ON THE REMAINDER TERM OF THE PRIME NUMBER FORMULA AND THE ZEROS OF RIEMANN'S ZETA-FUNCTION

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1. The aim of the present paper is to treat further developments compared with the author's earlier contribution [10] concerning irregularities in the distribution of primes. First we shall consider the oscillation of $|\Delta(x)|$ where we define (p always runs through the primes)

$$\Delta(x) = \sum_{n\leq x} \Lambda(n)-x = \sum_{p^m\leq x} \log p-x \ . \tag{1.1}$$

Supposing the existence of an arbitrary zeta-zero $\rho_o = \beta_o + i\gamma_o$, Littlewood (1937) raised the problem of explicit Ω-estimation of $\Delta(x)$ in terms of ρ_o. At that time the only existing relation, due to Phragmén, was *ineffective*, asserting for any $\varepsilon>0$

$$\Delta(x) = \Omega(x^{\beta_o-\varepsilon}) \ . \tag{1.2}$$

More generally one may ask for lower bounds for

$$S(x) = \max_{0\leq u\leq x} |\Delta(u)| \tag{1.3}$$

and

$$D(x) = \frac{1}{x}\int_0^x |\Delta(u)|du \ . \tag{1.4}$$

Littlewood's problem was solved in 1950 by Turan [15, part I] who showed

$$(1.5)\qquad S(x) \geq x^{\beta_0} \exp\left(-c_1 \frac{\log x}{\log_2 x} \log_3 x\right)$$

for $x > \max(c_2, e_2(|\rho_0|))$ where $\log_\nu x$ and $e_\nu(x)$ denote the ν-times iterated logarithmic function and exponential function, resp., and, as in the sequel, the c_ν denote explicitly calculable positive constants (eventually depending on some parameters indicated in brackets). The same inequality was proved by S.Knapowski [5] for $D(x)$ in place of $S(x)$.

The explicit prime number formula,

$$(1.6)\qquad \Delta(x) = -\sum_{\rho} \frac{x^{\rho}}{\rho} + O(\log x)$$

(where $\rho = \beta + i\gamma$ always runs through the non-trivial zeta-zeros) suggests, however, a larger oscillation for $\Delta(x)$ than that furnished by Turán's result (1.5). Using also Turán's power-sum theory it is possible to show (slightly improving the result of [10, part I]) the following.

THEOREM 1. *If* $\zeta(\rho_0) = 0$, $\varepsilon > 0$ *and*

$$(1.7)\qquad Y > \max\left(c_3, \left(\frac{|\rho_0|}{\varepsilon}\right)^8, \exp\left(\frac{4}{\varepsilon^2 |\rho_0|^2}\right)\right)$$

then there exists a value

$$(1.8)\qquad x \in [Y, Y^{6\log|\gamma_0|+60}]$$

such that

$$(1.9)\qquad |\Delta(x)| > (1-\varepsilon)\frac{x^{\beta_0}}{|\rho_0|} .$$

This yields a very weak (but non-trivial) lower bound for $S(x)$. However, using an entirely different method we can prove:

THEOREM 2. *If* ρ_0 *is a zeta-zero with multiplicity* ν, $x > e^{\gamma_0^2/20}$, $A(x) = 10^{-4} x/\log x$, *then*

$$(1.10) \qquad D(x) > \frac{1}{x}\int_{A(x)}^{x} |\Delta(u)|du > \frac{c_4|\zeta^{(\nu)}(\rho_o)|}{(\nu-1)!|\rho_o|^3} x^{\beta_o} - c_5$$

and a fortiori we have the same inequality for $S(X)$ *and* $\max_{A(x)\le u\le x} |\Delta(u)|$, *too.*

Choosing $\rho_o = 1/2+i\cdot 14.13....,$ the first zero of $\zeta(s)$ over the real axis, with some extra trouble we can show

COROLLARY 1. *For every* $x \ge 1$ *we have*

$$(1.11) \qquad D(x) \ge \frac{\sqrt{x}}{400} .$$

We remark that every improvement of (1.11) with a non-constant factor would already disprove the Riemann Hypothesis (RH) since Cramer [2] showed in 1922 that RH implies

$$(1.12) \qquad D(X) \le (\frac{1}{x}\int_{0}^{x}\Delta^2(u)du)^{1/2} \le c_6 x^{1/2} \qquad (x > c_7)$$

(and with some numerical computation one can choose even $c_6 = 1$) . Thus we have

COROLLARY 2. *Assume* RH. *Then for* $x > c_7$,

$$(1.13) \qquad \frac{\sqrt{x}}{400} \le D(x) \le \sqrt{x} .$$

If RH is true then one can easily infer from (1.11) and (1.12) a good lower bound for $|\Delta(x)|$ for a positive proportion of all positive numbers.

COROLLARY 3. *Assume* RH *and let* $|A|$ *denote the measure of the set* A. *Then for* $x > c_7$

$$(1.14) \qquad |\{0 \le u \le x;\ |\Delta(u)| > \frac{\sqrt{x}}{800}\}| > \frac{x}{800^2} .$$

If RH does not hold but there is a zero $\rho_o = \Theta+i\gamma_o$ where

$$(1.15) \qquad \Theta = \limsup_{\zeta(\rho)=0} (\mathrm{Re}\ \rho)$$

then we have a phenomenon similar to (1.14) in the stronger form. This is expressed by

COROLLARY 4. *Under the above conditions we have*

$$(1.16)\qquad c_8(\rho_o)x^{\beta_o} \le D(x) < S(x) \le c_9(\rho_o)x^{\beta_o} .$$

Finally we mention another result, seemingly weaker than Theorem 2, which, however, has important applications in the problems discussed in the following section.

THEOREM 3. *If* $\zeta(\rho_o) = 0$, $x > \max(c_{10}, \exp(\sqrt{|\rho_o|})$, $B(x) = x\exp(-40\log_2^2 x)$ *then*

$$(1.17)\qquad D(x) > \frac{1}{x}\int_{B(x)}^{x} |\Delta(u)|du > x^{\beta_o}\exp(-60\log_2^2 x) .$$

2. Our further investigations deal with the assertion of Riemann [11]

$$(2.1)\qquad \Delta_1(x) \overset{\text{def}}{=} \pi(x) - \text{li}\,x \overset{\text{def}}{=} \sum_{p\le x} 1 - \int_0^x \frac{dt}{\log t} < 0 \qquad (x > 2)$$

stated without proof in 1859. Although generally believed to be true for more than 50 years (and checked up to $x = 10^7$) this was disproved by Littlewood [9] in 1914: he showed that $\Delta_1(x)$ infinitely often changes sign. His theorem was completely *ineffective* and it took more than 40 years to give the explicit upper bound $e_4(7.705)$ for the first sign change of $\Delta_1(x)$ (Skewes [12]).

S.Knapowski was the first who succeeded in furnishing a lower estimate for the number $V_1(Y)$ of sign changes of $\Delta_1(x)$ in the interval $[2, Y]$. Applying Turán's one-sided power-sum method he proved in 1961-62 [6, 7]

$$(2.2)\qquad V_1(Y) > c_{11}\log_4 Y \quad \text{for} \quad Y > c_{12}$$

and the weaker *ineffective* inequality

$$(2.3)\qquad V_1(Y) > \log_2 Y \qquad \text{for} \quad Y > Y_1$$

where the Y_i denote ineffective absolute constants. These results were improved in 1974-76 by Knapowski and Turán [8]. They showed by Turán's power-sum method that (2.2) and (2.3) remain true with the functions $c_{13}\log_3 Y$ and $c_{14}\log^{1/4} Y/\log_2^4 Y$. The author was able to replace the above functions by $c_{15}\log^{1/2} Y/\log_2 Y$ and $c_{16}\log Y/\log_2^3 Y$ [10, parts III - IV] using also Turán's method.

Making use of (1.17) (better to say, its analogoue where $\Delta(u)$ is replaced by $\Delta_1(u)$) we can now show the *ineffective*

THEOREM 4. *$\Delta_1(x)$ changes sign in the interval*

$$[Y\exp(-500\log_2^3 Y),\ Y] \quad \text{if} \quad Y > Y_2 \quad . \tag{2.4}$$

This implies trivially the *ineffective* lower bound $c_{17}\log Y/\log_2^3 Y$ for $V_1(Y)$. But this can also be shown *effectively*.

THEOREM 5. $V_1(Y) > \dfrac{\log Y}{500\log_2^3 Y}$ *for* $Y > c_{18}$.

We remark that Theorems 4 and 5, unlike all the earlier *effective* results of this kind, were proved independently from Turán's method. A suitable *effective* result of type (2.4) needs, however, Turán's method.

THEOREM 6. *$\Delta_1(x)$ changes sign in the interval*

$$[Y^{c_{19}},\ Y] \qquad \text{if} \qquad Y > c_{20} \quad . \tag{2.5}$$

Finally we remark that in a recent work J.Kaczorowski [4] announced the ineffective inequality $V_1(Y) > c_{21}\log Y$ for $Y > Y_3$. We also remark that Riemann's assertion (2.1) had not only empirical background but was supported also by some theoretical arguments. The assertion

$$\int_1^x \Delta_1(u)du < 0 \qquad \text{for} \qquad x > x_0 \tag{2.6}$$

is e.g. equivalent with RH. But it is interesting to note that there is a relatively simple averaging procedure such that the statement "$\pi(x)$ - li x is negative on the average" is true without any conditions.

THEOREM 7. $\displaystyle\int_1^\infty \Delta_1(x)\exp(-(\log^2 x)/y)dx \to -\infty$ as $y \to \infty$.

3. All results of Sections 1 and 2 are based on the investigation of the zeros of $\zeta(s)$ (although in the formulations of Theorems 1-3 only one zero and in Theorems 4-7 no zero appears). The aim of the present section is to examine the connection of the order of magnitude of $|\Delta(x)|$ with the distribution of zeros of $\zeta(s)$.

A general theorem of this type was obtained by Ingham [3, Theorem 22]:

Suppose $(s = \sigma + it)$

(3.1) $$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1-\eta(t)$$

where $\eta(t) \in C^1[2,\infty)$, $\eta'(t) \leq 0$, $\lim\limits_{t\to\infty} \eta'(t) = 0$, $\eta(t) \gg \log^{-1} t$.

Let $0 < \varepsilon < 1$ be fixed and

(3.2) $$\omega_{(\eta)}(x) = \inf_{t\geq 1} (\eta(t)\log x + \log t) .$$

Then

(3.3) $$\Delta(x) \ll x\exp(-\tfrac{1}{2}(1-\varepsilon)\, \omega_{(\eta)}(x)) .$$

This implies e.g. that in case of

(3.4) $$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{c_{22}}{\log^{\alpha} t}, \quad t > t_0$$

one has

(3.5) $$\Delta(x) \ll x\exp(-c_{23}(\alpha) \log^{1/(1+\alpha)} x) .$$

Turán [15, part II] was the first to show that the inverse implication (3.5) ⇒ (3.4) is also true (with a $c'_{22} < c_{22}/40$, however). His result was later extended to more general domains by W.Staś [14]. The author [10, part II] succeeded in showing that the factor 1/2 can be deleted in (3.3) and that the slightly stronger assumption

(3.5) $$\Delta(x) \ll x\exp(-(1+\varepsilon)\, \omega_{(\eta)}(x))$$

already implies (3.1) (for $t > t_0$) if $\eta(t) = g(\log t)$, where

(3.7) $$g(u) \in C^1(1, \infty), \quad g'(u) \nearrow 0 \quad \text{as} \quad u \to \infty .$$

The above results suggest that perhaps there is a real function $\omega(x)$ depending in a simple way on the distribution of zeta-zeros (without using a hypothetical zero-free region in the formulation of the results) which describes the largest possible order of magnitude of $|\Delta(x)|$. In the favourite case we may hope that such a function $\omega(x)$ determines the functions $S(x)$ or $D(x)$ (see (1.3) - (1.4)) with considerable accuracy. It turns out that this is really possible by choosing

$$(3.8) \qquad \omega(x) \overset{\mathrm{def}}{=} \min_{\rho} ((1-\beta)\log x+\log|\gamma|) = \log\frac{x}{Z(x)}$$

where

$$(3.9) \qquad Z(x) = \max_{\rho} \frac{x^{\beta}}{|\gamma|}$$

is, up to an insignificant factor $|\rho|/|\gamma|$, the modulus of the largest error term in the explicit formula (1.6).

TEHOREM 8. *Using the notations* (1.3) - (1.4), (3.8) *we have*

$$(3.10) \qquad \log\frac{x}{S(x)} \sim \log\frac{x}{D(x)} \sim \omega(x) \quad \textit{as} \quad x \to \infty .$$

Theorem 8 includes

THEOREM 9. $\Delta(x) \ll x\exp(-(1-\varepsilon)\,\omega(x))$

and

THEOREM 10. $S(x) > D(x) \gg x\exp(-(1+\varepsilon)\,\omega(x))$

Consequently $\Delta(x) = \Omega(x\exp(-(1+\varepsilon)\,\omega(x)))$.

Taking into account that in case of (3.1) we have trivially $\omega(x) \geq \omega_{(\eta)}(x)$ (cf. (3.2)), Theorem 9 implies

COROLLARY 5. *If* $\eta(t)$ *is an arbitrary real function and* (3.1) *is true then*

$$(3.11) \qquad \Delta(x) \ll x\exp(-(1-\varepsilon)\,\omega_{(\eta)}(x)) .$$

We remark that Theorem 8 also implies

COROLLARY 6. *Using the notation* (1.15) *we have*

$$(3.12) \qquad \log S(x) \sim \log D(x) \sim \log Z(x) \sim \Theta\log x .$$

(3.12) is a sharpening of the well-known relation

$$(3.13) \qquad \Theta = \inf\{\vartheta\ ;\ \Delta(x) = O(x^{\vartheta})\} .$$

Although (3.12) is equivalent with Theorem 8 if $\Theta < 1$, it is much weaker in the sense that the crucial case in proving Theorem 8 is just

$\Theta = 1$. Corollary 6 already follows from (1.5) and from the corresponding result for $D(x)$. In case of $\Theta = 1$ Corollary 6 yields only $\log(x/S(x)) = o(\log x)$ whilst Theorem 8 gives an asymptotic relation for this quantity, the function $\omega(x)$. Theorem 8 can be considered also as a far-reaching extension of Wiener's result [16] *interpreted in this context* as

$$(3.14) \qquad S(x) = o(x) \Leftrightarrow \lim_{x\to\infty} \omega(x) = \infty$$

(although the main point there was the method used).

Furthermore we remark that the fact that $\omega(x)$ *itself* describes the asymptotic behaviour of *both* functions $S(x)$ and $D(x)$ implies that the maximal value $S(x)$ cannot be much larger than the mean value $D(x)$, a phenomenon not discovered before. Namely, Theorem 8 yields

THEOREM 11. $S(x) \ll D(x)(x/D(x))^{\varepsilon}$.

The interesting feature of the above result is that, unlike in Theorems 8-10, no zeta-zeros occur in the formulation (however, a direct proof seems to be hopeless).

4. In what follows we shall sketch the proof of Theorem 8. (The details of proof, as well as the proofs of the other theorems will appear in a series of papers entitled "Irregularities of prime distributions".) According to the remark following Corollary 6 we shall restrict ourselves to the case $\Theta = 1$.

Concerning the upper estimate of $\Delta(x)$ we obtain by Carlson's density theorem [1]

$$(4.1) \qquad N(1-\varepsilon, T) = \sum_{\substack{\beta\geq 1-\varepsilon \\ |\gamma|\leq T}} 1 \ll_{\varepsilon} T^{4\varepsilon}$$

and so we have for every natural number n

$$(4.2) \qquad \sum_{\substack{\beta>1-\varepsilon \\ e^n<|\gamma|\leq e^{n+1}}} \frac{x^{\beta}}{|\gamma|} \ll_{\varepsilon} e^{4n\varepsilon} \max_{e^n<|\gamma|\leq e^{n+1}} \frac{x}{e^{((1-\beta)\log x+\log|\gamma|)}}$$

$$\leq e^{-n\varepsilon} \max_{e^n<|\gamma|\leq e^{n+1}} \frac{x}{e^{((1-\beta)\log x+\log|\gamma|)(1-5\varepsilon)}} .$$

This implies by $\omega(x) = o(\log x)$ $(\Leftrightarrow \Theta = 1)$

$$(4.3) \qquad S(x) \ll_{\varepsilon} x^{1-\varepsilon/2} + \sum_{n=1}^{\infty} e^{-n\varepsilon} \frac{x}{e^{(1-5\varepsilon)\omega(x)}} \ll_{\varepsilon} \frac{x}{e^{(1-5\varepsilon)\omega(x)}} .$$

The lower estimate in Theorem 8 is the consequence of the following Lemma, which can be proved by Turán's power-sum method.

LEMMA. *Let* $0 < \varepsilon < c_{24}$, $\zeta(\beta_o + i\gamma_o) = \zeta(1-\delta_o+i\gamma_o) = 0$, $\gamma_o > 0$, $\delta_o < \varepsilon^{10}$. *Then for* $x > \gamma_o^{1/\varepsilon^{10}}$ *we have*

$$(4.4) \qquad \bar{D}(x) = \frac{1}{x} \int_{x^{1-\delta_o}\gamma_o^{-\varepsilon}}^{x} |\Delta(u)|du \geq \frac{1}{(x^{\delta_o}\gamma_o)^{\varepsilon}} \cdot \frac{x^{\beta_o}}{\gamma_o} .$$

It is really easy to check that if $\Theta = 1 \Leftrightarrow \omega(x) = o(\log x)$ and $\omega(x) = \delta_o \log x + \log\gamma_o$ then the conditions of our Lemma are satisfied and so we obtain

$$(4.5) \qquad D(x) \geq \frac{1}{(x^{\delta_o}\gamma_o)^{\varepsilon}} \cdot \frac{x}{x^{\delta_o}\gamma_o} = \frac{x}{e^{\omega(x)(1+\varepsilon)}} .$$

In order to sketch the proof of the Lemma we introduce the notations $(\varepsilon_1 = \varepsilon/24)$

$$(4.6) \qquad \varphi = \delta_o \log x + \log\gamma_o = \alpha\log x , \quad k = 5\varepsilon_1^2\alpha\mu ,$$

where the real number μ will be chosen later so as to satisfy

$$(4.7) \qquad \mu \in [\log x - 6\varepsilon_1\varphi , \log x - 5\varepsilon_1\varphi] .$$

Using the well-known relations

$$(4.8) \qquad \int_1^{\infty} \Delta(u) \frac{d}{du}(u^{-s})du = \frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} \overset{\text{def}}{=} H(s) \qquad (\sigma > 1)$$

$$(4.9) \qquad (2\pi i)^{-1} \int_{(2)} e^{As^2+Bs}ds = (4\pi A)^{-1/2} \exp(-B^2/4A)$$

we obtain our basic identity

$$(4.10)\qquad U(\mu) \overset{\text{def}}{=} (2\pi i)^{-1} \int_{(2)} H(s+\rho_o)\, e^{ks^2+\mu s}\, ds$$

$$= (4\pi k)^{-1/2} \int_1^{\infty} \frac{\Delta(u)}{u^{1+\rho_o}} \exp\left(-\frac{(\mu-\log u)^2}{4k}\right)\left(-\rho_o+\frac{\mu-\log u}{2k}\right) du \quad .$$

Since the weight function gives small weights if we are far from e^{μ}, it is relatively easy to show that the simple estimate $\Delta(u) \ll u$ implies

$$(4.11)\qquad U(\mu) = \int_{e^{\mu-5\varepsilon_1\varphi}}^{e^{\mu+5\varepsilon_1\varphi}} + O(e^{-\varphi/5}) \ .$$

Further we obtain from (4.7) and from (4.10)

$$(4.12)\qquad \left|\int_{e^{\mu-5\varepsilon_1\varphi}}^{e^{\mu+5\varepsilon_1\varphi}}\right| \le \frac{\gamma_o}{\varepsilon_1}\left(\frac{e^{11\varepsilon_1\varphi}}{x}\right)^{1+\beta_o} \int_{xe^{-11\varepsilon_1\varphi}}^{x} |\Delta(u)|du$$

$$\le \frac{\gamma_o}{x^{\beta_o}}\ \varepsilon_1^{-1} e^{22\varepsilon_1\varphi} \cdot \overline{D}(x) \ .$$

In order to obtain a lower estimation for $|U(\mu)|$ by a suitable choice of μ we shift the line of integration in (4.10) to $\sigma = -1-\beta_o$, thereby obtaining

$$(4.13)\qquad U(\mu) = \sum_{\rho} e^{k(\rho-\rho_o)^2+\mu(\rho-\rho_o)} + O(x^{-1}) \ .$$

Further we can trivially estimate the total contribution of all zeros which do not belong to the set

$$(4.14)\qquad M = \{\rho;\ |\gamma-\gamma_o| \le \varepsilon_1^{-1}\ ,\quad \beta \ge \beta_o-\alpha\} \quad .$$

Thus we get (cf.(4.6))

$$(4.15)\qquad E(\mu) \overset{\text{def}}{=} \sum_{\rho\in M} e^{\{5\varepsilon_1^2\alpha(\rho-\rho_o)^2+\rho-\rho_o\}\mu} = \cup(\mu)+O(e^{-\varphi/2}) \quad .$$

Now $|E(\mu)|$ can be estimated from below (under a suitable choice of μ) by the continuous form of the second main theorem of Turȧn's power-sum theory [13], which asserts for arbitrary complex numbers $a_1, \dots a_n$ with $\mathrm{Re}\, a_1=0$ and for any real positive b, d the inequality

$$(4.16)\qquad \max_{b\le v\le b+d} \Big|\sum_{i=1}^{n} e^{a_i v}\Big| \ge \Big(\frac{d}{8e(b+d)}\Big)^n \quad .$$

(For this form see e.g. [10, part I], Theorem A of the Appendix). In any application of this theorem a crucial role is played by the value of n . In our case the estimate of Korobov-Vinogradov

$$(4.17)\qquad \zeta(1-h+it) \ll t^{c_{25}h^{4/3}} \log^{c_{26}} t \qquad (t > 2)$$

and Jensen's inequality leads to

$$(4.18)\qquad 1 \le n = |M| \le \frac{c_{27}}{\varepsilon_1\alpha}(\alpha^{4/3}\log\gamma_o + \log_2\gamma_o) \quad .$$

From (4.18) we can infer $n \le c_{28}\varepsilon_1^{-1}\alpha^{1/3}\varphi$ and applying (4.16) we obtain a value μ satisfying (4.7) such that

$$(4.19)\qquad |E(\mu)| \gtrsim \exp\Big(-c_{29}\log\frac{1}{\varepsilon_1\alpha}\cdot\varepsilon_1^{-1}\alpha^{1/3}\varphi\Big) \ge e^{-\varepsilon_1\varphi} \quad .$$

Consequently, by (4.15) we have

$$(4.20)\qquad |\cup(\mu)| \ge \frac{1}{2}e^{-\varepsilon_1\varphi} \quad .$$

Combining this with (4.11) - (4.12) we obtain the assertion of the Lemma.

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