{L}{2} - c_{16} q \lambda \log^2 L \right\}. \end{aligned} \tag{2.27}$$

The assertion of the Theorem follows now with an application of Lemma 3 of [2] which we state in a slightly more general form (which can be proved similarly) as Lemma 3.

Lemma 3. *Let $d(n)$ be an arithmetical function satisfying $\sum_{n \leqq x} d(n) \ll x$ and let*

$$D_3(x) \overset{\text{def}}{=} \sum_{n \leqq x} d(n), \qquad D_1(x) \overset{\text{def}}{=} \sum_{n \leqq x} \frac{d(n)}{\log n}. \tag{2.28}$$

Then for positive K, μ with $\mu^{-1} \leqq K \leqq \mu/9$ we have

$$\begin{aligned} \left| \sum_n d(n) \exp\left(- \frac{(\mu - \log n)^2}{4K} \right) \right| &\ll \sqrt{\frac{\mu}{K}} \int_{e^{\mu - 3\sqrt{\mu K}}}^{e^{\mu + 3\sqrt{\mu K}}} \frac{|D_3(x)|}{x} \, dx + O(1), \\ \left| \sum_n d(n) \exp\left(- \frac{(\mu - \log n)^2}{4K} \right) \right| &\ll \mu \sqrt{\frac{\mu}{K}} \int_{e^{\mu - 3\sqrt{\mu K}}}^{e^{\mu + 3\sqrt{\mu K}}} \frac{|D_1(x)|}{x} \, dx + O(1). \end{aligned} \tag{2.29}$$

Choosing

$$d(n) = \begin{cases} \varepsilon(n) \Lambda(n) & \text{if } n = p^{2j+1},\ j = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \tag{2.30}$$

we obtain by (2.5)–(2.11), (2.27) and Lemma 3, for $i = 1, 3$

$$\int_{e^{\mu - 3\sqrt{\mu K}}}^{e^{\mu + 3\sqrt{\mu K}}} \frac{|D_i(x)|}{x} \, dx > \exp\left(\frac{L}{2} - c_{17} q \lambda \log^2 L \right). \tag{2.31}$$

This implies by (1.7), (2.14) and (2.18)

$$\frac{1}{Y}\int_{Y^{1-7/\lambda}}^{Y} |D_i(x)|\,dx > \exp\left(\frac{L}{2} - \frac{7\log Y}{\lambda} - c_{17}q\lambda\log^2 L\right)$$

$$> \sqrt{Y}\exp\left(-\frac{9\log Y}{\lambda} - c_{17}q\lambda^2\log^2 Y\right). \tag{2.32}$$

Since

$$D_1(x) = \sum_{p\leq x}\varepsilon(p) + O(x^{1/3}) = \Delta_1(x) + O(x^{1/3}),$$

$$D_3(x) = \sum_{p\leq x}\varepsilon(p)\log p + O(x^{1/3}) = \Delta_3(x) + O(x^{1/3}), \tag{2.33}$$

(2.32) proves our Theorem for $i = 1, 3$ if l_1 and l_2 are both quadratic residues.

3. Let us now consider the case when l_1 is a quadratic residue, l_2 a non-residue ($i = 1$ or 3). In this case we have no squares in the second progression and therefore the pole of $L(2s, \chi_0, q)$ at $s = 1/2$ gives an extra term in our power-sum. So we obtain (cf. (2.5))

$$S' = \frac{1}{2\sqrt{\pi K}}\left\{\sum_n \varepsilon(n)\Lambda(n)\exp\left(-\frac{(\mu-\log n)^2}{4K}\right) - \sum_n \varepsilon_1(n^2)\Lambda(n^2)\exp\left(-\frac{(\mu-\log n^2)^2}{4K}\right)\right\}$$

$$= \frac{1}{2\pi i}\int_{(2)}\{F(s) + F_1(s)\}\,e^{Ks^2+\mu s}\,ds$$

$$= -\frac{N}{2\varphi(q)}e^{\frac{K}{4}+\frac{\mu}{2}} + \sum_\varrho a_\varrho e^{K\varrho^2+\mu\varrho} + \sum_\varrho a'_\varrho e^{K\left(\frac{\varrho}{2}\right)^2+\mu\left(\frac{\varrho}{2}\right)} + O(1) \tag{3.1}$$

where (cf. (2.6)–(2.7))

$$a'_\varrho = \frac{1}{2\varphi(q)}\sum_{\substack{\chi\\ L(\varrho,\chi)=0}}\sum_{i=1}^{N}\bar\chi(\alpha_i^{(1)})\,m_\chi(\varrho). \tag{3.2}$$

Since the only property of a_ϱ^* used in Section 2 was ($\varrho = \beta + i\gamma$)

$$\sum_{n\leq\gamma\leq n+1}|a_\varrho^*| \ll \log(q(n+2)), \tag{3.3}$$

which holds also for a'_ϱ, writing always a'_ϱ in place of a_ϱ^* all formulas of Section 2 remain valid. Thus the only change is the appearance of the new term $b'(z')^\nu$, where

$$b' = \frac{N}{2\varphi(q)}e^{\frac{K_0}{4}+\frac{\mu_0}{2}}, \qquad z' = e^{\frac{Bw}{4}+\frac{B}{2}}. \tag{3.4}$$

Now the inequality (2.26) remains true if we include one term more and (2.25) is also true for z'. So we have only to control (2.24) where we have by

$$|b'| < \tfrac{1}{2}e^{\frac{K_0}{4}+\frac{\mu_0}{2}} < \sqrt{P} < \sqrt{D} \tag{3.5}$$

the inequality

$$\text{(3.6)} \qquad \operatorname*{Min}_{l \geqq h} \left(\left| \sum_{j=1}^{h} b_j \right| \right) > c_{11} D - \sqrt{D} - c_{14} > 1 .$$

Thus we have (2.24) unchanged valid and so all formulas (2.25)–(2.33) remain valid without any change. This proves our Theorem in the remaining case when l_1 is a quadratic non-residue, l_2 a residue.

References

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