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Irregularities in the distribution of primes in arithmetic progressions, I.

By

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Summary. Supposing that all the Dirichlet L-functions $\bmod q$ have no zeros in the region $(s = \sigma + it)$ $\sigma > 1/2$, $|t| \leqq c q^2 \log^6 q$, various lower estimations are proved for the mean value of $|\Delta_i(x)|$ $(i = 2, 4)$, where

$$\Delta_4(x) = \sum_{\substack{n \leqq x \\ n \equiv l_1(q)}} \Lambda(n) - \sum_{\substack{n \leqq x \\ n \equiv l_2(q)}} \Lambda(n)$$

(and $\Delta_2(x)$ is a similar difference, see (1.1)) for arbitrary pairs l_1, l_2 satisfying $(l_1, q) = (l_2, q) = 1$, $l_1 \not\equiv l_2(q)$.

1. In a series of papers [1, 2, 3] written in 1961–62, Knapowski investigated the oscillation of the functions (in case of $i = 1, 2, 4$)

$$(1.1)\qquad \begin{aligned} \Delta_1(x, q, l_1, l_2) &= \sum_{p \leqq x} \varepsilon(p, q, l_1, l_2), \\ \Delta_2(x, q, l_1, l_2) &= \sum_{n \leqq x} \varepsilon(n, q, l_1, l_2) \frac{\Lambda(n)}{\log n}, \\ \Delta_3(x, q, l_1, l_2) &= \sum_{p \leqq x} \varepsilon(p, q, l_1, l_2) \log p, \\ \Delta_4(x, q, l_1, l_2) &= \sum_{n \leqq x} \varepsilon(n, q, l_1, l_2) \Lambda(n), \end{aligned}$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \\ 0 & \text{otherwise} \end{cases}$$

and

$$(1.2)\qquad (l_1, q) = (l_2, q) = 1, \quad l_1 \not\equiv l_2 (\bmod q),$$

$$(1.3)\qquad \begin{aligned} &\varepsilon(n, q, l_1, l_2) = \varepsilon(n) = \varepsilon_1(n) - \varepsilon_2(n), \\ &\varepsilon_i(n) = \begin{cases} 1 & \text{if } n \equiv l_i (\bmod q) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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In these papers he proved lower bounds for

$$(1.4)\qquad \frac{1}{Y}\int_{A(Y)}^{Y} |\Delta_i(x,q,l_1,l_2)|\,dx.$$

More detailed, Knapowski showed for $i = 2,4$, $Y \geqq \max(c_1, \exp(q^{c_2}))$, the lower estimate

$$(1.5)\qquad \frac{1}{Y}\int_{A(Y)}^{Y} |\Delta_i(x,q,l_1,l_2)|\,dx > Y^{1/4}$$

with

$$(1.6)\qquad A(Y) = Y\exp(-\log^{0.9} Y)$$

where c_ν, as always in the following, denotes an explicitly calculable positive absolute constant.

Further, supposing the "Finite Riemann-Piltz conjecture",

$$(1.7)\qquad L(s,\chi,q) \neq 0 \quad \text{for } \sigma > 1/2,\ |t| \leqq \max(c_3, q^7),$$

he showed the stronger inequality

$$(1.8)\qquad \frac{1}{Y}\int_{A'(Y)}^{Y} |\Delta_i(x,q,l_1,l_2)|\,dx > Y^{1/2}\exp\left(-7\frac{\log Y}{\log_2 Y}\right)$$

for $i = 1,2,4$, $Y > Y_i$ where

$$(1.9)\qquad A'(Y) = Y\exp(-\log^{3/4} Y),$$

$$(1.10)\qquad Y_1 = \max(c_4, e^{e^q}),\quad Y_2 = Y_4 = \max(c_5, \exp(q^{40})).$$

Our purpose here is to show theorems similar to (1.7)–(1.10) with better localisation and with better lower estimates. In this first part we shall be concerned with the cases $i = 2,4$ only. If l_1, l_2 are both quadratic non-residues, the cases $i = 1,3$ are immediate consequences of the previous ones, but for general l_1, l_2 they present additional complications, and this will be the object of the second part.

We also point out that our requirements concerning the Finite Riemann-Piltz conjecture and the starting point are weaker than Knapowski's corresponding assumptions.

2. In our results, we shall always assume the truth of the finite Riemann-Piltz conjecture (FR-P) until a level

$$(2.1)\qquad D = c_0^2 q^2 \log^6 q$$

where c_0 is a sufficiently large, fixed positive absolute constant.

Moreover, without special mention we shall always assume the trivial condition (1.2).

Our results are the following ($\log_2 Y = \text{loglog}\, Y$):

Theorem 1. *Assume FR-P until D. Then, for every Y with*

$$Y > \exp(c_6 q^{10}) \tag{2.2}$$

we have, for $i = 2, 4$

$$\int_{A(Y)}^{Y} \frac{|\Delta_i(x)|}{x}\,dx \gg \sqrt{Y}\exp\left(-\frac{2\log Y}{\lambda} - c_7 q\lambda\log^2 Y\right) \tag{2.3}$$

with

$$A(Y) = Y\exp\left(-\frac{7\log Y}{\lambda}\right) \tag{2.4}$$

for every λ *satisfying*

$$\frac{\sqrt{\log Y}}{\sqrt{q}\,\log_2 Y} < \lambda < \frac{c_8\log Y}{q\log_2^2 Y}. \tag{2.5}$$

The following corollaries are obtained by Theorem 1 choosing the parameter λ as

$$\lambda = \frac{7\sqrt{\log Y}}{\sqrt{q}\,\log_2 Y} \quad\text{and}\quad \lambda = \frac{7\log Y}{\log_2^3 Y} \tag{2.6}$$

resp.

Corollary 1. *Subject to the hypotheses of the Theorem, for every*

$$Y > \exp(c_6 q^{10})$$

we have, for $i = 2, 4$

$$\int_{A(Y)}^{Y} \frac{|\Delta_i(x)|}{x}\,dx \gg \sqrt{Y}\exp(-c_9\sqrt{q}\sqrt{\log Y}\log_2 Y) \tag{2.7}$$

with

$$A(Y) = Y\exp(-\sqrt{q}\sqrt{\log Y}\log_2 Y). \tag{2.8}$$

Corollary 2. *Assuming the hypotheses of the Theorem, for every*

$$Y > \exp(\exp(c_{10} q))$$

for $i = 2, 4$ *the following estimation holds*:

$$\int_{A(Y)}^{Y} \frac{|\Delta_i(x)|}{x}\,dx \gg \sqrt{Y}\exp\left(-c_{11} q\frac{\log Y}{\log_2 Y}\right) \tag{2.9}$$

where

$$A(Y) = Y\exp(-\log_2^3 Y).$$

Corollaries 1 and 2 improve Knapowski's results (1.8)–(1.9) both with respect to $A(Y)$ and the lower estimate, whilst other choices of λ lead essentially to Knapowski's lower estimate, but with a much stronger localisation. In the course of the proof c denotes a generic (explicitly calculable positive) constant whose value might be different in various appearances. The constants implied by $\ll$ and 0 symbols are also explicitly calculable positive absolute constants.

3. The main tool in the proof of the results stated in Section 2 is a two-sided powersum theorem of Knapowski [1].

We need it in the following slightly modified form:

Lemma 1. *For $j = 1, \dots, n$, let b_j, z_j be complex numbers with*

$$|z_1| \geqq |z_2| \geqq \dots \geqq |z_n|.$$

Then, for every h with $1 \leqq h \leqq n$ and for every $m \geqq 0$, we have

$$\max_{m \leqq \nu < m+n} \left| \sum_{j=1}^{n} b_j z_j^\nu \right| \geqq \left(\min_{l \geqq h} \left| \sum_{j=1}^{l} b_j \right| \right) |z_h|^\nu \left| \frac{z_h}{z_1} \right|^n \left(\frac{n}{16 e^2 (m+n)} \right)^n. \tag{3.1}$$

P r o o f. The proof of the Theorem of Knapowski [1, Lemma I] quoted above implies the following inequality, after normalising with $|z_1| = 1$

$$\max_{m \leqq \nu < m+n} \left| \sum_{j=1}^{n} b_j z_j^\nu \right| > \left(\min_{l \geqq h} \left| \sum_{j=1}^{l} b_j \right| \right) (|z_h| - 4e\varepsilon)^m \left(\frac{\varepsilon}{4} \right)^n \tag{3.2}$$

for every $\varepsilon < |z_h|/4e$. Now choosing

$$\varepsilon = \frac{n}{4e(m+n)} |z_h| \tag{3.3}$$

instead of Knapowski's choice $\varepsilon = n/6e(2n+m)$ we obtain inequality (3.1) by an easy calculation.

We introduce the following series of Dirichlet:

$$F(s) = \sum_n \frac{\varepsilon(n) \Lambda(n)}{n^s}. \tag{3.4}$$

Then

$$F(s) = \frac{1}{\varphi(q)} \sum_\chi (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \frac{L'}{L}(s, \chi) \tag{3.5}$$

where $\sum_\chi$ always denotes summation over all the characters mod q.

Lemma 2. *Let $d(n)$ be an aritmetical function such that*

$$D(x) = \sum_{n \leqq x} d(n) \ll x. \tag{3.6}$$

Then, for positive k, μ, δ with $\delta > 4k$, we have

$$\sum_n d(n) \exp\left(-\frac{(\mu - \log n)^2}{4k}\right) \tag{3.7}$$
$$= -\int_{e^{\mu-\delta}}^{e^{\mu+\delta}} \frac{D(x)}{x} \exp\left(-\frac{(\mu - \log x)^2}{4k}\right) \frac{\mu - \log x}{2k}\, dx + 0(e^{\mu+\delta-\delta^2/4k})$$

where the constant implied by the 0*-symbol only depends on the constant implied by the* $\ll$ *symbol in* (3.6).

Proof. After performing partial summation, we are clearly left with the problem of estimating the remainder terms. Now, since $\delta > 4k$, we have

$$\int_{e^{\mu+\delta}}^{\infty} \exp\left(-\frac{(\mu - \log x)^2}{4k}\right) \frac{\log x - \mu}{2k}\, dx$$
$$= e^{\mu} \int_{\delta}^{\infty} e^{u - \frac{u^2}{4k}} \cdot \frac{u}{2k}\, du \tag{3.8}$$
$$\leqq 2e^{\mu} \int_{\delta}^{\infty} e^{u - \frac{u^2}{4k}} \left(\frac{u}{2k} - 1\right) du$$
$$= 2e^{\mu+\delta-\frac{\delta^2}{4k}},$$

and similarly we can estimate $\int_1^{e^{\mu-\delta}}$ too.

Lemma 3. *Let $d(n)$, $D(x)$ be defined as in Lemma* 2.
Then, for positive k, μ with $1/\mu \leqq k \leqq \mu/9$ we have

$$\left|\sum_n d(n) \exp\left(-\frac{(\mu - \log n)^2}{4k}\right)\right| \ll \sqrt{\frac{\mu}{k}} \int_{e^{\mu-3\sqrt{\mu k}}}^{e^{\mu+3\sqrt{\mu k}}} \frac{|D(x)|}{x}\, dx + 0(1). \tag{3.9}$$

Proof. This is an immediate consequence of Lemma 2, choosing $\delta = 3\sqrt{\mu k}$. The assumption $\mu > 9k$ assures $\delta > 4k$.

Lemma 4. *For $1/\mu \leqq k \leqq \mu/9$, $\mu \geqq \log q$ we have*

$$\frac{1}{2\sqrt{\pi k}} \sum_n \varepsilon(n) \Lambda(n) \exp\left(-\frac{(\mu - \log n)^2}{4k}\right) = \sum_{\varrho} a_{\varrho} e^{k\varrho^2 + \mu\varrho} + 0(1) \tag{3.10}$$

with

$$a_{\varrho} = \frac{1}{\varphi(q)} \sum_{\substack{\chi \\ L(\varrho, \chi) = 0}} (\bar{\chi}(l_2) - \bar{\chi}(l_1))\, m_{\chi}(\varrho), \tag{3.11}$$

where the sum is performed over all zeros of all Dirichlet L-functions (mod q) *in $\sigma \geqq 0$ and $m_{\chi}(\varrho)$ denotes the multiplicity of ϱ as a zero of $L(s, \chi)$.*

Proof. Using the integral formula

$$\frac{1}{2\pi i}\int_{(2)} e^{ks^2+As}\,ds = \frac{1}{2\sqrt{\pi k}}\exp\left(-\frac{A^2}{4k}\right), \tag{3.12}$$

we obtain

$$\begin{aligned}\frac{1}{2\sqrt{\pi k}}\sum_n \varepsilon(n)\,\Lambda(n)\exp\left(-\frac{(\mu-\log n)^2}{4k}\right) &= \sum_n \varepsilon(n)\,\Lambda(n)\cdot\frac{1}{2\pi i}\int_{(2)} e^{ks^2+(\mu-\log n)s}\,ds\\ &= \frac{1}{2\pi i}\int_{(2)}\left(\sum_n \frac{\varepsilon(n)\,\Lambda(n)}{n^s}\right) e^{ks^2+\mu s}\,ds\\ &= \frac{1}{2\pi i}\int_{(2)} F(s)\,e^{ks^2+\mu s}\,ds.\end{aligned} \tag{3.13}$$

The well-known estimate (see e.g. Prachar [6], Satz 4.1, p. 225)

$$\left|\frac{L'}{L}(s,\chi)\right| \ll \log(q(|t|+2)) \quad \text{for } s = -\frac{1}{2}+it \tag{3.14}$$

and our bounds on k and μ imply

$$\int_{(-1/2)} F(s)\,e^{ks^2+\mu s}\,ds \ll (\log q)\,e^{k/4-\mu/2}\int_{-\infty}^{\infty} e^{-kt^2}\log(|t|+2)\,dt \ll 1. \tag{3.15}$$

Taking into account (3.15), we obtain by Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_{(2)} F(s)\,e^{ks^2+\mu s}\,ds = \sum_\varrho a_\varrho e^{k\varrho^2+\mu\varrho} + O(1). \tag{3.16}$$

Now, (3.10) follows from (3.13), (3.16).

Lemma 5. *Assume FR-P until D. Then there exists a prime $P \equiv l_1 (\operatorname{mod} q)$ with*

$$\frac{D}{2} < P\log^2 P < D, \tag{3.17}$$

such that for

$$k_0 = \frac{1}{P^2\log^2 P}, \qquad \mu_0 = \log P \tag{3.18}$$

we have

$$\sum_\varrho a_\varrho e^{k_0\varrho^2+\mu_0\varrho} \gg P\log^2 P, \tag{3.19}$$

where the constant implied in the $\gg$ symbol is independent of the constant c_0 appearing in the definition (2.1) *of D.*

Proof. For the proof of this Lemma, due essentially to Knapowski and Turán [4], we refer for instance to [5]. Now, we shall choose the parameters as follows:

$$L > cq(\log^2 D)\, D^4, \quad B = \frac{1}{L^2}, \tag{3.20}$$

$$\frac{\sqrt{L}}{\sqrt{q}\,\log L} \leqq \lambda \leqq \frac{L}{\log^2 L} \quad \text{hence } \lambda \geqq D^2 > q, \tag{3.21}$$

$$v \in \left[\frac{L-\mu_0}{B} - cq\lambda\log\lambda, \frac{L-\mu_0}{B}\right] \quad \text{and so } Bv \in [L-\mu_0-1, L-\mu_0], \tag{3.22}$$

$$w = \frac{L/\lambda^2 - k_0}{L-\mu_0} \quad \text{therefore } |w| \leqq \frac{q\log^2 L}{L}, \tag{3.23}$$

$$k = k_0 + Bwv, \quad \mu = \mu_0 + Bv. \tag{3.24}$$

The main problem we will be concerned with in the sequel of the proof, is to estimate the modulus of the critical power sum

$$\sum_{\varrho} a_\varrho e^{k\varrho^2 + \mu\varrho}$$

from below, after cutting it at the suitable level $|\varrho| \geqq 2\lambda$. The localization of the result for the mean value of $|\Delta_i(x)|$ will depend essentially on the choice $\delta = 3\sqrt{\mu k} \sim \frac{3L}{\lambda}$ (cf. Lemma 3). The improvement of the localization (cf. Corollary 2) will require to choose λ as large as possible (at the condition of not completely destroying the lower bound given by Lemma 1) and this forces k to be very small. On the other hand, the starting point of Lemma 5, required for having coefficients with sum far from zero, implies k_0 having definite positive size. So, we are led to choose negative w, and, in this respect, to an unusual choice of the parameters.

Lemma 6. *Let us assume* (3.20) *to* (3.24) *hold. Then, there exists an integer*

$$v \in \left[\frac{L-\mu_0}{B} - cq\lambda\log\lambda, \frac{L-\mu_0}{B}\right] \tag{3.25}$$

such that with $k = k_0 + Bwv$, $\mu = \mu_0 + Bv$, *we have*

$$\Big|\sum_{\varrho} a_\varrho e^{k\varrho^2 + \mu\varrho}\Big| \geqq e^{L/2 - cq\lambda\log^2 L}. \tag{3.26}$$

Proof. First of all, we have

$$\Big|\sum_{|\varrho| \geqq 2\lambda} a_\varrho e^{k\varrho^2 + \mu\varrho}\Big| \ll \sum_{n=[2\lambda]-1}^{\infty} e^{\mu + k(1-n^2)} \log(qn) \ll 1. \tag{3.27}$$

Moreover

$$\Big|\sum_{|\varrho| > D} a_\varrho e^{k_0\varrho^2 + \mu_0\varrho}\Big| \ll \sum_{n=[P\log^2 P]-1}^{\infty} \log(qn)\cdot P e^{\frac{1-n^2}{P^2\log^2 P}} \ll 1. \tag{3.28}$$

In view of Lemma 5, this implies that

$$(3.29)\qquad \Big|\sum_{\varrho\in S} a_\varrho e^{k_0\varrho^2+\mu_0\varrho}\Big| \gg P\log^2 P$$

for every set S of zeros containing all zeros with $|\varrho| \leqq D$. (We remark that the constants implied by the $\ll$ symbols in (3.28)–(3.29) are independent from c_0.)

Now we define

$$(3.30)\qquad b_\varrho = a_\varrho e^{k_0\varrho^2+\mu_0\varrho},\qquad z_\varrho = e^{Bw\varrho^2+B\varrho}$$

and after numerating them according to Lemma 1 we choose the index h corresponding to the last term z_ϱ with $|\varrho| \leqq D$.

By Lemma 1, we get the existence of a v in the interval (3.25) such that, recalling (3.29),

$$(3.31)\qquad \Big|\sum_{|\varrho|<2\lambda} a_\varrho e^{k\varrho^2+\mu\varrho}\Big| = \Big|\sum_{|\varrho|<2\lambda} b_\varrho z_\varrho^v\Big| \geqq |z_h|^v \left|\frac{z_h}{z_1}\right|^n \left(\frac{n}{16e^2(m+n)}\right)^n.$$

In our case we choose

$$(3.32)\qquad m = \frac{L-\mu_0}{B}.$$

By Jensen's inequality we have

$$(3.33)\qquad n \leqq cq\lambda\log\lambda.$$

Recalling (3.22), (3.23), we have by FR-P

$$(3.34)\qquad |z_h| > e^{B/2 - BD^2|w|} \geqq e^{B/2 - B\lambda|w|},$$

$$(3.35)\qquad |z_1| < e^{B + BD^2|w|} \leqq e^{B + B\lambda|w|}.$$

Then we obtain by (3.20)–(3.24) and (3.31)–(3.35)

$$(3.36)\qquad \begin{aligned} &\Big|\sum_{|\varrho|<2\lambda} a_\varrho e^{k\varrho^2+\mu\varrho}\Big| \\ &> \exp\left\{\frac{Bv}{2} - L\lambda|w| - cq\lambda\log\lambda\left(\frac{B}{2} + 2B\lambda|w| + c + \log\frac{L}{Bq\lambda\log\lambda}\right)\right\} \\ &> \exp\left(\frac{L}{2} - cq\lambda\log^2 L\right). \end{aligned}$$

Q.E.D.

Proof of the Theorem. We take

$$(4.1)\qquad Y = e^{L(1+3/\lambda)};$$

we notice that (3.20) to (3.24) assure

$$(4.2)\qquad k \in \left[\frac{L}{\lambda^2}\left(1-\frac{1}{L}\right), \frac{L}{\lambda^2}\right],\qquad \mu \in [L-1, L].$$

Hence $3\sqrt{\mu k} \leqq 3L/\lambda$. By Lemmas 3.4 and 6 we obtain, choosing $d(n) = \varepsilon(n)\, \Lambda(n)$,

$$\int\limits_{A(Y)}^{Y} \frac{|\Delta_4(x)|}{x}\, dx \geqq \int\limits_{e^{\mu - 3\sqrt{\mu k}}}^{e^{\mu + 3\sqrt{\mu k}}} \frac{|\Delta_4(x)|}{x}\, dx$$

$$\geqq c\lambda^{-1} \left| \sum_n d(n) \exp\left(-\frac{(\mu - \log n)^2}{4k} \right) \right| + 0(1) \tag{4.3}$$

$$> \exp\left(\frac{L}{2} - cq\lambda \log^2 L \right) > \sqrt{Y} \exp\left(-\frac{3 \log Y}{2\lambda} - cq\lambda \log^2 L \right)$$

with

$$A(Y) = Y e^{-(7 \log Y)/\lambda} \leqq e^{L - 1 - 3L/\lambda} \tag{4.4}$$

from which the conclusion of the theorem is clear for $i = 4$.

For the proof in the case $i = 2$, we observe that, using partial summation

$$\begin{aligned} S &= \left| \sum_n \varepsilon(n)\, \Lambda(n) \exp\left(-\frac{(\mu - \log n)^2}{4k} \right) \right| \\ &= \left| \int\limits_1^{\infty} \frac{\Delta_2(x)}{x} \left(\frac{\log x - \mu}{2k} - \frac{1}{\log x} \right) \exp\left(-\frac{(\log x - \mu)^2}{4k} \right) dx \right| \end{aligned} \tag{4.5}$$

and, similarly to Lemma 3, we obtain

$$S \leqq \mu \sqrt{\frac{\mu}{k}} \int\limits_{e^{\mu - 3\sqrt{\mu k}}}^{e^{\mu + 3\sqrt{\mu k}}} \frac{|\Delta_2(x)|}{x}\, dx + 0(1). \tag{4.6}$$

Now, still using Lemmas 4 and 6, we get our theorem.

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