Studies in Pure Mathematics
To the Memory of Paul Turán

Oscillatory properties of the remainder term of the prime number formula

by

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1. The first oscillatory result for the remainder term

$$\Delta(x) \overset{\text{def}}{=} \Psi(x) - x \overset{\text{def}}{=} \sum_{n \leqq x} \Lambda(n) - x \overset{\text{def}}{=} \sum_{p^m \leqq x} \log p - x \tag{1.1}$$

of the prime number formula was proved in the last century by PHRAGMÉN [3]. He showed that if Θ denotes the least upper bound of the real parts of the ζ-zeros, then

$$\Delta(x) = \Omega(x^{\Theta-\varepsilon})^{1} \tag{1.2}$$

for any $\varepsilon > 0$. At the same time the proof could not substitute (1.2) with any explicit inequality.

LITTLEWOOD [2] wrote from this problem in 1937 the following lines.[2]

"Those familiar with the theory of the Riemann zeta-function in connection with the distribution of primes may remember that the interference difficulty arises with the function

$$f(x) = \sum_{\rho} \frac{x^{\rho}}{\rho} = \sum \frac{x^{\beta+i\gamma}}{\beta+i\gamma}$$

(where the ρ's are complex zeros of $\zeta(s)$).

There exist proofs that if Θ is the upper bound of the β's (so that $\Theta = \frac{1}{2}$ if Riemann hypothesis is true) then $f(x)$ is of order at least $x^{\Theta-\varepsilon}$ in x. But these proofs are curiously indirect: if $\left(\Theta > \frac{1}{2} \text{ and}\right)$ we are given a particular $\rho = \rho_0$ for which $\beta = \beta_0 > \frac{1}{2}$ they

[1] The notation $f(x) = \Omega(g(x))$ means $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} > 0$.

[2] His $f(x)$ is essentially $-\Delta(x)$.

provide no explicit X depending only upon β_0, γ_0 and ε such that $|f(x)|>X^{\beta_0-\varepsilon}$ for some x in $(0, X)$. There are no known ways of showing (for any explicit X) that the single term $\frac{x^{\beta_0+i\gamma_0}}{\beta_0+i\gamma_0}$ of f is not interfered with by other therms of the series over the range $(0, X)$."

So LITTLEWOOD asked for an explicit Ω-type estimation of $\Delta(x)$ depending on a single zero only. This important theoretical problem was solved by TURÁN [7] in 1950 who proved the following

Theorem A (TURÁN). *If $\rho_0=\beta_0+i\gamma_0$ is a zero of $\zeta(s)$ with $\beta_0 \geqq \frac{1}{2}$, further*

$$T \geqq \max(c_1, \exp(|\rho_0|^{60})) \tag{1.3}$$

then

$$\max_{1\leqq x\leqq T} |\Delta(x)| > \frac{T^{\beta_0}}{|\rho_0|^{10\log T/\log_2 T}} \exp\left(-c_2 \frac{\log T \log_3 T}{\log_2 T}\right) \tag{1.4}$$

where c_1, c_2 are explicitly calculable absolute constants.

2. The second problem is the connection between zero-free regions of $\zeta(s)$ and the remainder term of the prime number formula. The usual way is that assuming a zero-free domain we prove an upper estimate for the remainder term. Such a general theorem is due to INGHAM [1] (Theorem 22):

Theorem (INGHAM). *Suppose that $\zeta(s)$ has no zeros in the domain*

$$\sigma > 1-\eta(|t|) \tag{2.1}$$

where $\eta(t)$ is, for $t \geqq 0$ a decreasing function having a continuous derivative $\eta'(t)$ and satisfying the following conditions

$$0<\eta(t)\leqq\frac{1}{2} \tag{2.2}$$

$$\eta'(t)\to 0 \quad as\ t\to\infty \tag{2.3}$$

$$\frac{1}{\eta(t)} = O(\log t) \quad as\ t\to\infty. \tag{2.4}$$

Let ε be a fixed number satisfying $0<\varepsilon<1$ and let

$$\omega(x) \stackrel{\text{def}}{=} \min_{t\geqq 1} (\eta(t)\log x + \log t). \tag{2.5}$$

[3] $\log_\nu(x)$ denotes the ν-times iterated logarithm function.

Then we have

$$\Delta(x) = O\left(\frac{x}{e^{1/2(1-\varepsilon)\omega(x)}}\right). \tag{2.6}$$

If we choose here

$$\eta(t) = \frac{c_3}{\log^{\beta}(t+2)} \tag{2.7}$$

we get the following

Corollary. *If* $\zeta(s) \neq 0$ *in the domain*

$$\sigma > 1 - \frac{c_3}{\log^{\beta}(t+2)} \tag{2.8}$$

then

$$\Delta(x) = O(x \exp(-c_4 (\log x)^{\frac{1}{1+\beta}})). \tag{2.9}$$

Again a very important theoretical problem is (restricting ourselves for the special case (2.7)) whether it is possible (perhaps with finer analytic methods) to deduce from the zero-free region (2.8) a better estimate for the remainder term than given by (2.9). An equivalent formulation of the problem is whether assuming the upper estimate (2.9) for the remainder term we get a domain of type (2.8) to be zero-free. With other words: assuming there are perhaps infinitely many zeros in the domain (2.8) — is it possible to prove (2.9) with Ω instead of O. These problems (in the special case (2.7)) were affirmatively answered in 1950 by the following theorem of P. Turán [8].

Theorem B (Turán). *If for a* β *with* $0 < \beta < 1$ *we have*

$$\Delta(x) = O(x \exp(-c_5 (\log x)^{\frac{1}{1+\beta}})) \tag{2.10}$$

then $\zeta(s) \neq 0$ *in the domain*

$$\sigma > 1 - \frac{c_6}{\log^{\beta}(|t|+2)}, \quad |t| \geqq c_7(\beta). \tag{2.11}$$

An equivalent formulation of this is:

Theorem B′ (Turán). *If for a* β *with* $0 < \beta < 1$ *there are infinitely many* ζ*-zeros in the domain*

$$\sigma > 1 - \frac{c_8}{\log^{\beta}(|t|+2)} \tag{2.12}$$

then

$$\Delta(x) = \Omega(x \exp(-c_9 (\log x)^{\frac{1}{1+\beta}})). \tag{2.13}$$

Both Theorems A and B of TURÁN were proved by his powersum method, the main tool being the so-called second main theorem of the powersum theory (this will be formulated explicitly later). The treatment of both theorems were separate but similar. As he mentioned in his book [9] (after Satz XXX) from Theorem A it is possible to deduce a theorem of type Theorem B′, namely under the suppositions of Theorem B′ one gets

$$\Delta(x)=\Omega\left(x\exp\left(-c_{10}\frac{\log x}{(\log_3 x)^{\beta}}\right)\right) \tag{2.14}$$

which is however much weaker than (2.13). This is the reason why a separate treatment of Theorem B is needed.

3. As an improvement of Theorem A we shall prove

Theorem 1. *If* $\rho_1=\beta_1+i\gamma_1$ *is a* ζ*-zero with* $\beta_1 \geqq \frac{1}{2}$, $\gamma_1>0$, *then for*

$$T \geqq \max(\gamma_1^{400}, c_{11}) \tag{3.1}$$

there exists an $x\in[T^{1/4}, T]$ *for which*

$$|\Delta(x)| > \frac{c_{12}x^{\beta_1}}{\gamma_1^{50}}. \tag{3.2}$$

A comparison with Theorem A shows that assuming a much weaker condition we get a much better lower bound for $|\Delta(x)|$, and even the proof is more simple. A further advantage of Theorem 1 is that applying this instead of Theorem A for the conversion of INGHAM's theorem we get Theorem B to be valid even for very general domains and the deduction from Theorem 1 will be very simple. So Theorem 1 makes possible the unique and simple treatment of the two phenomenons dealt in Theorems A and B of TURÁN.

Theorem 2. *Suppose* $\zeta(s)$ *has an infinity of zeros in*

$$\sigma \geqq 1-\eta(t) \tag{3.3}$$

where $\eta(t)$ *is, for* $t\geqq 0$ *a continuous decreasing function and let*

$$\omega(x) \stackrel{\text{def}}{=} \min_{t\geqq 0}(\eta(t)\log x+\log t). \tag{3.4}$$

Then we have

$$\Delta(x)=\Omega\left(\frac{x}{e^{54\omega(x)}}\right) \tag{3.5}$$

where the constant implied by the Ω-symbol is explicitly calculable (in fact it is equal to the constant c_{12} appearing in Theorem 1).

Theorem 2 shows that the conversion of INGHAM's theorem is true in an explicit form for more general domains than dealt in INGHAM's theorem. However, we may note that supposing for $\eta(t)$ only that it is a continuous decreasing function, (2.6) is true even with the sharper estimate

$$\Delta(x)=O\left(\frac{x}{e^{(1-\varepsilon)\omega(x)}}\right) \tag{3.6}$$

and thus the suppositions (2.2)–(2.4) and that $\eta(t)$ has a continuous derivative can be omitted in the formulation of INGHAM's theorem (see [5], Theorem 1).

The proof of Theorem 1 is based on TURÁN's method, more precisely on the second main theorem of the powersum theory which we state here as

Theorem C (T. SÓS–TURÁN). *For arbitrary complex numbers z_j and $m>0$ the inequality*

$$\max_{m<\nu\leqq m+n} \frac{\left|\sum_{j=1}^{n} z_j^{\nu}\right|}{|z_1|^{\nu}} \geqq \left[\frac{1}{8e\left(\frac{m}{n}+1\right)}\right]^n \tag{3.8}$$

holds.

The proof is contained in [6]. If we choose here $m=a\frac{n}{d}$; $z_j=e^{\alpha_j\frac{u}{m}}=e^{\alpha_j\frac{d}{n}}$ we get

$$\max_{\frac{n}{d}a<\nu\leqq(a+d)\frac{n}{d}} \frac{\left|\sum_{j=1}^{n} e^{\alpha_j\frac{d}{n}\nu}\right|}{|z_1|^{\nu}} \geqq \left[\frac{1}{8e\left(\frac{a}{d}+1\right)}\right]^n \tag{3.9}$$

From this we get immediately the continuous form of the second main theorem, which we formulate as

Theorem C′ (T. SÓS–TURÁN). *For arbitrary complex numbers α_j, and for $a, d>0$ the inequality*

$$\max_{a<t\leqq a+d} \frac{\left|\sum_{j=1}^{n} e^{\alpha_j t}\right|}{|e^{\alpha_1 t}|} \geqq \left[\frac{1}{8e\left(\frac{a}{d}+1\right)}\right]^n \tag{3.10}$$

holds.

Finally we note that with far more complicated arguments but based also on TURÁN's method one can show the following sharper results:

Theorem I. *If $\rho_0 = \beta_0 + i\gamma_0$ is a non-trivial zero of $\zeta(s)$, $\varepsilon > 0$ and $T > c(\rho_0, \varepsilon)$ (effective lower bound depending on ρ_0 and ε), then there exist*

$$x_1, x_2 \in [T, T^{50 \log \gamma_0}] \tag{3.11}$$

such that the inequalities

$$\Delta(x_1) > (1-\varepsilon)\frac{x_1^{\beta_0}}{|\rho_0|} \tag{3.12}$$

and

$$\Delta(x_2) < -(1-\varepsilon)\frac{x_2^{\beta_0}}{|\rho_0|} \tag{3.13}$$

hold.

Theorem II. *If $\zeta(s)$ has infinitely many zeros in the domain*

$$\sigma > 1 - g(\log t) \tag{3.14}$$

where $g(u)$ is a continuous decreasing function and $g'(u) \nearrow 0$, $0 < \varepsilon < 1$, then

$$\Delta(x) = \Omega_{\pm}\left(\frac{x}{e^{(1+\varepsilon)\omega(x)}}\right) \tag{3.15}$$

with the $\omega(x)$ function defined by (3.4).

Theorem I is essentially optimal concerning the lower estimate, only the localisation in (3.11) is weaker than in Theorem 1. In Theorem II one has to require stronger conditions for the domain, i.e. for the $\eta(t) = g(\log t)$ function, but the Ω-type estimate is again essentially optimal in view of (3.6). A further advantage is that it gives $\Omega_{\pm}$ results, i.e. Theorems I and II assure "big positive" and "big negative" values too for the remainder term. The more elaborated proofs of Theorems I and II will appear in [4] and [5] resp.

4. For the indirect proof of Theorem 1 let μ be a real number, to be chosen later for which

$$\frac{\log T}{2} \leqq \mu \leqq \frac{2\log T}{3} \tag{4.1}$$

and let

$$k \overset{\text{def}}{=} \frac{\mu}{20}. \tag{4.2}$$

We shall start with the formula

$$\int_1^{\infty} \Delta(x)\frac{d}{dx}(x^{-s})\,dx = \frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} \overset{\text{def}}{=} H(s), \tag{4.3}$$

which can be proved easily by partial integration. Using the well-known formula

$$\frac{1}{2\pi i}\int\limits_{(2)} e^{As^2+Bs}\,ds = \frac{1}{2\sqrt{\pi A}}\exp\left(-\frac{B^2}{4A}\right), \tag{4.4}$$

which is valid for real $A>0$ and arbitrary complex B we get from (4.3)

$$\begin{aligned}
U &\overset{\text{def}}{=} \frac{1}{2\pi i}\int\limits_{(2)} H(s+i\gamma_1)e^{ks^2+\mu s}\,ds = \frac{1}{2\pi i}\int\limits_{(2)}\int\limits_{1}^{\infty} \Delta(x)\frac{d}{dx}(x^{-s-i\gamma_1}e^{ks^2+\mu s})\,dx\,ds = \\
&= \int\limits_{1}^{\infty} \Delta(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\frac{1}{2\pi i}\int\limits_{(2)} e^{ks^2+(\mu-\log x)s}\,ds\right\}dx = \\
&= \int\limits_{1}^{\infty} \Delta(x)\frac{d}{dx}\left\{x^{-i\gamma_i}\frac{1}{2\sqrt{\pi k}}\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\right\}dx = \\
&= \frac{1}{2\sqrt{\pi k}}\int\limits_{1}^{\infty}\frac{\Delta(x)}{x}x^{-i\gamma_1}\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\left\{-i\gamma_1+\frac{\mu-\log x}{2k}\right\}dx.
\end{aligned} \tag{4.5}$$

We split the integral U in (4.5) into three parts:

$$U_1 = \int\limits_{1}^{e^{\mu/2}}, \quad U_2 = \int\limits_{e^{\mu/2}}^{e^{3\mu/2}}, \quad U_3 = \int\limits_{e^{3\mu/2}}^{\infty} \tag{4.6}$$

Using $|\Delta(x)|<x$ (if $x>x_0$) for $T>T_0$ we get

$$\begin{aligned}
|U_3| &< \gamma_1\int\limits_{e^{\mu+10k}}^{\infty}\exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\left(\frac{\log x-\mu}{2k}-1\right)dx = \\
&= \gamma_1\int\limits_{10k}^{\infty}\exp\left(-\frac{r^2}{4k}\right)\left(\frac{r}{2k}-1\right)e^{r+\mu}\,dr = \gamma_1 e^{\mu+10k-\frac{100k^2}{4k}} = \gamma_1 e^{\frac{\mu}{4}}
\end{aligned} \tag{4.7}$$

and analogously for $T>T_0$ we get

$$|U_1|<\gamma_1 e^{\frac{\mu}{4}}. \tag{4.8}$$

Now assuming Theorem 1 to be false we have

$$|U_2| < \frac{(\gamma_1+5)c_{12}}{2\sqrt{\pi k}\gamma_1^{50}} \int_{e^{\mu-10k}}^{e^{\mu+10k}} x^{\beta_1} \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\frac{dx}{x} <$$

$$(4.9) \qquad < \frac{c_{12}}{\sqrt{\pi k}\,\gamma_1^{49}} \int_{-10k}^{10k} \exp\left((\mu+r)\beta_1 - \frac{r^2}{4k}\right) dr <$$

$$< \frac{e^{\mu\beta_1+k\beta_1^2}c_{12}}{\sqrt{\pi k}\,\gamma_1^{49}} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{r}{2\sqrt{k}} - \beta_1\sqrt{k}\right)^2\right\}dr = \frac{2c_{12}}{\gamma_1^{49}} e^{\mu\beta_1+k\beta_1^2}$$

Taking in account (3.1), (4.6)–(4.9) imply

$$(4.10) \qquad |U| < \frac{3c_{12}}{\gamma_1^{49}} e^{k\beta_1^2+\mu\beta_1}.$$

5. Shifting the line of integration in (4.5) to $\sigma = -\frac{1}{2}$ we get

$$(5.1) \qquad U = \sum_{\rho} e^{k(\rho-i\gamma_1)^2+\mu(\rho-i\gamma_1)} + \frac{1}{2\pi i} \int_{(-\frac{1}{2})} H(s+i\gamma_1)e^{ks^2+\mu s}\,ds.$$

Estimating here the integral I trivially we have

$$(5.2) \qquad |I| = O(\log \gamma_1 e^{\frac{k}{4}-\frac{\mu}{2}}) = O(1)$$

since $|H(s+i\gamma_1)| = O(\log(|t+\gamma_1|+2))$.

Further the contribution of zeros with $|\gamma-\gamma_1| \geqq 4$ to the infinite powersum is

$$(5.3) \qquad O\left(\sum_{n=4}^{\infty} \log(\gamma_1+n)e^{k(1-n^2)+\mu}\right) = O(\log \gamma_1 e^{\mu-15k}).$$

Applying Jensen's inequality for the circle $|s-(3+i\gamma_1)| \leqq 10$ we get for the number n of zeros with $|\gamma-\gamma_1| < 4$

$$(5.4) \qquad 1 \leqq n \leqq \frac{7.5\log\gamma_1 + c_{13}}{\log 2} < 10.83\log\gamma_1 + c_{14}.$$

Thus we get from Theorem C′ the existence of a μ, satisfying (4.1) for which

$$\left|\sum_{|\gamma-\gamma_1|<4}\left\{e^{\frac{1}{20}(\rho-i\gamma_1)^2+(\rho-i\gamma_1)}\right\}^{\mu}\right| \geqq \frac{e^{k\beta_1^2+\mu\beta_1}}{(32e)^{10.83\log\gamma_1+c_{14}}} > c_{15}\frac{e^{k\beta_1^2+\mu\beta_1}}{\gamma_1^{48.4}}. \tag{5.5}$$

Owing to (3.1) the formulae (5.2), (5.3), (5.5) imply

$$|U| > c_{15}\frac{e^{k\beta_1^2+\mu\beta_1}}{\gamma_1^{49}} \tag{5.6}$$

which contradicts to (4.10) if we choose $c_{12} = \frac{c_{15}}{3}$.

6. We note that Theorem 2 trivially follows from Theorem 1 (applied with the zero ρ_0 of minimal imaginary part, $\rho_0 = \frac{1}{2}+i\gamma_0 \approx \frac{1}{2}+14.13i$) if $\varliminf_{t\to\infty}\eta(t)\geqq 0.01$ since in this case $\omega(x)\geqq 0.01\log x$ and thus $xe^{-54\omega(x)}\leqq x^{0.46}$.

If $\varliminf_{t\to\infty}\eta(t)<0.01$, let $\rho_n=\beta_n+i\gamma_n$ $(0<\gamma_1<\gamma_2<\dots)$ be an infinite sequence of zeros with

$$\beta_n > 1-\eta(\gamma_n) \tag{6.1}$$

and let T_n be the unique real number defined by

$$\eta(\gamma_n)\log\sqrt[4]{T_n} = \log\gamma_n. \tag{6.2}$$

Then for $n>n_0(\eta)$ (3.1) is satisfied for ρ_n and T_n and therefore we have an $x_n\in[\sqrt[4]{T_n}, T_n]$ for which by Theorem 1

$$\begin{aligned}\frac{|\Delta(x_n)|}{x_n} &> \frac{c_{12}}{x_n^{1-\beta_n}\cdot\gamma_n^{50}} \geqq \frac{c_{12}}{e^{\eta(\gamma_n)\log x_n+50\log\gamma_n}} \geqq \\ &\geqq \frac{c_{12}}{e^{4\eta(\gamma_n)\log\sqrt[4]{T_n}+50\log T_n}} \geqq \frac{c_{12}}{e^{\omega(\sqrt[4]{T_n})\cdot 54}} \geqq \frac{c_{12}}{e^{\omega(x_n)\cdot 54}},\end{aligned} \tag{6.3}$$

since $\omega(x)$ is trivially monotonically increasing, further by (6.2)

$$\begin{aligned}\omega(\sqrt[4]{T_n}) &\geqq \min_{t\geqq 0}\{\max(\eta(t)\log\sqrt[4]{T_n}, \log t)\} \geqq \\ &\geqq \eta(\gamma_n)\log\sqrt[4]{T_n} = \log\gamma_n.\end{aligned} \tag{6.4}$$

(6.3) obviously proves Theorem 2.

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